EQUALITY INVOLVED IN 0.999… AND $(-8)^{\frac{1}{3}}$

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The equality sign that denotes 'equal' is one of the most frequently used and encountered mathematical symbols in school mathematics. The equality expressed by '=' can mean "to perform some operations" or "the same" in ordinary meaning, or "equality" that is mathematically defined (Freudenthal, 1983, pp. 477-482).

Students are more likely to interpret the sign in terms of its ordinary meaning than its abstract mathematical idea. In fact, students usually interpret the equality 2 + 3 = 4 + 1 as having the meaning that the two numbers 2 + 3 and 4 + 1 are equal because they both yield the same number 5, not because they are other names representing the number (Behr, Erlwanger and Nichols, 1976, 1980; Denmark, Barco and Voran, 1976; Sáenz-Ludlow and Walgamuth, 1998). Moreover, even college students use the equality sign as a link between steps and commit errors such as $x + 3 = 7 = 7 - 3 = 4$ (Byers and Herscovics, 1977; Kieran, 1981).

The concept of two numbers being equal is related to the mathematical structure underlying the numbers, which is more fundamental than an operator or another name.

Let us review a proof that $\sqrt{2}$ is irrational (Laczkovich, 2001). Supposing $\sqrt{2}$ is rational, in other words,

$$\sqrt{2} = \frac{q}{p}(p, q \in Z, p \neq 0),$$

we can get the equality $2p^2 = q^2$. If we express the numbers placed on both sides as a product of primes, the prime number 2 appears raised to an odd power in $2p^2$ and to an even power in q^2. This contradicts the Fundamental Theorem of Arithmetic. Therefore $\sqrt{2}$ is irrational. The equality $2p^2 = q^2$ connotes the mathematical idea that if two natural numbers are equal, the prime factorizations of the two numbers are also equal, which derives from the mathematical structure of integers as a unique factorization domain.

As exemplified above, equality involved in the equality sign contains formal and abstract mathematical ideas, as well as ordinary meanings comprehended through common sense.

Therefore, understanding the concept of equality involves using the equality sign in capturing mathematical ideas and related representations along with its ordinary meanings. The irrationality proof of $\sqrt{2}$ is a good example in which the mathematical idea underlying the concept that two integers are equal is properly applied. However, learners can encounter cognitive problems when they improperly understand the mathematical idea about two numbers being equal. In this article, we consider equality as a mathematical concept and investigate cognitive problems in understanding the concept 'two numbers are equal' from the viewpoint of extensions of number systems, especially, rational numbers to real numbers and real numbers to complex numbers.

Principles of extensions of number systems

To analyze different meanings of two numbers being equal and related cognitive problems, it is necessary to know which number system two numbers belong to and how the system is mathematically structured. To this purpose, we need to understand the nature of extensions of number systems in a clear way.

Defects in a number system provide a motivation to extend that number system. This is the substantial process of establishing extended number systems. We can express an extension of number systems from natural numbers through complex numbers in the following form:

Natural Numbers → *Integers* → *Rational Numbers* → *Real Numbers* → *Complex Numbers.*

The extension from natural numbers to integers can be said to be algebraic – it aims to have a closed system under addition and subtraction. The case of extension from integers to rational numbers can also be said to be algebraic – it aims to have a closed system under multiplication and division.

On the other hand, the extension from rational numbers to real numbers can be said to be analytic because it aims to have a closed system for limiting processes. Generally, the limit of a sequence in which every term is a rational number is not rational. For example, there exists a sequence $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, $\cdots$ whose limit $\lim_{n\to\infty} a_n = \sqrt{2}$ is not rational, while every one of its terms is a rational number. Such being the case, the rational number system is not closed for limiting processes, which forbids us to deal with limiting processes in the rational number system. Here we need an extended number system beyond rational numbers to address this defect. The real number system, derived from the Dedekind cuts of rational numbers, is analytically complete in the sense it is closed for limiting processes.

