When is a Symbol Symbolic?

JOHN MASON

These notes are written as if to a teacher on an inservice course about developing mathematical thinking who missed a session I have deliberately used a didactic rather than academic style in order to try to keep my feet on the ground by talking directly to experience That experience is provided by questions embedded in the notes, which *must* be attempted. In a classroom I pause and get people to work on them so that we can enrich our discussions about thinking, by direct reference *to* thinking

The questions I am addressing concern the roles of symbols, the generally absent icons that should support those symbols, and the mathematical processes of conjecturing and proving, or in a slightly more refined form, specializing, generalizing and reasoning The particular points I am trying to make have proved useful in classes in Thinking Mathematically in Canada and the U K.

Seeking patterns

> *In a warehouse you receive a 20% discount but you must pay 15% Value Added (sales) Tax. Which would you prefer to have calculated first, discount or VAT?*

How do you get to grips with such a question? The only sensible thing to do is to try it on some specific examples of prices. I hope that you have already done this, but if not,

DO SO NOW!

Surprised? Most people are, and it is that surprise which fuels mathematical thinking. The real question is whether it will always work, or whether there was something special about the prices you chose. Try some more examples, this time with an eye to trying to get a sense of the underlying pattern or structure.

A lot depends on the form in which you do your calculations The usual approach is to

— calculate the discount
— subtract the discount to get the discounted price
— calculate the VAT on the discounted price
— add the VAT to the discounted price to get the final price and similarly in reverse order

Try to find another way which reveals why it works You are looking for a form which is independent of the initial price This suggests calculating what percentage of a quoted price you pay when granted a 20% discount, and what percentage you pay when VAT is included.

With any luck you will have found that:
(i) subtracting 20% is the same as paying 80%, or 0.80 times the price,
(ii) adding 15% is the same as paying 115%, or 1 15 times the price.

Thus for any quoted price, say £100,

Discount first: you pay (100) (0 80) (1.15)
VAT first: you pay (100) (1.15) (0.80)

From this shape you can see that the initial quoted price is irrelevant to the order of calculation, since (0.80) (1 15) = (1.15) (0.80), or more technically, multiplication is commutative.

The point of going through this example in some detail is to focus attention on key processes at the heart of mathematical thinking:

SPECIALIZATION — *doing specific examples to try to find out what is meant and get a sense of what is going on*
GENERALIZATION — *trying to articulate the underlying general pattern*
REASONING — *producing an argument to verify that your articulation of the general pattern is valid.*

In the VAT question, enough specific prices had to be used to suggest that the end price is independent of the order of calculation. Articulating this then introduces a second phase of looking at examples to try to discover why the calculations always come out the same. Articulating this "why" very often suggests how to lay out the reasoning In this case the separation of the 0.80 and 1 15 leads to observing that (0.80) (1 15) = (1 15) (0 80) The fact that the order of calculation is independent of the quoted price is often expressed symbolically as:

For any quoted price P, the final price will be
$P(0\ 80)\ (1\ 15) = P(1.15)\ (0.80)$
so the order of calculation does not matter.

Notice that the final argument represents the crystallizing of experience on examples, but these examples are not mentioned. It is precisely for this reason that it is useful to be explicit about SPECIALIZING, since most written mathematics hides the pattern-revealing examples.

Before going on to another example, I hope you have already asked yourself what happens if the discount or VAT rate changes. Express it symbolically

Finger computing

> *Asked to multiply 9 by a single digit number like 7, I open my palms towards me. I fold down the seventh finger from the left, and read off a block of six figures and a block of three fingers as 63.*

I hope that there is enough in this description to have arise inside you the question 'What is going on?'. HAVE YOU SPECIALIZED — that is tried more examples? In order to do this, you must first GENERALIZE from the specific number 7 to the role of 7 and formulate a general rule, which must then be checked. Of course you can quite easily try all possible cases, but even so you will not have answered the question of why it works

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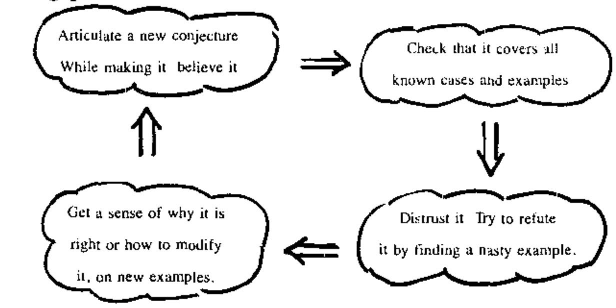
What is the general pattern or feature which makes finger computing work? A number of ideas may spring to mind from past experience. Perhaps you knew that the digits of 9 times x add to 9 if x is a single digit. Ultimately you may come to place value, and noticing that adding 9 is the same as adding 10 and subtracting one. The process is the same as with VAT: doing examples and looking for different forms to describe the examples until one form reveals an underlying structure.

