Looking at a Painting with a Mathematical Eye

MARION WALTER

It was at one of the U K Association of Teachers of Mathematics meetings, probably in the early or mid-1970s, that I first met David Wheeler. I am not sure how it came about that he always supported my interest in mathematics and the visual arts, but it was certainly he who gave me my first opportunity to give a course in this area when he invited me to teach a summer session for teachers at Concordia University in Montréal. He gave me free reign and much encouragement, but occasionally this came with a sprinkling of constructive suggestions and a twinkle in his eye

Unfortunately, I am not able to locate the correspondence dealing with the session on 'mathematizing' that he, Eric Love, John Trivett and I carried on as part of the preparation for our session at ICME IV in San Francisco in 1980. I do recall though that I was not successful in convincing him that I really did not fully understand what was meant by that term. But he assured me that I did it all the time, and it is true I did manage to give a talk on 'mathematizing with a piece of paper' [1]

From the launch of FLM in 1980 until issue 50 when he retired as editor, David was concerned with the visual aspects of the journal. A few quotations from some of his letters to me will indicate this:

> You see I printed one of the Dürer's. But failed to trace either publisher or author, in spite of several letters Hope I don't get sued! (December, 1982)
>
> The lack of a picture on the front of the latest issue wasn't planned as such. (July, 1983)
>
> When (if) you've any time to sit and think, I hope you will keep FLM's visual needs in mind. Volume 4 Number 1 is visually awful - hardly any diagrams/pictures and nothing on the cover again [..] But I'd also like the occasional diagram/picture inside, not necessarily connected with any article. (1984)

I was able to help out a little by obtaining some visual material for FLM and I am sorry I did not do more [2] In one letter, he asked me:

> Are you working on any problems, by any chance? If so - or you find yourself doing one - and you'd like to record your thoughts, I'd be interested in a protocol. Alternatively, if you have any introspections about your own processes - either generally or specifically in relation to a particular question - I'd be pleased if you'd jot them down. I ask you (with our research in mind, of course) because I get the impression you are more conscious of your steps than most people. Maybe I'm wrong. (November, 1984)

On occasion, I have recorded my thoughts, but I do not think I have been introspective in the sense that David wanted. Nevertheless, I contribute this piece in David's memory because:

- *it does deal with a visual situation;*
- *it might lead to some more visual pieces that readers might submit and that could be used to make FLM richer visually;*
- *I have tried to record some of the thoughts I had while developing the piece below (I just wish I had done this earlier).*

Introduction

When I first saw a reproduction of Theo van Doesburg's painting *Arithmetic Composition 1* (1930) [3], I was struck by its title. An outline of the painting is shown in Figure 1

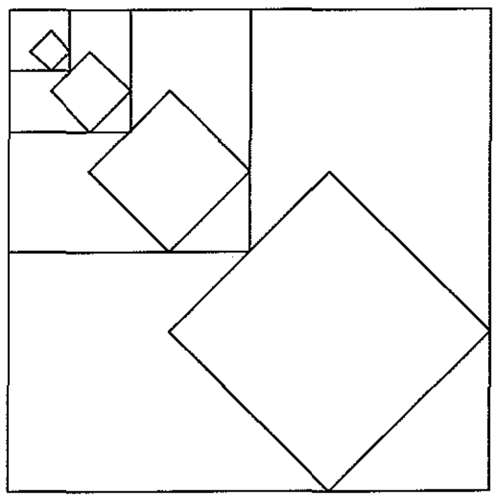

Figure 1

One side of each tilted square seems to lie on the diagonal of a successively smaller horizontal square. I wondered what was 'arithmetic' about this painting, and so I asked myself, 'Why did the artist call it *Arithmetic Composition 1*, when surely it, whatever 'it' may be, is geometric?' After all, not only is the plan geometric but the sequences we can obtain from it are geometric sequences

For the Learning of Mathematics **21**, 2 (July, 2001)
FLM Publishing Association, Kingston, Ontario, Canada

I focused first on the horizontal squares because they are so obviously in geometric progression.

