Fuzzy Sets and Mathematics Education

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1. Introduction

While it is still a general belief in society that mathematics is an "exact science", at the educational level it is generally accepted that it may be just as important to learn about the mathematics of approximations, errors, expectations, statistics, probabilities, etc. This can be seen today in mathematics curricula at all levels but was not so clear just a few years ago. It took a long time to convince mathematics educators about the need to teach some elementary facts of probability and statistics (in fact, it is amazing to realize that in some textbooks at the secondary school level, chapters devoted to such questions are located at the end, following the old teaching tradition that "in cases of lack of time forget about the last part").

Since 1965 the new theory introduced by Lotfi A. Zadeh in his celebrated paper, "Fuzzy Sets" (*Information and Control*, 8 [1965], 338-353), has generated a growing interest both in the theoretical mathematical framework as well as in its practical applications. The aim of this paper is to give some ideas on the interest that the mathematics educational process may have in the introduction of some basic facts arising from Fuzzy Sets theory.

2. What is a fuzzy set?

The key points of the theory are how to obtain fuzzy sets and how to use them. While the characteristic function is unique, given A it may be possible to associate many different fuzzy sets with it, so we select, depending on context, the one which gives us the more interesting information.

Let us consider an example which may clarify this notion. Let X be the set of mathematicians named in the 1990 edition of the World Directory of Mathematicians [AMS, USA]. The fuzzy set A of "young mathematicians" can have many representations:

(a) Define $\mu_A(m)=1$ - [age of m/age of the oldest mathematician in X]. This fuzzy set attains many different values and may be of interest to an insurance company mailing information about an offer of life insurance.

(b) Define

$$\mu_A(m) = \begin{cases} 1, \text{ if age of } m \le 40 \text{ years} \\ 0, \text{ if age of } m > 40 \text{ years} \end{cases}$$

In this case we have a classical set (you are young or

In a given universe X a *subset* of A is defined when for each element x of X we have that either x belongs to A or x does not belong to A. Thus any set A is characterized and represented by its characteristic function $\chi_A : X \to \{0,1\}$, defined by

$$\chi_A(x) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$$

In a given universe X a *fuzzy subset* A (a fuzzy set, for short) is given by a mapping $\mu_A : X \to [0,1]$, where [0,1] denotes the closed unit interval of real numbers. The number $\mu_A(x)$ is interpreted as the *degree of membership* of x in A. Usually, *fuzzy* sets appear as extensions of vague predicates over X. That is, given a vague predicate A on X, there is no bipartition of X that classifies the elements of X into both of the classes { $x \in X$ | proposition "x is A" is true } and { $x \in X$ | proposition "x is A" is false }, as is the case for crisp statement predicates. $\mu_A(x)$ represents some truth value; i.e. the truth value of proposition "x is A" is $\mu_A(x)$. Fuzzy set A represents in an elastic way the truth of such propositions.

For example, if $X = [0,1]$ and A = Large Number, then we can consider $\mu_A(0) = 0$, $\mu_A(1) = 1$ and $\mu_A(0.5) = 0.5$. Then, if we consider that μ_A should be continuous, we can accept $\mu_A(x) = x$. Notice that things change as soon as we enlarge [0,1] to, for example, [0,10]: then $\mu_A(0) = 0$, $\mu_A(10) = 1$, but now we have merely that $\mu_A(1)$ is a small number between 0 and 1. Then it becomes clear that μ_A is strongly dependent on X and that μ_A behaves like a compatibility function between A and X.

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not) which is of interest to the International Mathematical Union considering possible candidates for the Fields Medals.

