MUSINGS ON MATHEMATICAL CREATIVITY

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This conversation began when I introduced the two authors in a hotel lobby in Cap Roig (Spain) after the close of CERME4 [1]. The conversation continued by e-mail well after the evening had ended. (ed.)

Bharath: I think that the notion of mathematical creativity among mathematicians is distinctly different from the notion of creativity found in the mathematics education literature. What do you think? Do we even have an agreed upon definition of what mathematical creativity is? Is this important at all?

Peter: I too wish to discuss the defining of creativity. This certainly is important. The writings in my dissertation express a slightly different parsing of creativity than the one you propose (*i.e.*, among mathematicians, psychologists, teachers, mathematics educators).

Bharath: I would like to give some background to the AHA! moment that you talked about (Liljedahl, 2004). The literature seems to attribute this construct to Wallas (1926) and his famous book *The art of thought*. Some authors erroneously attribute it to Hadamard and Poincaré due to the popularization of their writings.

However, this construct was developed within Gestalt psychology in Germany in the very early part of the 20th century by Wertheimer, Koehler, and Koffka. Some historians push it back to the late 19th century to the writings of Mach (a physicist turned philosopher interested in the physiology of sensory perception). There is evidence of written communication between Poincaré, Hadamard and the Gestaltists, which could lead one to infer that Hadamard was influenced by the developments and the terminology within Gestalt psychology. Hence the use of the 3- or 4-stage model from then on.

I really liked your critique of the reductionist attempt of viewing creativity as a confluence of person, process and product, as well as your critique of the misconceptions that occur when using *creative* or *creativity*. Should we focus our conversation on the definition of creativity by considering each aspect (person, process and product) separately and then analyze whether this is consistent (or inconsistent) with various confluence models that combine the three along with societal variables proposed by researchers in psychology? Our attempt could perhaps be to construct a definition of creativity for the particular domain of mathematics and then seeing whether general definitions hold up?

Incidentally, Ervynck (1991) described mathematical creativity in terms of three stages. The first stage *(Stage 0)* is referred to as the *preliminary technical stage*, which consists of

> some kind of technical or practical application of mathematical rules and procedures, without the user having any awareness of the theoretical foundation. (p. 42)

The second stage *(Stage 1)* is that of *algorithmic activity*, which consists primarily of performing mathematical techniques, such as explicitly applying an algorithm repeatedly. The third stage *(Stage 2)* is referred to as *creative (conceptual, constructive) activity*; true mathematical creativity occurs and consists of non-algorithmic decision making:

> The decisions that have to be taken may be of a widely divergent nature and always involve a choice. (p. 43)

Although Ervynck (1991) tries to describe the process by which a mathematician arrives at the questions through his characterizations of *Stage 0* and *Stage 1*, his description of mathematical creativity is very similar to that of Poincaré and Hadamard. In particular, his use of the term "non-algorithmic decision making" seems to be analogous to Poincaré's use of the "choice" metaphor.

Peter: As I turn my mind back to our conversation thus far many ideas come to the fore. To begin with, you correctly identify my reductionist attempt at defining creativity as a critique. I find that many definitions (and authors of definitions) rely too much on rhetoric. The discourses that emerge are like a sieve – they are easily grasped but they don't hold much water. Attempting to fit actual instances of (perceived) creativity into these discourses, the definitions are often too confining (even rigid). One of the findings that emerged from my own work with mathematicians on their AHA! experiences (and I defined the AHA! experience in the four stage Gestaltist way of initiation-incubation-illumination-verification as discussed in Wallas, 1926) was that the experience was self defining. That is, I did not have to define the experience for the mathematicians in order for them to know that they had one. As it turned out, this was also true for undergraduate mathematics students, as well as pre-service elementary school teachers.