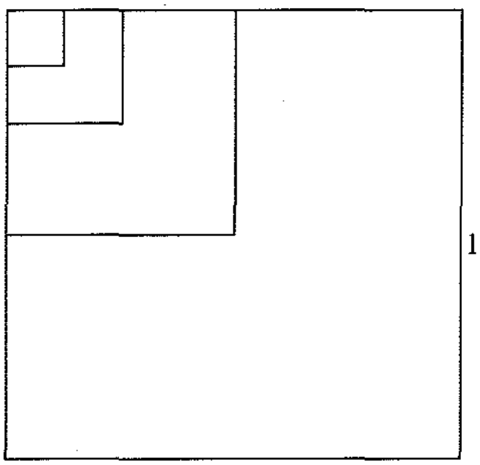

Figure 2

I let the outside square have side length 1 unit, and then the side lengths of successive horizontal squares (starting with the largest) are:

1, 1/2, 1/4, 1/8 units

If, instead of sides, I focus on the *areas* of these squares, I obtain areas of:

1, 1/4, 1/16, 1/64 square units

Though the artist, for his own reasons, stopped after four squares, we can theoretically continue the pattern obtaining two geometric sequences (As an aside, here is a nice opportunity for students to see a specific example of a more general statement: if two similar figures have corresponding sides in the ratio $m : n$, then their areas are in the ratio $m^2 : n^2$.)

Deciding on a problem

In the painting, the tilted squares are colored solid black. Figure 3 shows a sketch of the painting without its delicate alternating background coloring of the gnomon-shaped regions of decreasing size

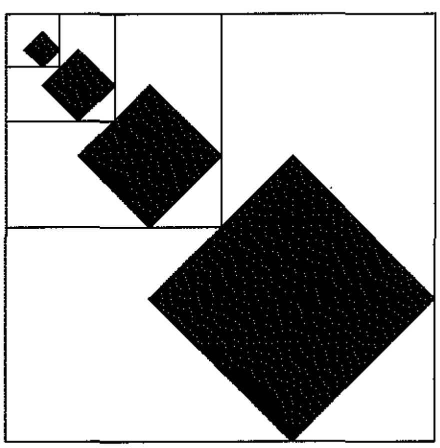

Figure 3

I looked at this very geometric painting again, starting from the bottom black square along the diagonal to the smallest. The next question that came to my mind was: how did the artist construct that *first* largest tilted black square? (You may want to stop reading now and explore how *you* would construct the first tilted square on the lower right.)

Once I know how to construct the large tilted square, I *could* use the same method for constructing each of the successive smaller tilted squares, because of the sense of self-similarity Ask yourself for a moment why I only say *could*.

I first decided to use a method discussed by Polya (1957, pp. 23-25). He starts by drawing a square at the corner of the triangle, as shown in Figure 4a One also could start by drawing a square symmetrically on the diagonal, with the midpoint of the diagonal the midpoint of the side, as shown in Figure 4b

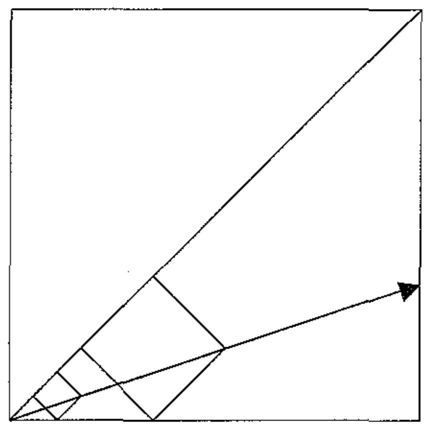

Figure 4a

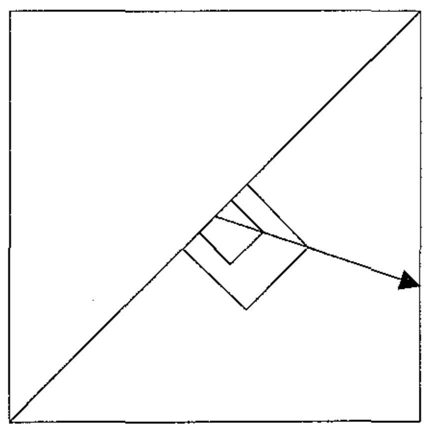

Figure 4b

In each case, by drawing in the indicated line, I obtained a corner of the required tilted square. The rest of the square can then be constructed.

At this point, before thinking further about the construction, I returned to thinking about areas. I realize that area is something I have always tended to think about, to attend to when posing problems or working with geometric situations

What is the area of the large tilted square, if the large horizontal square has side length 1?