(c) From an experiment we find that people distinguish the age intervals: up to 30 years old, from 30 to 40, from 40 to 50, from 50 to 65, and from 65 on, using in each case a particular linguistic term (very young, young, etc.) Then one could consider

$$\mu_A(m) = \begin{cases} 1, & \text{if age of } m \leq 30 \text{ years} \\ 3/4, & \text{if age of } m \in [30 \text{ years}, 40 \text{ years}] \\ 1/2, & \text{if age of } m \in (40 \text{ years}, 50 \text{ years}] \\ 1/4, & \text{if age of } m \in (50 \text{ years}, 65 \text{ years}] \\ 0, & \text{if age of } m > 65 \end{cases}$$

This may be of interest for a study concerning age and job opportunities in mathematics departments of universities.

Case (a) was based upon some measurable parameters, case (b) was decided by an arbitrary definition, and case (c) arose after some statistical procedure.

> The case of Large Numbers in [0,1] was obtained through an inductive process but admits obvious variations, e.g., $\mu_A(x) = 0$ if $x \leq 1/100$, $\mu_A(x) = 1$ if $x \geq 1 - 1/100$, and $\mu_A(x) = (100x - 1)/98$ otherwise represents the notion that we cannot "distinguish" numbers close to 0 or 1.

In view of its formulation, Fuzzy Sets theory may be controversial if its relations with other fields are not clarified. Let us point out the following: Fuzzy Sets theory uses classical set theory, i.e. it is not an alternative approach nor does it involve a revolution in the foundations of mathematics. Fuzzy Sets theory helps to approach problems which by their nature (imprecision, vagueness, etc.) cannot have a satisfactory description in a boolean picture. Fuzzy Sets theory may use probabilistic or statistical methods but the interesting points in the theory are those where the "fuzzy analysis" gives a better description than a random model. In other words: there is no need of fuzziness whenever the problem can be perfectly handled by boolean or random representations.

It should be pointed out that classical probability theory deals with events arising from questions formulated with crisp predicates: for example, if in flipping a die we ask for the probability of obtaining a multiple of 3, then we are actually considering the event represented by the elementary results contained in the *subset* $\{\{3\},\{6\}\}$.

> When we write Prob (A) it is assumed that A is a crisp subset of X. Of course, *any* probability name **Prob** defines a particular fuzzy set P(X); $\mu_{Prob}(A) = \mathbf{Prob}\ (A)$, which represents the membership of A in the fuzzy set Prob.

3. Do we need to teach fuzziness?

Something that mathematics educators have learnt over the last decades is that one must be careful about focusing education upon a single approach. The "modern mathematics" period is remembered, and is possibly best forgotten. Moreover, not all recent research is of interest for teaching purposes. Fuzzy sets should not become a foundation stone; some of the theoretical results may not be of interest at the elementary or secondary levels. But we are strongly convinced that some elements of fuzziness may offer a richer discussion and a more suitable modelling style in our classrooms.

> Why not introduce some elementary ideas that represent in an elementary way predicates frequently used in everyday life? Why not introduce a simple arithmetic that can help use such important mathematical functions as Min (x,y), Max (x,y), $x \cdot y$, $x + y - xy$ and Max $(0, x + y - 1)$, Min $(1, x + y)$ to combine vague predicates? Precisely, given **A** and **B**, one can represent "**A** and **B**" by $\mu_{A \cap B}(x)$ = Min $(\mu_A(x), \mu_B(x))$ and "**A** or **B**" by $\mu_{A \cup B}(x)$ = Max $(\mu_A(x), \mu_B(x))$. Depending on the context Min and Max may be replaced by other functions. Since there are so many examples closely linked to human everyday reasoning, why not use elementary mathematics to approximate as soon as possible reasoning about real life?

Most situations in real life are fuzzy. Our language is essentially fuzzy. The learning process is in some sense an evolution from fuzzy to crisp. These three statements are enough to justify the modest role in teaching that fuzzy sets may play by offering a broader development of many concepts.

4. Some examples of fuzzy questions

Following the well-known Socratic principle that questions are more important than answers, let us present here a list of fuzzy problems.