But an equation of which all the coefficients are real numbers (for example, $x^3 + 8 = 0$) does not always have solutions in the set of real numbers. Therefore, real numbers are not structurally complete in seeking solutions of polynomial equations. In this sense, real numbers are not algebraically closed. We need an extended number system to address this defect in the following way: all the polynomials whose coefficients are numbers in the extended number system must have solutions in the same system. This characteristic is referred to as 'algebraic closure.'

The real number system is extended to the complex number system to meet this condition. Every equation of which the coefficients are complex numbers has all its all solutions in the set of complex numbers. This is referred to as the Fundamental Theorem of Algebra. According to this argument it is possible to solve $x^3 + 8 = 0$ in the complex number system.

Equality involved in 0.999… = 1

Korean middle school mathematics textbooks explain the equality 0.999… = 1 in the following way:

Let $x = 0.999\ldots$ and multiply both sides by 10

$10x = 9.999\ldots$ So $10x = 9 + 0.999\ldots = 9 + x$

Therefore $9x = 9$

So $x = 1$.

Is 0.999… really equal to 1?

Most of the thirty-two 8th grade mathematically gifted students, aged thirteen or fourteen, who had been enrolled in the Seoul National University Gifted Education Center were able to get the equality 0.999… = 1 by the above processes.

However, all the students except 2 said that they do not believe 0.999… is equal to 1. Those who said they do not believe gave detailed reasons for their belief, while those who said they believe replied with the very brief statement, "I believe." Here are some examples of students' responses to the question "Do you believe the equality 0.999… = 1? Why?"

I don't believe. Because

- 1 = 0.999… + 0.000…01, so 0.999… is not equal to 1.
- Theoretically 0.999… can be equal to 1, but there exists a very fine tolerance between the two numbers.
- Just as 0.9, 0.99, 0.999 and so on are all less than 1, so 0.999… is less than 1.

I believe. Because

- Just I believe. I have learned so, and the textbook explains so.
- In fact I'm not confident. But I have to get the score. To do this, I have no choice but to believe that.

Interestingly and contrastingly, most of the students said they believe the equality $\frac{1}{3} = 0.333\cdots$. This can be interpreted as an inductive belief derived from their experiences in calculation, where the remainder is repetitively 1. The view that 0.999… is less than 1 is also found in one of Zeno's paradoxes, the story in which Achilles cannot catch up with a tortoise. Students with this view may have a dynamic image of a number running towards 1 on the real number line. If we observe each digit of 0.999… in terms of dynamic process, it is never equal to 1 while it comes nearer and nearer to 1. Likewise, even if each term of a series of rational numbers 1.4, 1.41, 1.414, … comes near to $\sqrt{2}$, it does not become an irrational number. This phenomenon occurs when local and finite nature turns into global and mathematical infinity and cannot be understood without a new paradigm. The concept of limit has been invented for this new paradigm and it functions as a bridge connecting the finite and infinite.

0.999… = 1 as a mathematical definition

A sequence $\{a_n\}$ is said to converge to a if there is a number a with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $|a_n - a| < \varepsilon$. In this case, we also say that the limit of $\{a_n\}$ is a, and write $\lim_{n\to\infty} a_n = a$.

According to the above definition (Rudin, 1964), 0.999… is equal to 1 in terms of the definition of limit, not in terms of common sense. To complete rational numbers and construct real numbers, we say two Cauchy sequences having rational number terms are equivalent if the difference between the terms of the two sequences converges to 0 and construct real numbers as the collection of equivalence classes. For example, sequences such as 0.9, 0.99, 0.999, … , 1.1, 1.01, 1.001, …, and 1, 1, 1, … are equivalent and all represent 1. In this sense, 0.999… is one way of representing the real number 1. The assumption in setting the equality $x = 0.999\ldots$ is that 0.999… is not a process of continuous increase, but a product of that process, that is, a static number. For a sequence or series, this means its limit value.

