Communications

Research Problems in Mathematics Education — II

Here follow five more responses to an enquiry about "research problems whose solution would make a substantial contribution to mathematics education." Five responses were published in Volume 4, Number 1, and some more will be published in the next issue. Anyone may join in.

JERE CONFREY

Your letter intrigued me, and I've been pondering what I think will be the telling questions in mathematics education. I think that the most significant questions will unite various branches of mathematics education. "New math" has passed; dissatisfaction with our "back-to-basics" curriculum is prevalent. A new vision of math instruction is due to fill those gaps.

As a preface to my first proposal, let me review what I think we have learned from the existing research. I have tried to create a list of "known" results that can be agreed on by cognitive psychologists, math educators, and researchers relying on information-procession models:

1. Students cannot solve nonroutine problems.
2. Student errors are not random or capricious; they are reasoned.
3. In particular areas, we have documented misconceptions and alternative conceptions.
4. Students rely on memorization and imitation to learn math.
5. Students possess weak strategies for approaching problems, including difficulties in representing problem, in generalizing, in abstracting, and in plugging in numbers.
6. Students hold conceptions of math which impede the development of processes; they focus on getting answers, manipulate blindly, and see mathematics as an alien, formal system on which they have no claim.
7. Students are unaccustomed to thinking about their own thinking processes.

From this list, I hope to construct a structure which will generate a guiding list of significant questions.

Learning

1. Most of these "facts" from research are negative—they tell us what students do not know as compared to our expectations. A question which needs to be addressed is: What is the structure which unites these alternative conceptions and misconceptions and which guides their development? These schemes are bound together into a structure which, although wrong, in some sense is cohesive and clever and is evidence of students' attempts at sense-making. It's the structures which bind these misconceptions together which I wish to understand.

2. I want to know the answer to Toulmin's question, "Under what conditions will a person or group change a concept?" Are examples the key? What type? Is a conflicting example essential? What part does the use of language play? I think Toulmin is correct that this question might replace for a time the question of what is a concept.

3. What outcome measures measure understanding? Problem-solving ability? Students who are taught constructively value flexibility (having alternative methods), they persist longer with a problem, they can justify their attempts and talk about their thought processes, and they are most autonomous in their explorations. Can we create measures of these characteristics?

Teaching

1. In reviewing the previous list of "research results" I find myself asking how these behaviors and attitudes evolved. It's easy to blame classroom teachers and textbook writers, but I think what is called for is an examination of the actual teaching-learning process at a level of detail which has not been done. The research on teaching has for the most part taken the achievement test as unproblematic and has ignored the peculiarities of subject matter. Classroom observation followed by clinical interviews of teachers and students seems to me to be essential. Thus, the question is: What is going on in the classroom setting between what a teacher intends to teach and what a student

actually learns in a particular subject matter, mathematics?

2. In optimal settings where we control the class size and allow for experimentation with curricular content and pace, what can students of *all levels* learn? I am calling for some demonstration projects. Teachers, mathematicians, and the general public all hold certain preconceptions about math ability. I believe these preconceptions are tied to the pace and social structure of the traditional classroom. Somehow this needs to be tested, and demonstration classrooms are the only way I can think to do it.

Sociological investigations

1. ETS reported 100-point differences between white and black students on the SAT and 50-point differences between males and females; very little focused concern has been generated by such a result. In my own studies with Latino students, I found differences in their approach to math and their sense of its purpose. We need to do some cross-racial, cross-sexual, cross-class, and cross-cultural studies of the learning of mathematics; and if these differences in scores are not the result of an acceptable difference in the "knowledge" (as problematic as that term is) of the two groups, a critical examination of that test needs to be undertaken.

2. The study of women and mathematics must enter the classroom. How the study of mathematics is portrayed in classrooms and how young women respond to it needs examination. The work on attribution theory begins to do this, but I would like to see one address a question posed by Kegan's definition of anxiety: If anxiety is "the sense of disintegration which occurs when a meaning-making organism finds itself unable to make meaning," could it be that women are not able to make sense of the mathematics learning environment? Georgia Sassen constructed this argument about success anxiety ("Success Anxiety in Women: A Constructivist Interpretation of its Source and Significance," *Harvard Education Review*, Vol. 50, No. 1, February 1980).