I marked all the 90° and 45° angles and the congruent segments that follow from that (see Figure 5)

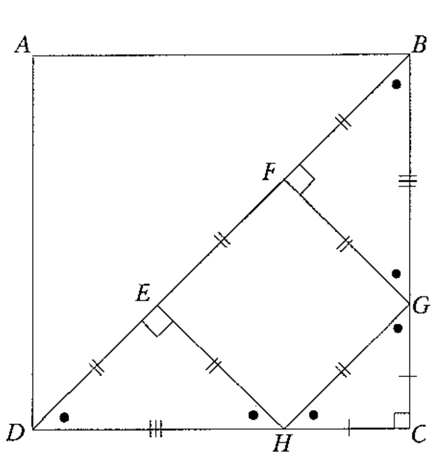

Figure 5

FB = FG = FE = EH = DE, etc and – a bonus – I saw that the side s of the tilted square is one-third of the diagonal of square ABCD (symmetrically placed). So s = DB/3= $\sqrt{2}/3$ and hence the area of the square EFGH = 2/9 square units.

So, the fact that the side of the tilted square = 1/3DB, the diagonal of the horizontal square, gave me a third way to construct the tilted square. I also noted that $BG^2 = s^2 + s^2 = 2/9 + 2/9 = 4/9$ So BG = 2/3 and hence GC = 1/3

So the side of the large square is divided by the vertex of the tilted square in the ratio 2 : 1. So here is a fourth (related) way to construct the tilted square – start by dividing BC in the ratio of 2:1

Next, to make trisecting the diagonal even easier, I got out graph paper and, wanting a number that after dividing by 2 several times would still be divisible by 3, chose a 24 x 24 grid in order to sketch the whole painting.

At this point, I looked back at the copy of a reproduction I had and examined it very carefully and noted to my surprise that the diagonal of the large square seemed not to be 100% on the side of the tilted square and I wondered if he had intended it to be In trying to find a clearer reproduction than the ones I was finding in books on van Doesburg, I corresponded with Dr. Michael White of York University, England, whose Ph D thesis (White, 1997) was on this artist He kindly sent me various papers and also a colored postcard entitled *Sketch for Forme Universelle II*, which, in spite of its different title, has the same structure as the painting *Arithmetic Composition 1*. I was delighted and relieved when I saw that this sketch was painted on graph paper – starting with a 12 x 12 grid which is refined as the squares get smaller So, lacking any further evidence to the contrary, I assumed that this was the way that van Doesburg had constructed his *Arithmetic Composition 1*

Looking once more at my graph-paper construction, I noticed that corresponding corners G, J, K, L of the tilted squares were collinear (see Figure 6a). It was only after I was asked how I came to think about looking at the drawing this way, that I actually became aware of how much Polya's construction using loci had influenced me.

I saw that if I now join corner A of the starting horizontal square to the points that divide the non-adjacent sides in the ratio 2 : 1 (G and H), I could construct two of the corners of each of the four tilted squares in one fell swoop! It was then straightforward to construct the remaining vertices of each of the squares.

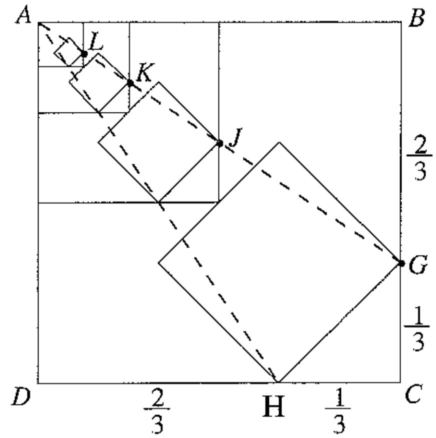

Figure 6a

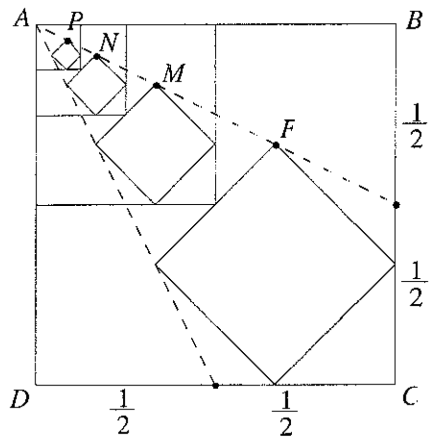

Figure 6b

I then realized that the other set of corners F, M, N, P were also collinear – lying on a line passing through A and the midpoint of BC (see Figure 6b). The fact that the two lines joining the mid-point of two adjacent sides of a square to the opposite vertex trisects the diagonal was a result I had encountered before.

