DUALITY IN COMBINATORIAL NOTATION

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Combinatorics problems are most commonly about counting; answers are cardinalities, and cardinalities are enumerations of elements in sets. Cardinalities—*i.e.,* solutions to counting problems—are often represented by numerical *expressions/formulas, e.g., C*(5, 3) × 3!, and elements in sets are often listed—or at least partially listed—as illustrative of *sets of outcomes, e.g.,* {*ABC, ABD, …, BAC, …*}. Although counting problems can be easy to state, they can be deceptively difficult to solve (*e.g.,* English, 1991; Martin, 2001; Tucker, 2012).

One interesting perspective to consider is not that of the combinatorial expert, but that of the combinatorial novice, or learner. Lockwood (2013) considered how students navigated between three important aspects of combinatorial problems: expressions/formulas and sets of outcomes, mentioned above, as well as counting processes—which are descriptions of an ordered procedure by which one could enumerate the desired solution. An interesting finding was that students had difficulty directly relating *expressions/formulas* and *sets of outcomes* (they often only mediated these two indirectly through counting processes). The key point is that in perhaps the most important relationship of all—of connecting expressions/formulas as *cardinalities* that themselves are enumerations of elements in sets of *outcomes*—students struggled. As a proposal, Lockwood (2014) advocated helping students develop a *set-oriented perspective* on combinatorial problems, because understanding problems in terms of sets (of outcomes) develops conceptual—and not just procedural—knowledge in combinatorics.

I argue here that one reason students struggle with connecting sets and the expressions/formulas that capture their cardinalities has to do with combinatorial notation; the mathematical symbols themselves give preference to an expert, not a learner, in ways that make recognition of this fundamental connection less transparent. That is, combinatorial symbolism itself might be an impediment to developing robust set-oriented perspectives. This is relevant to mathematics education—in particular, to the teaching and learning of combinatorics and ways to improve it. In accord with Lockwood (2014), the underlying pedagogical focus is helping develop set-oriented perspectives in learners. Throughout, one of my aims is to amplify the voice and perspective of the learner—as a means to contrast with an expert, and as a means to provide insight into the genuine challenges learners face. The combinatorial challenges elaborated here would be true for any learner—including secondary and undergraduate students; however, the notational solutions proposed might be more appropriate for undergraduate students, who would be more familiar with sets and set notation. As such, 'learners' in this context is best understood as referring to undergraduate students. Learners are regarded as having resources, not deficits, and as being sensible, not malicious, agents in the learning process—they are capable of, and trying their best to, make sense of the mathematics. Additionally, I also take an overtly mathematical approach to the issue of learning; looking carefully at the mathematical foundations as they relate to the field of combinatorics. The reason for doing so is that I consider mathematical symbolism—the mathematics—as part of the challenge, but also part of a solution for teaching and learning in the classroom. Attending to and analyzing the mathematical aspects is used to identify ways to make increasingly explicit the opportunities learners have in their mathematics education to develop a set-oriented, conceptual, perspective on combinatorics.

Background

A mathematical symbol is a character, sign, or mark used as a conventional representation of something else—of a mathematical object, idea, concept, process, *etc.* Mathematical notation is a system of such symbols. Because mathematical ideas are, by their very nature, abstract, notation and symbols are a necessary part of interacting with, and communicating about, mathematics. Indeed, part of the work of mathematicians throughout history has been the creation, development, and refinement of symbols and notation. These have not only served as a vehicle—a language—for communication, but have also afforded insights into mathematics itself. Use of decimal notation for numbers, for example, afforded great insights into the structure of the real numbers; algebraic symbolism greatly facilitated the development of algebraic methods; common symbols for derivatives and integrals helped communicate the fundamental connections between the two; *etc.* In some sense, one might regard the creation of symbols and notation as affording communication about mathematical ideas amongst mathematicians—amongst experts. But, as discussed in previous articles in this journal, such as Tillema and Hackenberg (2011), allowing for additional symbolism that notates important developments for learners can be a powerful tool for teaching and learning.