Generalizing takes several forms. Initially, in this example, a general rule was inferred, then an underlying pattern was sought. Later the question 'What If ...' needs to be asked to try to set the question in a more general context. In this case there are several 'What If's ...

What if I want to multiply 9 by two digit numbers?
What if I use my toes?
What if I want to multiply by some other number like 6 instead of 9?

Singly and in combination these 'what if' questions open a door to other bases (hinted at by place value), and to articulating more advanced rules involving toes.

Specializing also arises several times in any question of reasonable difficulty. Not only is it useful to examine specific examples in order to try to understand a question, but whenever a conjecture is articulated, it should be checked against some examples before trying to find out how an argument might go. You should also try some nasty looking examples, that is, examples which might refute your conjecture. Conjecturing is a sort of swaying or oscillating process:



Trying to refute a conjecture is a very useful way of finding holes in your conjecture, as well as revealing why the conjecture might be valid. The examples we have looked at so far have been too straightforward to illustrate this process — indeed no question is certain of demonstrating it because it depends on how carefully and systematically you specialize while seeking pattern. The next example might be helpful in this respect.

Consecutive sums

Observe that $3 = 1 + 2$

$5 = 2 + 3$

$6 = 1 + 2 + 3$

$9 = 4 + 5 = 2 + 3 + 4$

What numbers can be expressed as the sum of consecutive positive integers. What ones can be done in more than one way?
In how many ways?

Systematic calculations present you with an obvious pattern, indeed several patterns, though you probably found that "it will be clearer if I do a few more examples". Emphasis should be placed on "systematic", which is often an essential ingredient in useful specializing. In this case at least two systematizing ideas are suggested. Trying 1, 2, 3, 4, 5, ..., and writing down all numbers that are the sum of two consecutive sums, three consecutive sums and so on. Answering the final question, in how many ways, is rather more challenging than the previous questions. It leads us into the domain of proof and reasoning, which I will postpone to another session. I hope however that you found yourself making little guesses — conjectures, and that, at least once, further examples caused you to modify your conjecture. If not — do some more examples, systematically!

The examples of specializing have so far been of a straightforward kind — routinely trying specific cases — but a lot more is lying beneath the surface. I want now to explore the relation between specializing/generalizing, and Bruner's three modes of representation. This will lead me to the distinction between symbol and symbolic representation which is the aim of this session.

In the questions posed so far, I have assumed that you have a natural, immediate impulse to turn to specific examples, a process called specialization. I have often noticed that faced with more difficult questions, questions which involve ideas or notation that are not entirely familiar, many people seize up, staring at the paper in front of them but with unfocused attention. A possible example is the following, though its effectiveness depends on your experience of the symbols.

Zeller's congruence

To determine the day of the week on which a given date falls, compute

$$\{[2.6m - 0.2] + d + y + [y/4] + [c/4] - 2c\} \text{ modulo } 7$$

where d = *day of month*
m = *month number* (*March* = 1, *February* = 12)
y = *year*
c = *century*
Sunday = *day* 0
and $[x]$ *means the greatest integer less than or equal to* x

If you are not totally familiar and confident with $[x]$, then it seems to me that the obvious thing to do is to ignore the question, and become familiar with $[x]$ by doing examples. In other words, you turn from something unfamiliar which induces queaziness, to examples involving entities with which you are fully familiar. You might therefore consider:

$[3.1] = 3$
$[3.2] = 3$ looks straightforward
$[\pi] = 3$
$[3] = 3$
$[-\frac{1}{2}] = -1$ AHA!
$[-1] = -1$
$[-\pi] = -4$
o.k, I think I see it now

This is only successful because 3.1, π, $-\frac{1}{2}$, etc., are totally confidence-inspiring, concrete entities for you.