In this way, the equality 0.999… = 1 can be properly explained from the standpoint of real numbers. The equality sign here has the meaning of "mathematically equal" in the real number system, not "to perform some operation" or "the sameness" in ordinary meaning. That is, a definition of convergence of a sequence is necessary to understand the meaning of 0.999… = 1 and 0.999… = 1 is a mathematical expression based on this mathematical definition.

Equality involved in $(-8)^{\frac{1}{3}}$

Debates over two different perspectives on $(-8)^{\frac{1}{3}}$ were developed in a series of articles (Even and Tirosh, 1995; Goel and Robillard, 1997; Tirosh and Even, 1997). Considering the equalities

$$-2 = (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = \left((-8)^2\right)^{\frac{1}{6}} = 2\ ,$$

Even and Tirosh (1995) argue that $(-8)^{\frac{1}{3}}$ is an undefined operation and show how Israeli teachers, who mistakenly understand that $(-8)^{\frac{1}{3}}$ is equal to −2, react to learners' diverse responses in understanding $(-8)^{\frac{1}{3}}$. However, Goel and Robillard (1997) contradict what Even and Tirosh argue and explain that teachers' understanding of $(-8)^{\frac{1}{3}}$ being equal to −2 is correct using an example of a mathematics textbook of the US (Dugopolski, 1995).

$(-8)^{\frac{1}{3}}$ as a complex number

The issue underlying these two perspectives to understanding $(-8)^{\frac{1}{3}}$ is derived from solving the equation $x^3 + 8 = 0$. In other words, the mathematical structures underlying two expressions are substantially the same and thus the principle to apply to them is the same. The Fundamental Theorem of Algebra is that principle.

Each complex number represents a point on a plane. We can represent a point on a plane with polar form made up of its absolute value, r and argument θ. In other words, we can represent a point on a plane (x, y) as a complex number $z = x + y$, which can be represented as its absolute value

$$r\left(r = \sqrt{x^2 + y^2}\right) \text{ and argument } \theta\left(\tan\theta = \frac{y}{x}\right).$$

Here the absolute value r is uniquely determined, while the value of θ is not. The polar form of z is represented as $re^{i(\theta+2n\pi)} (n \in Z)$, and $\theta(-\pi < \theta \leq \pi)$ is referred to as the principal value. $z^{\frac{1}{k}}$ is defined as the set of all the complex numbers whose k-power is z and $z^{\frac{m}{k}}$ is defined as the m-power of $z^{\frac{1}{k}}$. Therefore, the rational power of a complex number $z\left(z = re^{i(\theta+2n\pi)}, n \in Z\right)$ is defined as follows (Churchill and Brown, 1990):

$$z^m = \left(re^{i(\theta+2n\pi)}\right)^m = r^m e^{i(m\theta+2n\pi)}, \qquad z^{\frac{m}{k}} = \left(re^{i(\theta+2n\pi)}\right)^{\frac{m}{k}} = r^{\frac{m}{k}} e^{i\left(\frac{m\theta}{k}+\frac{2nm\pi}{k}\right)}.$$

By the above definition, we can calculate:

$$(-8)^{\frac{1}{3}} = \left(8e^{i(\pi+2n\pi)}\right)^{\frac{1}{3}} = 2e^{i\left(\frac{\pi}{3}+\frac{2n\pi}{3}\right)} = (-8)^{\frac{2}{6}}, \quad \left((-8)^2\right)^{\frac{1}{6}} = \left(8^2 e^{i(2\pi+2n\pi)}\right)^{\frac{1}{6}} = 2e^{i\left(\frac{\pi}{3}+\frac{n\pi}{3}\right)}.$$

The absolute value of $(-8)^{\frac{1}{3}}$ and $(-8)^{\frac{2}{6}}$ is 2 and the principal value of the argument has three different values according to n. $(-8)^{\frac{1}{3}}$ and $(-8)^{\frac{2}{6}}$ get located on the real axis when the principal value of the argument is π and they are equal to −2 (see Figure 1). However, in the case of $\left((-8)^2\right)^{\frac{1}{6}}$, it has six different values depending on n, while its absolute value is always 2. $\left((-8)^2\right)^{\frac{1}{6}}$ gets located on the real axis when the principal value of the argument is 0 or π (see Figure 1). In the former case, it is equal to 2; in the latter, −2.