1. Fix a scale in order to define the fuzzy set of tall women.
2. Fix a scale in order to define the fuzzy set of rich people in your country.
3. What is the difference between a street and an avenue?
4. How many people can stand on a square of area $S\ m^2$?
5. How can you classify the set of colours of all the dresses present in the classroom?
6. Can you give the degrees of kinship of all your relatives with respect to you?

7. One "countryman hour" is the distance that can be walked in one hour. Can you measure it in meters?
8. How do you understand the phrase "she is always late"?
9. Do you believe that friendship can be quantified?
10. At what time do you take an "early breakfast"?

The reader may immediately come up with many more ideas. Some of the previous questions may have a *crisp answer* ("a person is rich who has more than one million dollars", "a lady is late if she is more than twenty minutes late for any appointment", etc.). But frequently fixing a crisp answer makes one lose the possibility of giving a more suitable and convenient description. Some questions may admit a *statistical answer* (data from newspapers, data from a sample of opinions, etc.). In such cases it is important to retain the information in order to specify degrees of difference and not to reduce everything to simplified parameters like means and expectations.

It is important not to forget that fuzzy answers are usually more informative, in some contexts, than crisp answers. In any case the ability of the human brain to manage vagueness is clear: most times we take important decisions based on incomplete or not totally reliable information translated by vague questions and vague answers.

Let us emphasize that we have restricted ourselves to fuzzy questions which in many ways admit countable or measurable answers, i.e. where mathematics may play a role. In other directions, e.g. at the language level, one may consider fuzzy questions that cannot necessarily be quantified ("how nice is that dress?").

5. Some examples of activities related to fuzzy sets

In this last section we would like to indicate some mathematical activities that were experimented with by 15-17 year old students during workshops on space geometry [Alsina, C., Burgués, C. and Fortuny, J.M., *Visca la Matemàtica,* CIRIT, Barcelona, 1990]. While geometry and measure were the main goals, several questions of a fuzzy nature were faced.

Activity 1: The proportions of the body
The famous divine proportion $(1+\sqrt{5})/2$ appears (statistically) in the relation of the height of a person and the distance from the navel to the feet. This is an unexpected result for everybody. The problem is to find alternative body regularities, i.e. to introduce possible evaluations of the fuzzy set A "well-proportioned body" in the universe X of the classroom. The following alternatives were studied:

$$\mu_A(x) = \frac{\text{Armlength of } (x)}{\text{Height of } (x)} \qquad \mu_A(x) = \frac{8 \cdot (\text{Height of face of } (x))}{\text{Height of } (x)}$$

$$\mu_A(x) = \frac{\text{Height of } (x)}{1.618 \cdot (\text{Navel height of } (x))} \qquad \mu_A(x) = \frac{4 \cdot (\text{Forearm of } (x))}{\text{Height of } (x)}$$

$$\mu_A(x) = \frac{3 \cdot (\text{Hatband of } (x))}{\text{Height of } (x)} \qquad \mu_A(x) = \frac{24 \cdot (\text{Noselength of } (x))}{\text{Height of } (x)}$$

Activity 2: The beauty of rectangles
A classical psychological experiment is to present a paper with a numbered collection of rectangles drawn on it and ask for an evaluation (e.g. from 0 to 10) of the subjective beauty attributed to each rectangle. Squares and golden rectangles invariable get the top marks. (It's interesting to remark that people accept to do such evaluations!) If A is the fuzzy set: "beautiful rectangles" on the universe of the rectangles of the plane, two ways of defining it are:

$$\mu_A(R_{a,b}) = \frac{\text{Min }(a,b)}{\text{Max }(a,b)}$$

$$\mu_A(R_{a,b}) = 1 - \left| 1 - \frac{\text{Min }(a,b)}{\text{Max }(a,b)^2}(a+b) \right|$$

In the first case squares are the most beautiful. In the second case golden rectangles become the nicest [Alsina, C. and Trillas, E., *Lecciones de algebra y geometria,* Gustavo Gili. Barcelona, 1985].