Gray and Tall (1994) used the term *procept* to describe the duality that mathematical symbols often convey. Consider the symbolism of 3 + 2. In one sense, it represents a process of addition—of adding two to three, which yields five. One might clarify this symbol as representing (3) + (2), which emphasizes the process of addition between two numbers. In another sense, however, the symbolism also expresses the sum itself, the concept—for which we actually need not know the sum's value. That is, the symbol represents (3 + 2), which emphasizes the singular nature the symbol also conveys—an object in its own right. Gray and Tall's argument was that for experts the symbolism is typically a non-issue,

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but for learners, the dual nature with which we use symbols poses challenges. Ask a young student if 3 + 2 = 2 + 3, and you might get an answer of no, because the expressions are not identical, or you might get a yes, because they both equal five. The key point is that even for those students who ultimately answer yes, their justification typically involves interpreting the symbols of 3 + 2 and 2 + 3 as a sign to engage in the process of addition, resulting in five for both—a reason for their equality. For an expert, however, even this is to some degree unsatisfactory. We want students to recognize both of the symbols, 3 + 2 and 2 + 3, as expressions of the sum itself; and the equality representative of the fundamentally important commutative property of addition that relates these two sums. The key point is that the dual way that symbols are sometimes used in mathematics poses challenges to learners, even if it is not problematic for experts.

This conversation about mathematical symbols and notation can be further grounded in the field of semiotics. Semiotics, generally, is the study of how people use and make sense of signs. Signs are that which represent an object; a mathematical symbol being a kind of sign. Here, I do not attempt to exhaust the literature on semiotics, nor reconcile different theoretical perspectives. Instead, I leverage a basic semiotic distinction, of differentiating between the *signified* (the object being represented) and the *signifier* (the sign or representation for the object). Mathematical objects are by their very nature *not* accessible by perception, but rather *only* accessible via signs and semiotic representations (Duval, 2006), and so we *need* signifiers to represent mathematical objects—part of the reason semiotics is argued to be of import in mathematics education (Presmeg *et al.,* 2016). In other words, the act of learning mathematics itself is an activity of making meaning from particular signs and symbols, which are used as representations of the mathematical objects. Particularly challenging for students is the fact that the same mathematical idea or object can have several (very) different interpretations—as exemplified previously. That is, the *same signifier* might represent *different signifieds.* In the example of 3 + 2 = 2 + 3, the sign signified to the learner the process of addition, but to the expert, the concept of sum. When signs might signify, or be intended to signify, different things, there is the potential for confusion for the learner because of the possible miscommunication.

Combinatorial symbols and notation

In this section, I explore common symbols in combinatorics—$P(n,r)$, $C(n,r)$, $n!$, *etc.*—and I use them to explain why learners might have fundamental difficulty directly relating *cardinalities* and *sets*. The key thrust is that we use the same symbols to signify both. Having signs that are explicitly defined as cardinalities, and only implicitly understood in relation to sets, is problematic for teaching and learning, if a primary aim of combinatorial education is for learners to develop robust set-oriented perspectives, because learners can interpret symbols and statements without ever having to consider sets.

Consider the mathematical symbol used to denote a combination. A not uncommon introduction to the idea is taken from Tucker (2012): "An *r*-**combination** of *n* distinct objects is an unordered selection, or *subset,* of *r* out of the *n* objects. We use […] $C(n, r)$ to denote the number of […] *r*-combinations […] of a set of *n* objects" (p. 189–190). After a few more sentences of explanation, the following equation is given:

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!} = \frac{n!}{r!\,(n-r)!}$$