I wonder if we cannot say the same thing for creativity – creativity is self-defining. Unfortunately, such a position does not advance our emerging (or at least attempts at emerging) definition, but it does allow us to include all things that are creative and exclude all things that are not. One of the downfalls of such a reliance on self-definition is the curse of discourse (or as I call it, discurse). How much of mathematicians' understanding of creativity/invention/discovery is a product of the discourse on these ideas within their domain? For example, you cite a mathematician (in Sriraman, 2004) as stating that "Opportunity knocks but you have to be able to answer the door" (p. 32). I had a mathematician state "Chance will favour those who are prepared." Aren't both of these mathematicians parroting Louis Pasteur's phrase "Chance favours the prepared mind"? For another example consider the following two statements:

> Perhaps I could best describe my experience of doing mathematics in terms of entering a dark mansion. One goes into the first room, and it's dark, completely dark. One stumbles around bumping into the furniture, and

> gradually, you learn where each piece of furniture is, and finally, after six months or so, you find the light switch. You turn it on, and suddenly, it's all illuminated. (Wiles, 1997)
>
> It is like going into an unfamiliar hotel room late at night without knowing even where to switch on the light. You stumble around in the dark room, perceive confused black masses, feel one or the other piece of furniture as you are groping for the switch. Then, having found it, you turn on the light and everything becomes clear. (Polya, 1965, p. 54)

My final point is that whatever definition we arrive at it needs to be relativistic. As Hadamard points out,

> Between the work of a student who tries to solve a difficult problem in geometry or algebra and a work of invention there is only a difference of degree. (p. 104)

If we use the word creation instead of invention in this quotation, the implications become interesting. Creation implies creativity. Does this mean that invention presupposes creativity? I think so. But does it also imply that creativity presupposes invention? I don't think so, and this leads to my stance on relativistic as opposed to absolutist views of creativity.

Bharath: Your thoughts on having a relativistic definition as opposed to an absolutist definition are interesting and I will comment on this shortly.

First, I will comment further on the Gestaltists and the fact that redundancies abound in extant descriptions of creativity. I was recently re-reading Schoenfeld (2002) and, interestingly enough, he points out that insight and structure were central concerns of the Gestaltists. This article also recalls the famous story of Poincaré taking a day trip after having struggled on a problem for a while and his experience of stumbling upon the solution just as he boards the bus. However, Schoenfeld's comments after this anecdote are:

> Poincaré's story is typical, both in substance and methods. With regard to substance the outline of the story is the basic tale of Gestalt discovery: One works as hard as possible on a problem, lets it incubate in the subconscious, has an insight, and verifies it. Similar stories are told concerning the chemist Kekulé's dreaming of a snake biting its tail, and realizing that benzene must be ring-like in structure, and of Archimedes (in the bath) solving the problem of how to determine whether King Heron's crown is pure gold, without damaging the crown itself. With regard to method, what Poincaré offered is a retrospective report. (Schoenfeld, 2002, p. 438)

In other words, the research carried out by the Gestaltists was by and large retrospective and relied on mathematicians and scientists reporting on their thinking after the fact. Numerous critiques are available on the unreliability of this research method. However, the features of creative thinking as proposed by the Gestaltists were meaning, insight and structure. Any definition should include these elements (whether they are 'measurable' or how they are identified is a completely different matter).

If we move away from the domain of mathematics to the general literature on creativity, numerous definitions are found. Craft (2003) used the term "life wide creativity" to describe the numerous contexts of day-to-day life in which the phenomenon of creativity manifests. Other researchers have described creativity as a natural "survival" or "adaptive" response of humans in an ever-changing environment (Gruber, 1989; Ripple, 1989).

Craft (2003) points out that it is essential we distinguish "everyday creativity" such as improvising on a recipe from "extraordinary creativity", which causes paradigm shifts in a specific body of knowledge. It is generally accepted that works of "extraordinary creativity" can be judged only by experts within a specific domain of knowledge (Csikszentmihalyi, 1988, 2000; Craft, 2003). For instance Andrew Wiles' proof of *Fermat's Last Theorem* could only be judged by a handful of mathematicians within a very specific subdomain of number theory.