How many different ways can I now construct the drawing in Figure 1? This is why I chose to write *could* in regard to using the same method to construct subsequent squares along the diagonal – I could potentially use a different method for each one.

I then looked back at Polya's method of constructing the tilted square in *any* right triangle by using the locus of a fourth vertex. Actually, he shows his method for any scalene triangle Suddenly, I had a *déjà vu* experience. I felt I had worked on this problem before. Then I recalled that I had indeed written a short piece before in which I approached the problem from a different starting point.

In that note (Walter, 1970), starting with a locus problem, I asked myself how the areas of the squares R and S in a right triangle - not necessarily isosceles (see Figure 7) - compared with each other

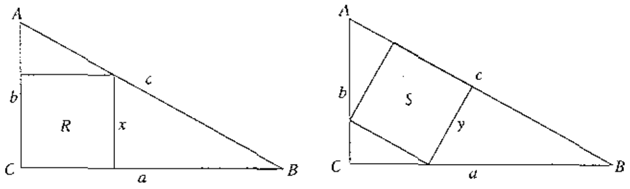

Figure 7

I looked at the difference $R - S$. That seemed to involve a bit of messy algebra, so I considered $1/S - 1/R$ Readers evidently fail to check all the computations in a journal, because no one wrote in to point out that there was a misprint (at least, I hope it was a misprint and not an error on my part - I do not have my original paper to check)

What was misprinted is:

$$\frac{1}{S} - \frac{1}{R} = 1$$

But if $R = 1/4$ and $S = 2/9$, you can check that $1/S - 1/R = 1/2$ not 1

The 'error' arose I think because there I took a special case of a right triangle ABC with hypotenuse AB to be 1, while here the hypotenuse is $\sqrt{2}$ The interested reader can check that, for the general right triangle, using the formulae $x = ab/(a + b)$ and $y = abc/(ab + c^2)$, that:

$$\frac{1}{S} - \frac{1}{R} = \frac{1}{c^2}$$

Readers who do not want to derive these formulae themselves may care to look at Lieske (1985), in which formulae for the lengths of the sides of the inscribed squares R and S for any right triangle and then also for any scalene triangle are derived.

Then I looked at the *ratio* of their areas in this isosceles right triangle case.

Area of $R = 1/4$
Area of $S = 2/9$

so area of R : area of $S = 9 : 8$

Readers who like Pythagorean triangles - right triangles with whole number sides that is, triangles whose sides form Pythagorean triples - might like to explore which Pythagorean triangles have squares of the tilted type discussed here also have integer sides. Yocom (1990) supplies an answer.

I am not finished! I could now start to problem pose by looking at the painting, listing some of its attributes and varying them (For a full discussion and use of this idea, see Brown and Walter, 1990, 1993.) I could start by noting that the outside shape is a *square*. It has certain shapes *inscribed*. These shapes are inscribed on *diagonals*. *Each* of these shapes is also a *square*.

But now I can invite you to start work by picking one of these attributes, say "each inscribed shape is a *square*". What if we do not inscribe squares? Can you make a painting choosing inscribed semi-circles instead? Equilateral triangles? Other shapes? List a few more of the many other attributes of the painting and vary some of them by asking *What-If-Not*? What mathematics is now lurking there?

I encourage some readers to provide new drawings based on van Doesburg's basic idea and their own *What-If-Not*? explorations. I think David Wheeler would have been very happy to receive such drawings, because he then would have had some visual pieces to use on the front cover and in the body of *his* journal that he cared so much about

Afterword

After I had 'finished' this piece, I had a further surprise which I want to add While working, I had drawn several times the diagram in Figure 6b as suggested by the van Doesburg painting with just the big square present and without the other squares (see Figure 8a) I had also drawn in, for symmetry's sake, the pair of lines from the opposite diagonal (see Figure 8b) But, although I had done that many times in the past, I did not on this occasion complete the diagram by drawing the corresponding lines from the remaining two vertices.