Having become happier with an embedded concept or symbol, you can turn to the formula itself and try it out. Today's date seems the most sensible, and perhaps the meaning of 'modulo 7' will become clearer with the example. Today's date for me is 12/3/80, so, being tidy so that I can see the pattern, I write

$d = 12$
$m = 1$ (that's convenient!)
$y = 1980$
$c = 20$

I want $\{[2.6 \times 1 - 0.2] + 12 + 1980 + [1980/4] + [20/4] - 2 \times 20\}$ modulo 7
$= \{[2.4] + 1992 + 495 + 5 - 40\}$ modulo 7
$= \{2454\}$ modulo 7
= remainder on dividing 2454 by 7
= 4 modulo 7
= Wednesday (which is correct! phew!)

I am now thoroughly intrigued as to why it works (it appears to!). What should I do now to investigate?

Having done one example (today's date), and being lazy, I am curious to see how it copes with tomorrow's date. This way I can hope to see if it works and how. I am seeking the underlying pattern. Quickly I find that the significant feature is how it copes with months of varying length, since advancing d by one is going to keep it perfectly in step until d suddenly flips back to one at the end of March. I leave further investigation to you.

Icons and symbols

Looking back at the processes involved, it seems to me that they can be described as a cyclic process of doing examples to get a sense of an underlying pattern, trying to articulate that sensed pattern, then testing or checking the articulated pattern to see if it fits, modifying or building upon this until reaching both a sense of "I see" and an adequate articulation of what is seen. Specifically in the case of Zeller's congruence, this meant seeing that it works in one random instance, then that it will continue to work as long as d advances; then seeing how the change of month is coped with, then the change of year, and finally the change of century. Even then, despite seeing how it all fits together, there may be lingering doubts about the formula as a whole, so to appreciate it fully, I try to modify it or improve it in some way. Can I change the formula neatly to get rid of the large numbers in twentieth century calculations?

I hope that a similar sequence occurred when you looked at "Consecutive sums". Doing examples leads to a developing pattern which seems to be there but is hard to capture. Successive articulations are quickly contradicted and need to be modified (unless of course the underlying pattern arises inside you). This oscillating can be very frustrating if you are determined to solve the problem, but exciting and stimulating if you view it as exploration into unknown territory. Do not forget that an underlying pattern may be very subtle, or may not even exist!

I wish to draw your attention specifically to the search, while doing examples, for a SENSE of underlying pattern. Vastly underrated and usually ignored, it is only a sense of pattern which can eventually come to articulation. As you know, I put a good deal of stress on achieving a sense of pattern or structure whenever I introduce a new topic, and it is the same idea in thinking about problems. Indeed it is part of THINKING of any kind.

The description of conjecturing, involving specializing and generalizing, conforms very closely with some ideas of Bruner, and now it is appropriate to make that more precise. To do that, let me summarize the conjecturing process in slightly different words.

1 Do examples (Specialize) using entities with which you are entirely confident, which you can manipulate easily while part of your attention remains focused on your primary goal.
2 Try to get a sense of underlying pattern or relationship. Often diagrams or metaphors will help here — have you seen a similar problem or idea? an analogous one?
3 Try to articulate the pattern you sense. Keep refining the articulation until it can be checked on examples (back to 1 again).

In this form it begins to look very much like Bruner's use of ENACTIVE, ICONIC and SYMBOLIC.

Bruner found it useful to distinguish three modes of internal representation which seem to describe stages in children's thinking. When asked a question, children seem to make use of the following internal representations:

ENACTIVE — *able to respond only by recourse to previous practical experience. The classic example is a number question for which the child turns to a balance and physically performs the required acts. Here the response is by the musculature.*

ICONIC — *able to respond by recourse to mental images of physical objects or to an inner sense of pattern or structure. In the case of numbers, having a balance in sight, or a drawing, can assist the work by extending the mental screen. Icons need no articulation because within a culture they need no definition.*

SYMBOLIC — *able to respond by using abstract symbols whose meaning must be articulated or defined. In the case of number, $3 + 4 = 7$ now has meaning, and no recourse to the balance or balance image is needed.*

Because Bruner was looking at stages in children's development, giving a slightly different perspective to Piaget's work, people seem to have identified

ENACTIVE with physical toys
ICONIC with drawings and pictures
SYMBOLIC with words and letters

or, worse,

ENACTIVE with primary school
ICONIC with middle school
SYMBOLIC with upper school

and missed the essential qualities which I describe as

ENACTIVE — confidently manipulable
ICONIC — having a sense or image of
SYMBOLIC — having an articulation of.