At this stage, we have been given a nice introduction to combinations, described as a counting process, and to the symbol $C(n, r)$, which is clarified to denote the number—*i.e.,* cardinality—of a set. Indeed, we even get to a fundamentally essential statement in combinatorics: for given natural numbers n, r (with $0 \leq r \leq n$), the number of *r*-combinations is easily quantifiable, $n! \,/\, (r!(n - r)!)$. Additionally, this statement about combinations also uses, and references, factorials and *r*-permutations, *i.e.,* $P(n, r)=n \cdot (n - 1) \cdot \ldots \cdot (n - r + 1) = n! \,/\, (n - r)!$. Unpacking the meaning of the statement that $C(n, r)$ is equal to $n! \,/\, (r!(n - r)!)$ rests, of course, on one understanding the factorial operation for a natural number, signified by !, and its boundary case definitions, *e.g.,* $0! = 1$.

The key point is that the symbol $C(n, r)$, by *definition*, signifies a numerical value—a *cardinality*. This symbol is, of course, important; indeed, the combination symbol is used often in expressions/formulas as part of the solution to a combinatorial problem. One might even argue that combinations, and our ability to express the cardinality of such a set, has been essential for development in the field of combinatorics—*e.g.,* combinations are the coefficients in the binomial expansion $(x + y)^n$, they are the way we determine the cardinality of multi-combinations, $MC(n, r) = C(r + n - 1, r)$, etc.

Yet, while $C(n, r)$ is a cardinality, we do not have common symbolism to signify the set. That is,

$$C(n,r) = |\text{some set we do not use a symbol to describe}|$$

Indeed, experts likely need no other symbol; experts can navigate the dual-nature of the sign $C(n, r)$ as representing the cardinality and also the set without much problem. For learners, however, this duality may provide some insight into why students often struggle to directly connect expressions/formulas with sets of outcomes: there is nothing in combinatorial notation that i) differentiates these dual uses, nor ii) connects them. In what follows, I explore two examples, contrasting a learner's and an expert's perspective on combinatorial interpretation and meaning.

A combination identity

Consider, as a first example, the identity:

$$C(n,r) = C(n,n-r)$$

This identity is fundamentally important in combinatorics and can be justified in many ways.

Learner's perspective: One interpretation of the identity $C(n, r)=C(n, n - r)$ is as an algebraic exercise. My experience in the classroom suggests this is often what the identity signifies to learners. That is, the semiotic chaining in which a learner often engages with respect to the identity is that the symbol $C(n, r)$ signifies a process of enumeration which can be accomplished by calculating $n! / (r!(n - r)!)$, and $C(n, n-r)$ signifies a similar process, accomplished by calculating $n! / ((n - r)!(n - (n - r))!) = n! / ((n - r)!r!)$. Hence, they are equal (because multiplication is commutative). One thing I am trying to suggest is that this 'algebraic-exercise' interpretation is extremely sensible; the symbolism for $C(n, r)$ in fact has been *defined* as the quantity $n! / (r!(n - r)!)$. Yet the algebraic exercise is extremely trivial. Moreover, the algebraic exercise in no way calls for students to grapple with the connection between sets and cardinalities; that is, this sensible, learner's interpretation of the symbolism and the identity does not serve to cultivate developing a set-oriented perspective in combinatorics, which makes the identity feel trivial and meaningless.

Expert's perspective: Experts would recognize this identity as a non-trivial insight. Despite the definition of the symbol $C(n, r)$ as a quantity, the statement of the identity is not interpreted by experts as the inane 'algebraic exercise':