Conceptions of math and math learning

1. A reexamination of the reasons for teaching mathematics must occur, especially in light of the extraordinary innovations in technology. This means the investigation of what teachers think mathematics is; and specifically, I'd advocate that from an epistemological perspective. Do teachers think of math as a rigid formal system — are they aware of alternative concepts and profound changes in method and standard? Tied to this could be an examination of the assumptions they hold about the learning of mathematics and the role of the teacher.

Teacher education

1. Research on the process of change of teachers from students, to student-teachers, to first-year teachers, to experienced teachers needs to be undertaken. Specifically, the question might be: What are the conflicting demands and the supportive influences made on teachers as they enter the profession?

Curriculum

1. I've avoided this one and will only say it's a fundamental blind spot for me. I think curricular reform will occur in a dramatic way, but its outlines elude me. However, the demands on the traditional curriculum are too great; the result is the extraordinary gaps in students' knowledge. I expect that geometry as axiomatics will disappear, that alternatives to calculus will develop, and that a problem-driven curriculum model might emerge. Judgement and decision-making will gain in prestige as computers and computer software perform the routine procedures. Courses on reflective thinking and perhaps logic will be developed.

Well, David, contrary to the evidence given beforehand, I do know the difference between a question and a proposal. I hope even in their primitive form some of these will be helpful. I'm sorry this is late; perhaps you can still use it if it's at all worthwhile. For myself, I enjoyed it.

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I am writing in reply to yor request for some Hilbertian problems. First some thoughts about problems. Problems in mathematics education are not like problems in mathematics — for a start they cannot be solved! For me they represent "pools of ignorance" and they operate as goals and as foci for activity. They are *problem areas* rather than problems.

So here are my most important problems areas.

1) *How can we discover more about children's goals in mathematics classrooms?*
   We talk always of goals as teacher goals but in considering the negotiation of mathematical meaning in classrooms what are the learners striving for? Perhaps we could understand their actions better if we had more knowledge of their goals? Research has recently shown us much about child methods and actions but we can only do anything to shape these if we know what the children feel they are achieving. Educating children's goals if it is done at all at present, is done inferentially. We should seek ways to do this more directly.

2) *What role does the media play in shaping mathematical knowlege and metaknowledge?*
   Children of today are very influenced by TV and yet we know little of how this influence affects their mathematical meaning. (By media I mean general publishing media, not the narrow use of media for teaching purposes). I am more worried about the uncontrolled mass media and its effect on knowledge, goals, attitudes, ideologies, etc.

3) *What are the essential differences between learning mathematics from a text and from another person?*

Interpersonal learning and textual learning are both used in education but what shaping and transfer effect do they both have? This would be a problem area worth developing in a social psychological research way. Most psychological research only considers learning which is not from another person.

4) *How can we release computers from the mathematical trap?*
Computers offer possibilities for exciting educational advances, provided they are not taken over by mathematics educators (unless those are educators first and mathematicians second). This is not to suggest that computers should take over mathematics education either but rather that those involved with computing need to develop an educationally beneficial maths/computing relationship.

5) *How to limit the generality of mathematics?*
In a similar way, there is an assumption that because mathematics has invaded every sphere of life, it should do so. Mathematics offers one way to construe and interpret reality but its value should not be overestimated. How do we in mathematics education define the limits of value of our own subject?

6) *How can we eliminate the construct 'mathematical ability' from teachers' vocabulary?*
This may be a problem local to the UK! Children are placed early on a dimension from more to less-able and they stay there. In comparison with children in other countries I feel that many of ours are literally disabled by this definition.

7) *How can mathematics teaching become more educational and less training?*
How can the examinations and texts be made to emphasize *educational* goals, to which mathematics might contribute? At present maths in schools is almost entirely a "doing" subject with little "reflective" possibilities.