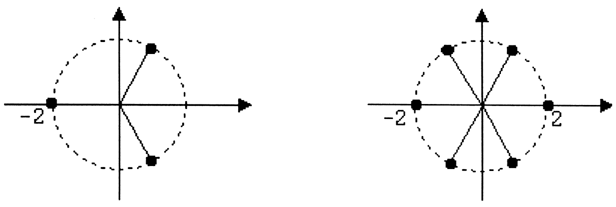

Figure 1: Geometrical representations of $(-8)^{\frac{1}{3}}$ *(left) and* $\left((-8)^2\right)^{\frac{1}{6}}$ *(right)*

In the equalities

$$-2 = (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = \left((-8)^2\right)^{\frac{1}{6}} = 2\ ,$$

−2 and 2 can be understood as one of a real number or a complex number. However, $(-8)^{\frac{1}{3}}$, $(-8)^{\frac{2}{6}}$, and $\left((-8)^2\right)^{\frac{1}{6}}$ must be understood in the complex number system. That is, we should view the equalities in terms of the complex number system, which is the most comprehensive one including both sides of equality signs. In terms of complex numbers, each side of the equalities can be understood as the solution set to an equation and the equalities here can be comprehended as the equalities of solution sets. We can represent each term of the above equalities with a set in the following way:

$$(-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = \left\{-2,\ 1 \pm \sqrt{3}i\right\}, \quad \left((-8)^2\right)^{\frac{1}{6}} = \left\{\pm 2,\ 1 \pm \sqrt{3}i,\ -1 \pm \sqrt{3}i\right\}.$$

And we can represent the given equalities with the inclusion relations of sets as follows:

$$-2 \in (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} \subseteq \left((-8)^2\right)^{\frac{1}{6}} \ni 2.$$

In the long run, the very attempt to solve the problems underlying $(-8)^{\frac{1}{3}}$ in the set of real numbers is incomplete. The fundamental settlement is possible only in the complex number system. Considering this, the debates on two conflicting perspectives dealing with $(-8)^{\frac{1}{3}}$ (Even and Tirosh, 1995; Goel and Robillard, 1997; Tirosh and Even, 1997) is incomplete in that it sought its solution in the real number system. The mathematical structure underlying this phenomenon is the principle of algebraic closure inherent in the complex number system.

Summary

Each mathematical concept in school mathematics contains mathematical ideas as well as ordinary meanings and it is often the case that one concept has different representations. To understand and analyze a concept means understanding and analyzing its mathematical idea, ordinary meaning, diverse representations, and how these are interrelated. The lack of balance of these three elements can cause learners' cognitive problems. In effect, 0.999…, an infinite decimal number, is not a concept existing in everyday life, but has a mathematical idea, one representation of the real number 1, abstracted by the decimal system invented by human beings. Therefore, it is impossible to understand the equality 0.999… = 1 as an ordinary meaning in terms of dynamic image. The meaning of the equality sign here can be understood correctly through the mathematical structure of real numbers (see Figure 2). As we pointed out above, discussion about the equalities

$$-2 = (-8)^{\frac{1}{3}} = (-8)^{\frac{2}{6}} = \left((-8)^2\right)^{\frac{1}{6}} = 2$$

in terms of real numbers is incomplete and, therefore, not adequate. The mathematical idea underlying the equalities is the structure of occupants of both sides, that is, the structure of complex numbers (see Figure 2).