$$C(n,r) = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,r!} = C(n,n-r)$$

The reason experts regard the identity as insightful is because it is a statement about the relationship between the cardinalities of two *sets*—two sets which are *different*, but which have the *same* cardinality. That is, the semiotic chaining in which an expert often engages with respect to the identity is that the symbol $C(n, r)$ signifies a set of r-combinations, $C(n, n - r)$ signifies a set of $(n - r)$-combinations, and the equality signifies that their cardinalities are the same. In other words, the combinatorial insight of the identity as understood by an expert demands one interpret the combination symbols as actually signifying two *sets*, not *just* their cardinalities—a duality of the use of the symbol. This duality is not problematic for experts; but for a learner, the 'algebraic-exercise' interpretation illustrates how this duality might not help foster a robust set-oriented perspective in leaners—one that could connect cardinalities, as expressed by expressions/formulas, and their associated sets. For example, for the set {1,2,3,4,5,6}, the statement $C(6, 2) = C(6, 4)$ is actually signifying three things to an expert (depicted in Figure 1): i) the set of 2-combinations (Set 1); ii) the set of 4-combinations (Set 2); and iii) the bijective mapping between these two different sets, which explains why their cardinalities are *equal*. Because this mapping is easily generalizable (*i.e.,* a single r-combination (in Set 1) has r elements, and maps uniquely to the $(n - r)$-combination (in Set 2) that has none of those r elements), the identity works in the general case.

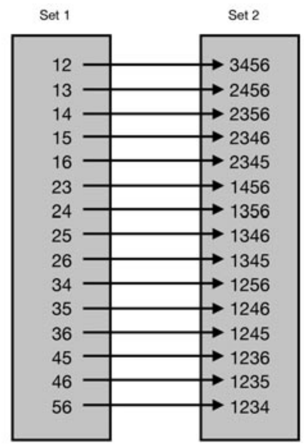

Figure 1. The two sets, and the bijective mapping, implicit in the identity C(6, 2)=C(6, 4).

Multiplication by 1

Consider, as a second example, the following statement relating factorials and permutations: $n! = P(n, n - 1)$. This statement amounts to the following:

$$n \cdot (n-1) \cdot \ldots 3 \cdot 2 \cdot 1 = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2$$

These are equivalent because they differ only by multiplication by 1. And although I use this particular example to discuss multiplication by 1 in combinatorics, there are many other combinatorial expressions that exemplify the same idea, *e.g.,* $C(8, 6) \cdot C(2, 2) = C(8, 6)$.

Learner's perspective: One interpretation of this statement, already mentioned, is that there is no meaningful difference between these two expressions because 1 is the multiplicative identity—a numerical exercise. My classroom experience suggests that this is a common sentiment amongst learners. That is, the semiotic chaining in which a learner often engages with respect to factorials and permutations is that these symbols signify a process of finding the product of several numbers. In this one example, they are equal because multiplication by 1 does not change the product. Indeed, writing down the multiplication by 1 with this 'numerical-exercise' interpretation feels unnecessary; the product does not change. This is also a sensible interpretation of the statement, because factorials and permutations have been defined as products and because the solutions to counting problems often involve operating on numbers, including multiplication. Yet such a numerical exercise feels trivial; moreover, it does not necessitate considering sets in any meaningful way.

Expert's perspective: Although multiplication by 1 does not make any difference in terms of the numerical product—the cardinality—an additional consideration of the sets provides some insight. Indeed, although the cardinalities of the two sets are the same, the sets themselves are different. That is, the semiotic chaining in which an expert often engages with respect to the statement is that the symbol $n!$ signifies a set of permutations of n elements, whereas $P(n, n - 1)$ signifies a set of $(n - 1)$-permutations, which is a set of permutations of $n - 1$ elements *(i.e.,* a different set, with different elements); the equality signifies that their cardinalities are the same. In other words, a set-oriented perspective yields what might be interesting about multiplication by 1—the sets are different but the cardinalities the same. Recognizing this, however, requires a dual-interpretation of the symbolism; the factorial and permutation symbolism, to an expert, represent not just numerical products but also particular sets. For example, for the set {1,2,3,4,5,6}, the statement $6! = P(6, 5)$ is actually signifying three things to an expert (depicted partially in Figure 2): i) the set of