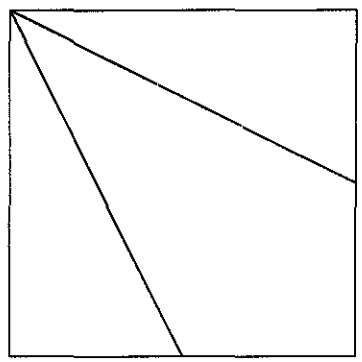

Figure 8a

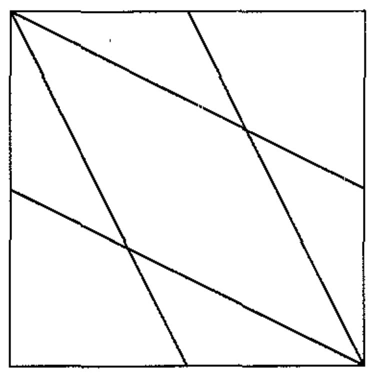

Figure 8b

Even given my obsession with squares for the past forty years, I got a lovely surprise when I finally looked up a reference that Michael White had sent me (Milner, 1996). I just wish I had found this result out on my own! Among other things, Milner shows and discusses this diagram completed with lines from all four vertices to form the familiar eight-pointed star (see Figure 9).

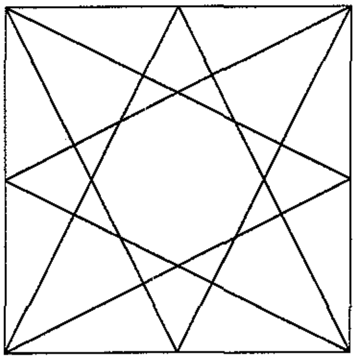

Figure 9

He indicates that there are significant intersection points in the diagram that allow us to draw vertical (and horizontal) lines that divide the square into two, three, four or five parts all of equal area. I had not known of the three- and five-part division before this I thank Milner for pointing this out and, in customary manner, leave it to the reader to find nice proofs for the 'three' and 'five' cases.

Acknowledgements

I would like to thank David Pimm not only for making helpful and constructive suggestions about this article but also for taking an interest in the subject matter, which among other things resulted in the subsequent article consisting of art-historical material. I would also like to thank Michael White for sending me parts of his thesis, copies of relevant pages of several references, helpful other comments and the colored postcard of *Sketch for Forme Universelle II* I am grateful to Dave Wagner for producing the diagrams which appear in this article using *Geometer's Sketchpad*.

Notes

[1] The order of the paragraphs of this piece were completely garbled in the ICME proceedings (Walter, 1983), but were correct in the ATM journal *Mathematics Teaching* (Walter, 1980)

[2] These include the Dürer, the drawings produced by Richard Rasala found on the contents pages of **5**(1) and **5**(2), the cover images on **8**(2) and **8**(3) produced by Steve Romero and various designs by Jennifer McGee, David P. Hardesty and John Sharp, which appeared in **17**(2), David's fiftieth and last issue as editor.

[3] A reproduction of the actual painting appears in several books: see, for example, Baljeu (1974, p. 99) or Lemoine (1987, p. 68; 1990, p. 223)

References

Baljeu, J. (1974) *Theo van Doesburg*, New York, NY, Macmillan

Brown, S. and Walter, M (1990, 2nd edn) *The Art of Problem Posing*, Hillsdale, NJ, Lawrence Erlbaum

Brown, S and Walter, M (eds) (1993) *Problem Posing Reflections and Applications*, Hillsdale, NJ, Lawrence Erlbaum.

Lemoine, S. (1987) *Mondrian and De Stijl*, New York, NY, Universe Books

Lemoine, S. (ed.) (1990) *Theo van Doesburg*, Paris, Philippe Sers

Lieske, G S. (1985) 'Right triangles II', *Mathematics Teacher* **78**, 498-499

Milner, J (1996) *Kazimir Malevich and the Art of Geometry*, New Haven, CT, Yale University Press

Polya, G (1957) *How to Solve It*, New York, NY, Dover Publications

Walter, M (1970) 'A few steps down the path of a locus problem', *Mathematics Teaching* **53**, 23-25.

Walter, M (1980) 'Mathematising with a piece of paper', *Mathematics Teaching* **93**, 27-30

Walter, M (1983) 'Mathematizing with a piece of paper: an introspection', in Zweng, M *et al.* (eds), *Proceedings of the Fourth International Congress on Mathematical Education*, Boston, MA Birkhäuser, pp. 287-290.

White, M. (1997) *Theo van Doesburg* (1924-31): *Towards Concrete Art*, Unpublished Ph.D thesis, Colchester, Essex, Essex University

Yocom, K. L (1990) 'Square in a Pythagorean triangle: solution to problem 390', *The College Mathematics Journal* **21**(2), 154-155