	$0.999\cdots=1$	$(-8)^{\frac{1}{3}}$
Ordinary meaning	$0.999\cdots$ is less than 1 and comes nearer and nearer to 1	
Representations (Algebraic)	$0.9 < 0.99 < 0.999 < \cdots < 0.999\cdots < 1$ $0.999\cdots=1$	$x^3+8=0$ $\{-2,\ 1+\sqrt{3}i,\ 1-\sqrt{3}i\}$
Mathematical idea	The limit of rational sequence $0.999\cdots$ is the real number 1	$(-8)^{\frac{1}{3}}$ is the set of complex numbers satisfying $x^3+8=0$

Figure 2: Table showing three aspects of a mathematical concept.

Conclusion

We have pointed out that the fundamental problem about two numbers being equal is related to the mathematical ideas inherent in extensions of number systems, and have tried to provide the perspective of "knowing why" to teachers. Equality involved in 0.999… and $(-8)^{\frac{1}{3}}$ cannot be understood properly unless it is interpreted in terms of underlying mathematical structure.

The issue in understanding the equality concept is one about a teacher's subject-matter knowledge as well as a learner acquiring a mathematical concept. A teacher cannot explain a mathematical concept properly when she or he does not understand the subject knowledge to be taught (Ball, 1990; Even, 1990, 1993; Skemp, 1976; Shulman, 1986). His or her reaction to learners' different understanding and diverse responses depends on how clearly he or she grasps the subject. While students have difficulties in understanding the concept of two numbers being equal because it often requires advanced knowledge beyond the learners' present knowledge level, it is essential for a teacher to clearly understand the mathematical idea of the concept in order to help students' learning.

[The references for this article can be found on p. 36. (ed.)]

- "Only awareness is educable"; I am working on and with students' awareness.
- That we all learn in the context of a social situation (education is a social activity) and, particularly young learners find it difficult to separate what they are learning from the situation that they are learning it in.
- Activity without reflection on that activity rarely produces deep learning.
- Learning is often optimised if I can train and then call upon a set of shared behaviours.
- Learning is a personal thing; it is bound up with emotion. To be a successful teacher you need to tap in to this and provide learners with the motivation to learn ("Why do I need/want to think about this?").

Now, this may be made mathematically specific when we talk about these beliefs and principles using the example of mathematics:

- Mathematics is a way of thinking, not a body of knowledge to be transmitted from expert to novice.
- Mathematics isn't just being able to perform certain techniques; it's about understanding big ideas.
- Seeing connections between concepts and actively making links between different mathematical topics is important.
- The symbols aren't the mathematics; there is a need to approach the symbols through activities that enable the learner to contact the mathematical idea and then to work on symbolising it.

But these statements are not really specific to mathematics. You could replace "mathematics" with something else and they would still work.

It is only when I am asking myself questions of the "What is the essence of the mathematical idea, here?" type that I entertain subject-specific considerations:

- Fractions – 3/5 (for example) means '3 of those things called fifths' and 3/5 + 2/5 is a particular example of 3 things add 2 of the same thing.
- Algebra is generalised arithmetic and so a technique like multiplying out brackets [*e.g.* $(x + 3)(x + 2)$] emerges from seeing the general in a number of particular examples where two numbers are multiplied together: 23×22, 33×32, 43×42, ….
- Ratio – any number can be transformed into any other by multiplying.
- Graphs – a rule that transforms numbers into other numbers can be represented as a picture (using coordinates). If there is link in the numbers (*i.e.*, there is a rule) then this will show itself in the position of the plotted coordinates.

… and I think that it is in these sorts of areas that the idea of subject-specific pedagogy exists.

Notes

[1] National College for School Leadership, *'Making mathematics count' in school networks*, NCSL, 2005: a pack of 12 booklets available free: e-mail nlc@ncsl.org.uk.

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[These references follow on from page 15 of the Younggi Choi and Jonghoon Do article beginning on page 13. (ed)]

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