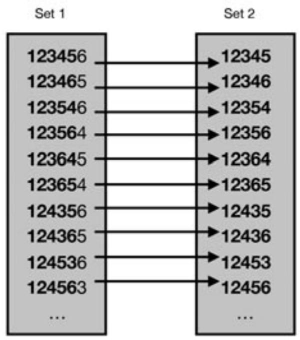

Figure 2. The two sets, and the bijective mapping, implicit in multiplication by 1.

permutations of 6 elements (Set 1); ii) the set of permutations of 5 (of the 6) elements (Set 2); and iii) the bijective mapping between these two different sets. In the case of multiplication by 1, this mapping is easily generalizable; an element (in Set 2) corresponds to the same element, but with one additional, unique, addition (in Set 1). That is, rearranging n elements has the same number of possibilities as rearranging $n - 1$ elements, because any rearrangement of n elements can be considered a rearrangement of $n - 1$ elements along with the one (unique) leftover element indicated/added at the end. In general, multiplication in combinatorics is related to a Cartesian product of sets; multiplication by 1, then, is the Cartesian product between a set A and a singleton set $\{a\}$, which produces a *different* set, but that has the same cardinality.

In sum, what we see from these two examples is that combinatorial expressions and statements (*e.g.,* $C(n, r) = C(n, n - r)$) can be interpreted differently by learners and by experts. For one interpretation, it is trivial; for the other, non-trivial. The difference is in the interpretation of the symbolism. Experts are able to interpret the dual nature of the combinatorial signs (*i.e.,* $C(n, r)$, $n!$, $P(n, r)$). They recognize these signs as signifying not just quantity, or cardinality, but also as signifying a set itself. Yet, technically, the symbols explicitly signify only a cardinality; there is no symbol that signifies the set. If we aim for students to develop robust set-oriented perspectives, because doing so fosters acquiring the kinds of meaningful and conceptual understanding that is desirable in the field of combinatorics, it is important that the symbols used help signify both of these important interpretations. In these two examples, such symbolism would help learners recognize the profundity of the combinatorial statements, which rests on recognizing the duality that i) the *sets* are different, but that, despite this difference, ii) their *cardinalities* are the same.

Addressing the duality of combinatorial symbols

Essentially, I argue that while, for experts, the dual nature of combinatorial symbols as used to signify both cardinality *and* set may not pose a substantial problem, for learners, it does. Indeed, this observation can be related to observed phenomena in educational research with students, in which learners have difficulty directly linking expressions/formulas to sets of outcomes (*e.g.,* Lockwood, 2013; Lockwood, Wasserman & McGuffey, 2018; Wasserman & Galarza, 2019). Pedagogically, helping students develop a robust set-oriented perspective in combinatorics would be valuable—largely, because it helps make combinatorically conceptual what might be considered very algebraically/numerically procedural. To do so, I argue that we need to find ways to disentangle the dual use of current combinatorial symbolism; one way to accomplish this, perhaps most appropriate in an undergraduate context where students would be familiar with sets and set notation, would be to have separate symbols to signify notions of set.

Combination, permutation, and factorial sets

Thus far, we have considered three foundational combinatorial symbols, $P(n, r)$, $C(n, r)$, $n!$, all of which, by definition, represent cardinalities, and all of which can be understood as a specific kind of operation on numerical quantities. In this section, first, I propose symbolism for these combinatorial ideas that explicitly defines signifiers for the associated sets. Although the following would likely not be the appropriate way to describe this to a learner, the new notation essentially can be understood as defining and symbolizing various sets and set operations (not unlike other binary set operations also common in combinatorics, *e.g.* $A \times B$ and $A \cup B$). As part of discussing the proposed symbolism, I explicitly address some of the mathematical aspects; such commentary is meant to provide clarity on the mathematical ideas, but is not being proposed as necessarily useful for learners. Second, I elaborate on two ways that having symbolism for both cardinality (which already exists) and set (the new notation proposed) might be productive for learners. And despite the sometimes mathematical way these two are presented and discussed, the key points being made are about teaching and learning—they provide a rationale for why such symbolism might be productive for accomplishing important pedagogical aims in combinatorics education.