Notice too that SYMBOLIC expression must ultimately become ENACTIVE if the idea is to be built upon or become a component in a more complex idea. Thus to a pre-school child 1, 2, 3 are truly symbolic, having little or no meaning. With time and extensive encounters a sense of one-ness and two-ness develops which underpins the symbols and provide a source of meaning when 1, 2 and 3 are encountered in a new context. To proceed with arithmetic it is essential that 1, 2, 3 become enactive elements, become friends. If they remain as unfriendly symbols then arithmetic must be a source of great mystery.

Moving along the spiral in which ENACTIVE elements provide an ICONIC representation of some pattern or relationship, to a SYMBOLIC articulation, to ENACTIVE elements and so on, consider the role of "symbols" like x and y. In the content of number work x and y begin by acting as placeholders for some yet-to-be-determined numbers. Without a sense of general pattern, as in the VAT example, and a wish to record a general pattern, the use of x and y is quite mysterious. Further manipulation of these in simple equations already assumes that x and y have become, or are well on the way to becoming, enactive elements representing an arbitrary number, or an unknown number.

The same spiral continues as students encounter sets of numbers described by properties, for example $\{2x + 1: x \text{ an integer}\}$, functions on those sets, sets whose elements are sets, sets whose elements are functions, functions whose domains are functions or even sets of functions, and so on. In other words, the E-I-S spiral is relevant to presenting mathematics at all levels. The relevance is based on releasing the word ENACTIVE to describe elements which are confidently manipulable, and the word ICON to describe mental images, feelings or intuitions which are pre-articulate. I suggest that in the problems you work on, and in mathematics classes, you try to identify these modes of representations — indeed they are almost modes of confidence or security. Having begun to distinguish them, you can then make use of the distinctions to trigger suggestions for useful activity:

Turning to enactive elements to explore the meaning of symbols or concepts; Using enactive elements to try to get a sense of pattern; Asking for images, metaphors, diagrams to illustrate what is going on; Crystallizing understanding in symbolic form; Practising with examples to move the symbolic form into enactive elements.

Of course the same ideas impinge on your own classroom behaviour where you can decide to put emphasis on "getting a sense of" before pushing your students to recording in symbolic form.

When is a symbol symbolic?

Now I can address the main question of this session. A symbol in common parlance is hard to define in the abstract. Its primary quality is that it requires explanation because its meaning is not immediately clear. In Zeller's congruence, d, m, y and c can be guessed at, but only if you already know what the expression is trying to compute. Computer programmes have moved away from the mathematicians predeliction with single letters towards short forms which require less explanation. For example,

$$\{[2.6\ \text{Month} - 0.2] + \text{Day} + \text{Year} + [\text{Year}/4] + [\text{Century}/4] - 2\ \text{Century}\}$$

seems clearer, but if extensive manipulation is called for, then m, d, y and c require less ink and time. More importantly, they reduce the clutter and make complicated manipulation tenable. However, as an intermediate stage in moving from symbolic representation to enactive elements, the longer expression seems helpful. It seems to me that this example captures precisely the difference between symbol and symbolic representation. A symbol is symbolic if it describes or expresses or stands for an idea but has not yet become an enactive element. It can only become an enactive element if it has meaning, in other words, if there is associated with it at least one icon, an image, metaphor, picture or sense with itself captures a pattern or relationship. Thus the state of being symbolic is highly relative.

How does the awareness of the relativity of symbolic experience help us as thinkers or as teachers? I have already suggested that the transitions from ENACTIVE to ICONIC, ICONIC to SYMBOLIC, and SYMBOLIC to ENACTIVE need care and preparation. Rapid movement to symbolic expression without the support of icons to fall back on can only result in trouble later. I suspect most people who study mathematics recognise the feeling of mental saturation as new ideas and symbols arise in a seminar or lecture. Awareness of E-I-S can help you as teacher to be more helpful in presenting mathematical ideas. I find that the transitions are the significant states to look for. They are very similar to physical phase transitions like ice to water and water to steam. There come times when energy in the form of a strong wish to find out "what is going on" generates lots of examples and questions, but nothing seems to come of it, and this seems to correspond to the physical events when energy pumped in goes not towards visible rise in temperature, but towards change of state.