8) *How can we develop more worthwhile problems, which satisfy the criteria both of realism and relevance to the children and of mathematical credibility?*
If mathematics is to illuminate reality then one must choose that reality carefully when teaching mathematics, otherwise the reality will either not be recognisable or it won't be illuminated. Both of these happen at present in large measure.

9) *How is mathematical meaning shared?*
Given that each individual constructs his own mathematical meaning how can we share each other's meaning? It is a problem for children working in groups, and for teachers trying to share their meaning with the children individually. Additionally how can the teacher help children in groups to share their own meanings?

10) *How can we improve children's communication skills in mathematics classrooms?*
If meanings are to be shared and negotiated then all parties must communicate. The teacher is trained, which can help, but the teacher must train the children. Also communication is more than just talking! It is also about relationship.

11) *How can we sustain methodological advances?*
Many methodological ideas exist in the literature and in the corporate wisdom of maths educators, yet one sees extremely *limited* methodologies in schools at present. How do we keep alive other possibilities? I do not see *growth* in our methodological sophistication in schools, only change and difference. Does this have to be the case?

Well, good luck! A very interesting project and I look forward to hearing the outcome.

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EFRAIM FISCHBEIN

Now, considering your questions. I see three main categories of problems:

- General problems, the answer to which may be given through serious theoretical analyses.
- Specific psycho-didactical problems which require systematic, complex, extensive empirical investigations.
- Problems raised by the research methodology of mathematics education.

A. *General problems*

1. What are the basic, general objectives of mathematics education?

Should mathematics education develop in pupils mainly the understanding, the taste, the interest of mathematics as a part of our modern culture?

Or, should the main objective of math education be to inculcate in every person a minimum of mathematical skills and knowledge necessary for everyday life and professional activities or, in some cases, for further, more advanced mathematics instruction?

Should the main role of mathematics education be that of training certain intellectual capacities such as systematic, logical, self-controlled reasoning, and the use of abstractions?

All these aims may be considered important but then a discussion has to be made about a possible hierarchy among them and how to correlate them in devising mathematical curricula.

2. A second, more specific theoretical problem is that of the organization and presentation of different topics. Should one maintain a clear cut distinction between the different branches of mathematics (arithmetic, geometry, trigonometry, etc.) or would it be better to integrate, as far

as possible, the different topics?

B. *Research problems*

a. Is it possible to identify specific aptitudes which may guarantee the success of a person in mathematics? If yes, is it possible to develop such aptitudes by specific training?

Is it true that an adequate mathematical training may enable every normal person to acquire basic mathematical skills and knowledge? (We have first to define what we mean by basic mathematical skills and knowledge.)

b. About mathematical reasoning:

What is the role of *intuitive* incentives, *intuitive* models in understanding, learning, creating in mathematics? The problem, in this formulation, is a general one. As a matter of fact it has to be investigated in relation to various specific concepts and topics—arithmetic, geometry, calculus, topology, etc. One has to consider the specific conflicts which arise between the formal, the algorithmic and the intuitive components of mathematical thinking.

Is it possible to train, successfully, the capacity of a person to solve problems? We have, again, a general problem, the answer to which one may only get by specific investigations. The crucial aspect of this problem is that of the *transfer* of intellectual abilities. It includes the idea of generalized approaches when coping with various types of problems.

c. Is it possible to overcome the following basic contradictions:

i. On the one hand, the pupils have to get an insight of what mathematics really is—a system of statements deductively organized, dealing with ideal, fully conceptualized mental objects and operations, the truth of which is only internally guaranteed (that is, internal to the system). On the other hand mathematics represents a powerful instrument for solving practical problems, the source of which is very often, reality itself.

ii. On the one hand mathematics is a system of formal, abstract statements, the validity of which is guaranteed only by logical means. On the other hand, for learning, understanding and creating in mathematics the role of intuition (self-evident, non rigorous, non formalized, interpretation) is absolutely essential.