For a given finite [1] set A, the set of r-sized combinations, the set of r-sized permutations, and the set of factorials are *unique*. This is important because it means that we can actually produce these new sets and we can *use symbolism to explicitly signify these (unique) sets.* Table 1 provides an example of these combination, permutation, and factorial sets, and includes the symbolism proposed as signifiers for them. Mathematically-speaking, we can consider r-combinations, r-permutations, and factorials to be set operations on a given set A. An r-combination is an r-sized subset, or unordered selection, of n distinct objects. Combinations are commonly referred to as 'choosing' because they indicate a selection (unordered) of some of the n distinct objects—these n distinct objects can be considered as forming a given set A. (Although in this description it is the case that $|A| = n$, I sometimes use $|A|$ to be more broadly representative of the size of the given set.) For a given set $A=\{a_1, a_2, \ldots, a_n\}$, and a given natural number r ($0 \leq r \leq |A|$), we can define an external binary set operation on A to determine a unique r-combination set; this set is composed of r-combination elements. Providing symbolism for this r-combination set, such as $C(A, r)$, allows us to explicitly reference this set. We could even give formal definition to such a set operation [2]. In a similar manner, one can define an external binary set operation on set A to determine a unique r-permutation set,

Table 1. Examples of combinatorial set operations, sets, and their elements.

Combinatorial set operation	Combinatorial set and its elements *
Factorial set operation, $A!$	$A!$= {(♠,♡,♣,♢), (♠,♡,♢,♣), (♠,♣,♡,♢), (♠,♣,♢,♡), (♠,♢,♡,♣), (♠,♢,♣,♡), (♡,♠,♣,♢), (♡,♠,♢,♣), (♡,♣,♠,♢), (♡,♣,♢,♠), (♡,♢,♠,♣), (♡,♢,♣,♠), (♣,♠,♡,♢), (♣,♠,♢,♡), (♣,♡,♠,♢), (♣,♡,♢,♠), (♣,♢,♠,♡), (♣,♢,♡,♠), (♢,♠,♡,♣), (♢,♠,♣,♡), (♢,♡,♠,♣), (♢,♡,♣,♠), (♢,♣,♠,♡), (♢,♣,♡,♠)}
Permutation set operation, $P(A,3)$	$P(A,3)$= {(♠,♡,♣), (♠,♡,♢), (♠,♣,♡), (♠,♣,♢), (♠,♢,♡), (♠,♢,♣), (♡,♠,♣), (♡,♠,♢), (♡,♣,♠), (♡,♣,♢), (♡,♢,♠), (♡,♢,♣), (♣,♠,♡), (♣,♠,♢), (♣,♡,♠), (♣,♡,♢), (♣,♢,♠), (♣,♢,♡), (♢,♠,♡), (♢,♠,♣), (♢,♡,♠), (♢,♡,♣), (♢,♣,♠), (♢,♣,♡)}
Combination set operation, $C(A,3)$	$C(A,3)$= {{♠,♡,♣}, {♠,♡,♢}, {♠,♣,♢}, {♡,♣,♢}}

*For A={♠,♡, ♣, ♢}. Curly brackets { } indicate unordered r-tuples. Parentheses ( ) indicate ordered r-tuples

composed of r-permutation elements. One might symbolize this as $P(A, r)$; and even give formal definition [3]. A factorial set might be described as a particular permutation set; namely, $P(A, |A|)$. Given the broad use of factorial notation in combinatorial expressions/formulas, however, I find value in providing symbols for it in relation to set notation. Indeed, in this case, we can define a unary set operation on set A to determine a unique factorial set, which we could symbolize as $A!$. The key distinction here is that, rather than the ordered r-tuples that are the r-permutation elements, factorial elements are ordered $|A|$-tuples (or n-tuples).

There are two ways the set-signifying symbolism would be productive for learners.