Not only does E-I-S provide a framework for thinking about mathematical presentation, it also provides an underpinning for the processes of specializing and generalizing. By making these explicit, and bearing E-I-S in mind, new perceptions of your own thinking activity becomes possible. The only proof of this is for you to try it. Certainly it has been true for me.

Finally, let me draw your attention to the role of attention. If you are tackling Zeller's congruence without having met it before, then it is probable that all of your attention focuses on those square brackets. If this happens, there is no spare attention to be providing an overall direction, and to be watching out for unexpected emerging patterns. Turning to enactive elements is natural (though people seem to need a lot of prodding!) precisely because manipulation of enactive elements does not require full attention. Instead, there is sufficient spare attention to be seeking patterns. Try bearing this in mind as you watch a young child labour to write down a question, unable to think about anything but the process of writing the words Try bearing it in mind when you invite students to solve physics problems requiring ratio and proportion as well as physical ideas. In many cases of mathematical paralysis, some component skill requires full attention, thus derailing the overall plan of attack. In other words, too many elements of the problem are symbolic not enactive Being only vaguely aware of this, the student remains frozen. By bringing these matters to the student's attention, other options become available.

I end, as usual with a problem. This one seems to me to illustrate pretty fully the qualities of specializing and generalizing and E-I-S transitions.

Reflecting

Denote lines in the plane by lower case letters, $a, b, c, \ldots$, and points by upper case letters $P, Q, R, \ldots$ To each line, say a, there corresponds a transformation of the plane, reflection in that line, which it is convenient to denote by a also. To each point, say P, there corresponds reflection in that point, denoted by P also

Transformations can be composed. Sometimes the result will be to leave all points of the phase fixed: denote this identity transformation by I For example,

reflection in a followed by reflection in a again is denoted by aa or a^2 and is always I

Similarly $ab = I$ if and only if $a = b$, and $PQ = I$ if and only if $P = Q$. Investigate the geometrical meaning of the following statements:

$(PQR)^2 = I$; $(ab)^2 = I$ but $ab = I$;
$(aP)^2 = I$; $(abc)^2 = I$;
$PQRS = I$; and so on.

Investigate how to express symbolically, geometrical statements of the form:

R is the mid point of PQ ($P \neq Q$);
a and b are perpendicular lines meeting at P;
and so on.

Stuck?

Do you understand the question? Try some specific examples! Have you tried various interpretations for a, b, P and Q? Have you tried special cases when $a = b$, etc?

I wish to express my thanks to the EM235 Course Team: Leone Burton, Nick James, Ann Floyd, Jean Nunn and Tim O'Shea for constituting such a creative environment that some fuzzy notions became clearer and more useful

Reference

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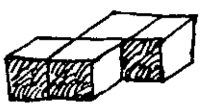
Four-Cube Houses

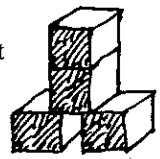
HANS FREUDENTHAL

The following is a summary of an experiment undertaken by E.J. Wijdeveld in a third grade class. It has been published in full*, with 38 colour pictures, taken in the classroom, didactical remarks, and critical comments by a teacher who repeated the experiment. Meanwhile the experiment has been repeated many times, even at PTA meetings. It has become a classic in our primary education

Paulus the Forest Midget is a wellknown feature on Dutch children's TV. The teacher tells a story about Paulus and the midgets. There is restlessness in the midget town. Some houses are more beautiful than others. Paulus is called in as a troubleshooter. He proposes to rebuild the town The midgets will live pairwise in houses, each consisting of a drawing room, a kitchen, and two bedrooms. All rooms are to be (congruent) cubes, and each house will be built from four cubes, which touch each other along complete faces, thus

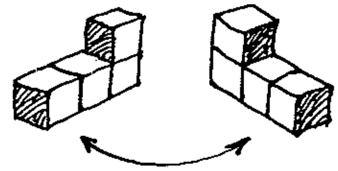
 or  and so on

*Edu Wijdeveld, Vierkubers — een onderwijsleerpakket voor de basisschool *Wiskobas-Bulletin* 6 nr 2 (1977) IOWO (Instituut Ontwikkeling Wiskunde Onderwijs) (Out of print)

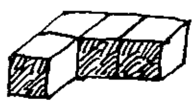
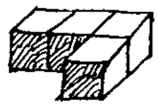
But not  or  and so on

It would be a dull town if all the houses were the same shape So it is understood that they should build as many different houses as possible The children will help them to design such four-cube houses. They are sitting at tables in a circle around the teacher who acts as Paulus, each child with four cubes Each has built one house.