Activity 3: Classifying shapes
This is always a successful activity: for students to walk around the place with paper and pencil drawing various shapes (1, 2 or 3 dimensions). The problem arises when all the papers are put together and a classification of shapes is required. While there are many clear classes (polygons, circles, etc.), many other classes are needed

Activity 4: The mystery trail
This is a game that can be played by teams. Each team begins in one place and must reach the finish passing a number of previously fixed intermediate points. The instructions found at each point in order to arrive at the next contain references either to measurable characteristics (turn 90° North, walk 5 meters North West, etc.) or fuzzy data (walk in the direction of the most attractive tree, etc.). Additional questions to be solved may be located at all the stop points.

Activity 5: Classifying rigidity
In the usual geometry course, the rigidity of figures is a bizarre topic (*all* drawn figures are rigid!), but by organising a workshop on 3-dimensional geometrical figures made with many different classes of materials the problem of rigidity arises naturally. Restricting attention to polyhedra, Fulton's measure $3A + V - C$ becomes useful.

Activity 6: Advertisements in a newspaper
Any collection of newspapers or magazines is beautiful mathematical material! The problem posed here was to define the fuzzy set describing levels of publicity. The best solution found was to measure, for each publication, the area devoted to publicity with respect to the total area printed in the publication.

Activity 7: Linguistic modifiers
Linguistic modifiers like *very, less, a lot, almost,* etc. appear everywhere in ordinary language but they do not play a role in classical mathematical predicates. In fuzzy sets theory one takes, for example, $\mu_A(x)^2$ as an expression that evaluates "very A". How should we define the fuzzy set of "very big numbers" in N? Note that for crisp predicates the modifiers do not affect the predicate, but this is not the case with vague statements.

Activity 8: Fuzzy relations and graphs
All classical equivalence relations, graphs, and partitions have an extremely interesting fuzzy approach: we can have degrees of relations $R(x_i, x_j) = a_{ij}$ in [0,1], matrices keeping this information, numbers incorporated to the graphs, non-classical "overlapping partitions", ... etc. Examples dealing with distances ("how far apart are the homes of x and y") constitute good materials.

Activity 9: Fuzzy circuits
An interesting way to visualize fuzzy connectives is to modify the usual electric circuits endowed with bulbs by adding rheostats, allowing us to consider different degrees of intensity. Thus we go from a boolean model to a fuzzy model more related to the realm of hardware and to multi-valued logics.

6. A last remark

We would like to think that the message of this paper is clear. One of the characteristic of human beings is the enjoyment of precision, exactness, randomness,—and also of vagueness. We may quote here the famous statement of L.A. Santaló: "to deal with vague data and results may require more precision than dealing with exact data". The scope of mathematics education must be to offer an attractive panorama of all procedures, concepts, descriptions, and representations which may be useful for our future citizens. Exact reasoning and approximate reasoning are both useful if they follow some rules of correctness. Perhaps it is time that fuzzy sets and fuzzy logic are taken into consideration as powerful tools for modelling, mathematically, many interesting facts integrating aspects of human knowledge.

Since World War II the discoveries that have changed the world were not made so much in the lofty halls of theoretical physics as in the less-noticed labs of engineering and experimental physics The roles of pure and applied science have been reversed; they are no longer what they were in the golden age of physics in the age of Einstein Schrödinger Fermi and Dirac Readers of *Scientific American* nourished on the Wellsian image of science will recoil from even entertaining the idea that the age of physical "principles" may be over The laws of Newtonian mechanics quantum mechanics and quantum electrodynamics were the last in a long and noble line that appears to have somewhat dried up in the last fifty years As experimental devices (especially measuring devices) are becoming infinitely more precise and reliable the wealth and sheer mass of new and baffling data collected by experiments greatly exceeds the power of human reason to explain them Physical theory has failed in recent decades to provide a theoretical underpinning for a world which increasingly appears as the work of some seemingly mischievous demiurge

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