Is it possible to overcome these (apparent) contradictions by adequate didactical means? If yes, how? Is it a problem of "a dose of each" (constructive and deductive, intuitive and formal) or we need, in fact, a completely new strategy, a completely new approach in mathematics education?

Two other problems are related to the above questions:

d. The problem of how to introduce and how to develop the concept and the correct use of mathematical proof. The natural tendency seems to be that of proving empirically. The idea of a formal proof, which is able to guarantee, once and for all, the universal validity of a statement, has no meaning from a behavioural-empirical point of view. How can we develop in pupils this new mental scheme which, fundamentaly, contradicts the natural, the intuitive meaning of the notion of proof? And this, without destroying their investigative ability (which includes inductive procedures, empirical checks, intuitive representations and confirmations) always needed when solving a problem and looking for a formal proof.

In a more general perspective the same basic question leads to the problem of the axiomatic structure of mathematics. Would it be possible, would it be useful to familiarize high school pupils with the concepts and the techniques of the axiomatic organization of a branch of mathematics?

e. It is a frequent practice to use various pictorial representations, structured materials and audio-visual techniques in order to stimulate the interest of pupils for mathematics and their understanding of mathematical concepts and operations. As a matter of fact, very little systematic evidence is available concerning the didactical effects of such means. It may be supposed that many of them do more harm than good. It is of major importance to get a serious evaluation of these categories of didactical techniques.

A similar problem is that concerning the use of pocket calculators and computers. If misused, they may do much harm to the learner, and this effect may be irreversible. When and how to introduce in the process of mathematical education pocket calculators or micro computers? It seems to me that this is a very urgent and very important problem which deserves careful and thorough investigation. It is, in my opinion, a great danger that commercial pressure combined with the ignorance (in psycho-educational problems) of curriculum designers will contribute in the near future to the ruin of mathematical education.

C. The last category of problems I would like to mention is that of educational research itself in mathematics. Only very recently we became aware of the fact that by simply applying general psychological or educational concepts and investigation techniques we are not able to advance very much in improving the teaching of mathematics. Complaints coming from math teachers and from researchers as well, emphasize the fact that the teaching of mathematics has profited very little from the enormous quantity of investigations carried out in developmental and educational psychology, and, more specifically in mathematical education.

It may be supposed that the basic approach is, at least, non efficient if not completely wrong. It is the general problem of the strategy of applicative investigations in behavioral science which is at stake here.

I wrote much more than was my initial intention. I hope that you will find one or two useful suggestions for your enquiry.

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WILLEM KUIJK

There are principally two ways along which one can approach mathematics learning, viz. along (i) the macroscale "environmental way" of sociology, psychology, epistemology, etc. (which provides insights into motivational drives and external stimuli) and (ii) the microscale "physiological way" which asks for the "internal" conditions of the brain and body that make possible (or impossible) a student's progress in mathematics. The following problems have an implicit reference to what is known about the relation between these two ways.

> *PROBLEM 1.* Take any elemental mathematical activity (digial counting, counting tones, symbol picturing, spatial picturing, mental calculation, etc.). Find in the brains of people of different age categories (from youngsters to old people) the subnetworks of the brain (or the so-called "focal areas" of the neo-cortex) that show a significant increase of chemo-electrical activity (and a concomitant increased blood-flow) when that selected elemental activity obtains, excluding the interference of all other activities. More advancedly, find the cerebral pathways that connect these focal areas if and when people are engaged in the dynamical process which we call discovery, problem solving, mathematical creativity and learning.

Teachers in math often hit upon students who cannot do certain mathematical things, while doing other (mathematical) things well enough. E.g. mental calculations are O.K., but there is no appetite for geometry, etc. Sometimes "blocks" of this type disappear with age or effort.

> *PROBLEM 2.* If blocks of this type persist then a special study of the student's brain might be undertaken with PET-scans or even better instruments (NMR). Brain-damaged children or children with learning difficulties could teach us a lot about healthy children. What do these scans teach us?