But are they all different? No, quite a lot are the same. It is easy to see whether two of them are the same, but not so easy to tell *why* they are so You can show it by turning them

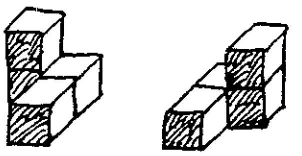


By chance Eloy and Guus, at the same table, had built the "same" house When Eloy noticed it, he changed his house a bit

Now they cannot any more be turned into each other, he said.

On a table in the center Paulus rebuilt each new house he noticed. Again and again it had to be discussed whether it was really a new one.



Aren't they the same? No, Guus said, climbing to the second floor you must turn left in the one and right in the other. They were "mirror" rather than "turn" houses (But some mirror houses may be turn houses too.)

When Paulus had finished assembling, there were 13 houses on the central table. Isn't it possible there are more?

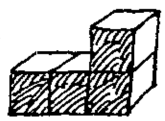
The children tried unsystematically. All the houses they constructed had already been found before. Then Marc discovered a new one. And by mirroring a house one is likely to create a new one. Finally they had got 15 houses, just as many as there were pairs of midgets.

But how should they describe them in order to get them built? "We can take them from the table and colour the place where they stood", a child said. Was it a good idea? No, it was not. There may be different houses standing on the same kind of ground floor.

Guus had a better idea: writing on each ground square a number that tells how many cubes were standing on it. Then

1	1	2

can only mean



Each child made a diagram of his or her own house, and finally all the diagrams were put on one big sheet.

The next day the children reconstructed their houses on the ground floor of the classroom, and finally they had again 15 different ones. But were there really no more than 15?

Eloy said: "Let us start with four rooms in a row." There is only one.

1	1	1	1

"Then all with three in a row"

1	2	1

2	1	1

1	1	1
	1	

1	1	1
1		

1	1	1
		1

They continued with two in a row, and finally the tower

1	
1	1
	1

	1
1	1
1	

2	2

1	1
1	1

2	1
	1

1	2
1	

2	1
1	

3	1

4

Paulus advanced a new problem: the building lots have to be bought and the walls painted. A square of ground floor is worth 100 florins, and painting a side square costs 10 florins. Some houses were more expensive than others. [2|1|1] and [1|2|1] were of course equally expensive. But why did [3|1] and [4] differ by 90 florins?

Brigitte could explain it. If you lift the top cube from the tower and put it by the side of the ground cube, there are two faces put together that need no painting, which makes a difference of 20 florins, but then you have to paint a new one on the top, which is 10 florins, and 100 florins for the new ground square.

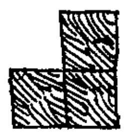
The midgets were disappointed. It was not a good idea to pay different prices. Paulus then proposed they build one big building of sixty cubes.

So the children started designing 60-cube houses by making diagrams. It yielded tremendous activity, a display of buildings, from trivial solutions to sophisticated arithmetical architecture.

The third lesson started with a recapitulation of the preceding ones. The children looked at a sequence of 30 slides that had been made while they worked. Then a new problem was posed.

The midgets had in fact built the original 15 four-cube houses. One night a burglar searched for one of the houses. In the twilight he could discern something that looked like three cubes in a row. Which house was it?

Anyway, the house the burglar meant was two stories high. When he went further, he noticed something like



Which one could it be? He went around to see it from the side and then it looked like

Paul made the house

	1
1	2

But

1	2
1	

was also right. Lisette added

1	2
1	

Everybody said she was wrong. The front was correct but seen from the right side it was wrong. Lisette laughed. "How can you tell whether the burglar turned right? I say he turned left to look again, and then it is correct."

The lesson finished with some more problems of the same kind.