First, giving a unique symbol to combination, permutation, and factorial sets *provides symbolism that explicitly allows us to signify the relationship between cardinality*—as an expression/formula—*and set.* In other words, having symbolism for set, along with symbolism for cardinality, not only provides a way to differentiate these two important facets of combinatorics, but doing so also allows for us to signify how the two are related. Namely, mathematically, we can state: $C(n, r) = |C(A, r)|$; $P(n, r) = |P(A, r)|$; and $n! = |A!|$. Such statements, whether formally expressed or informally described, can be used to help learners explicitly recognize that existing signs (*e.g.,* $C(n, r)$) are not just numerical quantities or algebraic expressions, but are in fact i) *cardinalities*, and ii) cardinalities of *particular* sets. The ability to signify these particular sets also provides an explicit opportunity to think about the set and its elements, in a way that can be separate from its cardinality. Handily, this symbolism also gives the following mathematical results:

$$|C(A,r)| = C(|A|,r) = C(n,r)$$

$$|P(A,r)| = P(|A|,r) = P(n,r)$$

$$|A!| = |A|! = n!$$

In relating *r-combinations* and *r-permutations,* we also get the following relationship: $|C(A, r)|=1/r! \cdot |P(A, r)|$. This equality provides symbolism to express how the cardinalities of the r-combination and r-permutation sets are related. Although I do not expand here, what interests me, again, is thinking about how we might be explicit about these in relation to *sets* and *set operations.* That is, $1/r! \cdot |P(A, r)|$ is an operation on quantities; what set operation, on $P(A, r)$ (not A), might we use to describe and symbolize what this product is accomplishing in terms of sets? Essentially, it boils down to an equivalence-relation (or quotient set) operation on the set $P(A, r)$ (*i.e.,* $P(A, r) / \sim$), with $|C(A, r)|$ equivalence classes each of size $r!$.

Second, and perhaps most importantly, having a way to differentiate set from cardinality *provides an opportunity to clarify what is interesting about the combinatorial statements* in the previous two examples. The symbolism provides a way to communicate, to learners, why such statements are in fact not-trivial combinatorically, even though they might be trivial algebraically/numerically. In the combinatorial identity, $C(n, r) = C(n, n - r)$, what is of interest is that even though $C(A, r) \neq C(A, n - r)$, it turns out that $|C(A, r)| = |C(A, n - r)|$. Having a way to symbolize set, whether stated in formal mathematical terms or not, allows one to communicate explicitly that the sets are different, but the *cardinalities* are the same. Similarly, in terms of multiplication by 1, what is of interest is that even though $A! \neq P(A, n - 1)$, it turns out that $|A!| = |P(A,n - 1)|$; and, more generally, for a set A and a singleton set $\{a\}$, that $A \neq A \times \{a\}$, but $|A| = |A \times \{a\}|$ (because $|A \times \{a\}| = |A| \cdot |\{a\}| = |A| \cdot 1 = |A|$). Again, although the *sets* are different, the *cardinality* does not change. Symbolism that explicitly captures both what is different and what is the same can help communicate what the expert already knows to be non-trivial, and can help foster in learners a set-oriented perspective toward combinatorics—a primary aim of their combinatorial education. Having notation to symbolize sets allows us, quite literally, to draw attention to sets more explicitly, providing a natural opportunity for learners to focus on sets and the conceptual aspects of combinatorics.