From general psychology and pedagogy it is known that certain things have to be learnt first in order that other things may be learnt without fail. There is a kind of *flexible hierarchy* making certain things more fundamental than other things, e.g. since they can be picked up more spontaneously than others.

> *PROBLEM 3.* Is there a similar hierarchy in the construction of the body of mathematics?

In a rough sense it seems there is; this shows in particular if one looks at people whose mathematics education contains big "holes". Their education misses the purpose(s) for which it had been given, and reduces at any rate the efficiency which could have been there. On the other hand, it seems that, on the contrary, *new* mathematical *view-points* — in particular those that primarily (and tentatively) pursue mathematics for its *own purpose* thrive on the flexibility aspect.

> *PROBLEM 4.* To what extent can, in mathematics education, the hierarchy aspect be "tampered with" without undermining the stated purpose of that education? To what extent is a hierarchy, woven into a curriculum, a determinant of special types of mathematics education (e.g. vocational training of sorts, technology etc.)?

It is obvious from history that there is a covariant relationship between needs of a precise and exact nature within a society (e.g. demography, the requirements of the army and of industries etc.) and the development of mathematical theories.

> *PROBLEM 5.* To what extent can the "axiomatic method" and the mathematical requirement that things have to be *proved* on the basis of axioms, be connected with the nature of those needs (mechanization, automation, efficiency etc.)?

The introduction of "new math" in secondary schools in many western countries meant the replacement of proofs in geometry (Euclidean) by proofs in set theory, Boolean algebra, etc. Classical formula manipulation and elementary number theory had to give way to making acquaintance with "structures", and "proof" became more formal. The student did not any more have to gain a considerable insight into any theory *showing at the same time* a high hierarchy of deductive complexity as well as a great direct intuitive content (as formerly students had to in Euclidean geometry). Why then do students having had new math show a lesser understanding of what proof is? Or do they not?

> *PROBLEM 6.* (i) Is the present newest of trends, namely computerized Turtle Geometry, to be interpreted as a compensation for the lack of concrete imagery (gestalten, etc.) in new math, or can it serve as such?
> (ii) Is the kind of imagery of abstract set theory not less "fundamental" (in the sense of Problem 3 and 4; e.g. since it comes about less spontaneously) than those in classical (and even Turtle) geometries? Should, in education, what physiologically comes first, not come first? And is the deductive order not merely methodical, and not psychological or "developmental"?

In a sense "new math" means a shift in the training of the brain, viz. a shift to the left hemisphere, in the sense that holistic and concrete images in geometry and otherwise are played down. In the process, more interest is taken in that part of all possible mathematics that is computerizable, automatizable, (recursively) decidable, linear, non-standardizable and learnable at a relative low level of mental concentration, and by rote.

> *PROBLEM 7.* Does "new math" not pull the rug out from under all those mathematical subjects that are not computerisable and recursive but analytical, topological, non-linear, and requiring a high degree of tense imagination and mental concentration?

If the core of mathematics creativity is the joining and separation of (i.e. an interplay between) formal-lingual and pictorial contents of the brain, then mathematics educators do best when they present students with learning situations wherein both types of content are clearly recognizable.

These questions I put to myself and now to you. They do not all concern education directly. However, how can one answer educational questions without probing into the nature of mathematical thinking?

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GERARD VERGNAUD

Your game of "problems to be solved in maths education" is a promising one. Although I think maths education research has not yet developed precisely enough, and therefore has not come to the stage of well-defined problems, I will try to play. There are three problems:

1) How do children's and adolescents' conceptions about addition, subtraction, multiplication and division change? Through which situations? Through which steps?
2) How can we relate algebra and functions (and especially the concepts of unknown, variable and parameter) to meaningful situations? and still keep the mathematical core of algebra and function theory?
3) How does computer science help and eventually disturb mathematics education?

I will stop here because my questions are very general. May I hope that, from 60 persons, you don't get 60*n* different questions (*n* being the average number of problems given).

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