Throughout school levels or even the beginning undergraduate level, I normally do not expect works of extraordinary creativity. However, I do think it is feasible for students to offer new insights/solutions to a mathematics problem. Getting back to definitions, in psychology the literature defines creativity as the ability to produce unexpected original work, which is useful and adaptive (Sternberg and Lubart, 2000). Other definitions usually impose the requirement of novelty, innovation or unusualness of a response to a given problem (Torrance, 1974).

Numerous confluence theories of creativity define creativity as a convergence of knowledge, ability, thinking style, motivational and environmental variables (Sternberg and Lubart, 1996, 2000), an evolution of domain specific ideas resulting in a creative outcome (Gruber and Wallace, 2000). Most recently, Plucker and Beghetto (2004) offered an empirical definition of creativity based on a survey and synthesis of numerous empirical studies in the field. They defined creativity as

> the interplay between ability and process by which an individual or group produces an outcome or product that is both novel and useful as defined within some social context. (p. 156)

Could a synthesis of these numerous definitions of creativity lead to a working definition of mathematical creativity at both the professional and school levels? Would one definition work for both levels? At the professional level mathematical creativity can be defined as:

1. the ability to produce original work that significantly extends the body of knowledge (which could also include significant syntheses and extensions of known ideas)

2. opens up avenues of new questions for other mathematicians. [2]

To illustrate the first point, Wedderburn's theorem that a finite division ring is a field is one instance of a unification of apparently random fragments because the proof involves algebra, complex analysis and number theory. Hewitt's (1948) paper on rings of continuous functions led to unexplored possibilities and questions in the fields of analysis and topology that sustained other mathematicians for decades. This would be an example of the latter.

Most descriptions of creativity also include the element of chance. Should this be included in the definition? Although Poincaré attributed his particular breakthrough in Fuchsian functions to chance, he did acknowledge that there was a considerable amount of previous conscious effort, followed by a period of unconscious work. Hadamard (1945) argued that even if Poincaré 's breakthrough was a result of chance alone, chance alone was insufficient to explain the considerable body of creative work credited to Poincaré in almost every area of mathematics.

One question then is how does (psychological) chance work? I conjecture that the mind throws out fragments (ideas), which are products of past experience. Some of these random fragments can be juxtaposed and combined in a meaningful way. For example, if one reads a complicated proof consisting of a thousand steps, a thousand random fragments may not be enough to construct a meaningful proof. However, the mind chooses relevant fragments from these random fragments and links them into something meaningful (such as Wedderburn's proof).

I agree with your relativistic stance on any definition of creativity. It would be difficult (but not impossible) for a student (at the school levels) to meet the criteria in the proposed definition. Instead, could we define mathematical creativity at the school levels as:

1. the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems, and/or
2. the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle (this bears resemblance to Kuhn's ideas). (Sriraman, [2])

Peter: This final definition that you offer is more palatable than others I have seen; it has the relativistic quality that I have been seeking. It also has an absolutist overtone with regards to "solution(s)". Perhaps this is necessary. I don't know, but I would like to explore it for a moment.

Almost all of the definitions make some reference to a product, a solution, or an outcome. What is it that a product/solution/outcome gives us that makes it so important? I suggest that the answer to this question lies in Schoenfeld's observation/criticism presented above. Without a product all that remains is a story; a story without an ending. I think that the centrality of product has to do with having a tangible and objective artifact by which to judge the art – by which to judge the *process* – by which to *study* the process. But, creative process isn't about product, it is about process. Our inability to access this most private of processes does not change this fact. Can we be creative without creating something? I don't think we can, but this does not make them one and the same. The creative process is both inseparable and unrecognizable within the creative product. So, the question remains – what is mathematical creativity?

Bharath: Perhaps this very enterprise of trying to define this construct falls within your description of a sieve – we are engaged in the discourse, using words that allude to it, but are failing to define it adequately. Given the reductionist pitfalls of our enterprise here, I have to resort to the words of the practical yet mystical Chinese Taoist, Chuang Tzu (1968):

> Heaven and earth were born at the same time I was, and the ten thousand things are one with me. We have already become one, so how can I say anything? But I have just said that we are one, so how can I not be saying something? The one and what I said about it make two, and two and the original one makes three. If we go on this way, then even the cleverest mathematician can't tell where we'll end, much less an ordinary man … Better not to move, but to let things be! (p. 43)

Notes

[1] *Fourth Congress of the European Society for Research in Mathematics Education*, Sant Feliu de Guíxols, Spain, 17-21 February, 2005.

[2] An article entitled 'Are mathematical giftedness and mathematical creativity synonyms? A theoretical analysis of constructs', by B. Sriraman is in press with *The Journal of Secondary Gifted Education*.

References

Chuang Tzu (1968, Watson, B., translator) The complete works of Chuang Tzu, New York, NY, Columbia University.

Craft, A. (2003) 'The limits to creativity in education: dilemmas for the educator', *British Journal of Educational Studies* **51**(2), 113-127.

Csikszentmihalyi, M. (1988) 'Society, culture, and person: a systems view of creativity', in Sternberg, R. (ed.), *The nature of creativity: contemporary psychological perspectives*, Cambridge, UK, Cambridge University Press, pp. 325-339.

Csikszentmihalyi, M. (2000) 'Implications of a systems perspective for the study of creativity', in Sternberg, R. (ed.), *Handbook of creativity*, Cambridge, UK, Cambridge University Press, pp. 313-338.

Ervynck, G. (1991) 'Mathematical creativity', in Tall, D. (ed.), *Advanced Mathematical Thinking*, Dordrecht, The Netherlands, Kluwer Academic Publishers, pp. 42-53.

Gruber, H. (1989) 'The evolving systems approach to creative work', in Wallace, D. and Gruber, H. (eds), *Creative people at work: twelve cognitive case studies*, Oxford, UK, Oxford University Press.

Gruber, H. and Wallace, D. (2000) 'The case study method and evolving systems approach for understanding unique creative people at work', in Sternberg, R. (ed.), *Handbook of creativity*, Cambridge, UK, Cambridge University Press, pp. 93-115.

Hadamard, J. (1945) *Essay on the psychology of invention in the mathematical field*, Princeton, NJ, Princeton University Press.

Hewitt, E. (1948) 'Rings of real-valued continuous functions', *Transactions of the American Mathematical Society* **64**, 45-99.

Liljedahl, P. (2004) *The AHA! experience: mathematical contexts, pedagogical implications*, Unpublished doctoral dissertation, Simon Fraser University, Burnaby, British Columbia, Canada.

Polya, G. (1965) *Mathematical discovery: on understanding, learning and teaching problem solving (vol. 2)*, New York, NY, Wiley.

Plucker, J. and Beghetto, R. (2004) 'Why creativity is domain general, why it looks domain specific, and why the distinction does not matter', in Sternberg, R., Grigorenko, E. and Singer, J. (eds), *Creativity: from potential to realization*, Washington, DC, American Psychological Association, pp. 153-168.

Ripple, R. (1989) 'Ordinary creativity', *Contemporary Educational Psychology* **14**, 189-202.

Schoenfeld, A. (2002) 'Research methods in (mathematics) education', in English, L. (ed.), *Handbook of international research in mathematics education*, Mahwah, NJ, Lawrence Erlbaum Associates, pp. 435-487.

Sriraman, B. (2004) 'The characteristics of mathematical creativity', *The Mathematics Educator* **14**(1), 19-34.

Sternberg. R. and Lubart, T. (2000) 'The concept of creativity: prospects and paradigms', in Sternberg, R. (ed.), *Handbook of creativity*, Cambridge, UK, Cambridge University Press, pp. 93-115.

Sternberg, R. and Lubart, T. (1996) 'Investing in creativity', *American Psychologist* **51**, 677-688.

Torrance, E. (1974) *Torrance tests of creative thinking: norms-technical manual*, Lexington, MA, Ginn.

Wallas, G. (1926) *The art of thought*, New York, NY, Harcourt Brace.

Wiles, A. (1997, NOVA, 1997, updated 2000) *The proof*, Aired on PBS on October 28, 1997, http://www.pbs. org/wgbh/nova/proof/wiles.html, accessed January 5th, 2003.