Conclusion

As Bass (2017) described, mathematics has continued to develop throughout history by increasing abstractions; because mathematical knowledge is cumulative—*i.e.,* we do not 'discard' ideas—the field develops by reorganizing and rescaling so that what were multiple phenomena become compressed as manifestations of a single unifying theory. That is, there is utility for the field of mathematics in compression. Teaching, however, often requires the reverse process of decompression—of unpacking mathematical ideas into their constituent components for the purposes of learning. The signs, symbols, and notation used to represent mathematics are no different; for learners, they need to be unpacked and utilized to differentiate, to develop, and to connect important mathematical meanings. In analysis, for example, symbolism exists to differentiate the sum of sequences, $(a_n) + (b_n)$, from the sum of particular elements (which are numbers) in the sequence, $a_n + b_n$; the symbolic distinction helps to differentiate, develop, and connect important facets of sequences in an analysis course. (In some sense, the distinction is about defining the operation addition on the set of sequences.) In this article, the duality of combinatorial symbols as signifying cardinalities *and* sets is part of the unpacking process that is important for learning; because of apparent student difficulties with connecting expressions/formulas directly to sets of outcomes (Lockwood, 2013; Lockwood, Wasserman & McGuffey, 2018; Wasserman & Galarza, 2019), further incorporation of combinatorial symbols that explicitly differentiate cardinalities from sets is potentially beneficial, from a learner's perspective, and might be used as one way to help foster a set-oriented perspective towards combinatorics.

Now, whether the specific symbols described here—$A!$, $P(A, r)$, $C(A, r)$—are optimal symbols from the perspective of a learner is unclear. It is possible that the notation could be confusing, in that the symbol A might in one context represent a number, in another a set; perhaps more clearly symbolizing 'set' might help—*e.g.,* $\{A!\}$, $\{C(A, r)\}$; yet this, too, would be unusual in describing a set operation—*i.e.,* we use $A \times B$, not $\{A \times B\}$, to represent the Cartesian product of sets. In addition, the set operation definitions in the footnotes privileged a particular representation for an outcome (also evident in Table 1); for example, a combination outcome listed the selected elements. Alternate representations for combination outcomes exist and could be used, such as $\{(1,1,1,0,0,0), (1,1,0,1,0,0), \ldots\}$, for which outcomes are representative of which of the n elements are in or out of the selection (*e.g.,* Lockwood, Wasserman & McGuffey, 2018; Wasserman & Galzarza, 2019). The specific point here is that the exact symbols and/or representations introduced may or may not be optimal for signifying the desired mathematical meanings. Yet, the broader point remains: the symbols we use in mathematics and mathematics education should consider not only the communication between mathematicians, but also foster the mathematical development of learners. Sometimes, these might be one and the same; at other times, they might differ.

Although I have only explored basic combinatorics ideas (*i.e.,* factorials, permutations, combinations), and have not addressed other potentially important distinctions in combinatorics (*e.g.,* distinguishable versus indistinguishable elements), the purpose of this article was to start a conversation about combinatorial notation—in particular to surface the duality of current symbolism with respect to representing both cardinality (as signified by expressions/formulas) and set (as signified by the set of outcomes). In the long term, with the aim of helping students connect expressions of cardinality and expressions of set, a goal might be that students be able to take *any* combinatorial expression/formula, *e.g.,* $C(5, 3) \cdot 2^3 \cdot 3!$, and have these mathematical signs and operations on quantities become *signifiers* for various sets and set operations—ones that could be used to generate an outcome set that has a cardinality exactly equal to the expression/formula. Nonetheless, improving combinatorial notation to help explicitly signify and connect both cardinality and set might be one step forward for helping learners realize a set-oriented perspective in combinatorics.

Notes

[1] I discuss sets and set operations thinking specifically about finite sets because we are in the realm of combinatorics.

[2]

$$C(A,r) = \left\{\{a_{i_1}, a_{i_2}, \ldots a_{i_r}\} \text{ with } i_j \in \mathbb{N}, i_j \leq |A| \text{ and } i_j \neq i_k \text{ for } 1 \leq j,k \leq r\right\}$$

The { } indicate unordered *r*-tuples

[3]

$$P(A,r) = \left\{(a_{i_1}, a_{i_2}, \ldots a_{i_r}) \text{ with } i_j \in \mathbb{N}, i_j \leq |A| \text{ and } i_j \neq i_k \text{ for } 1 \leq j,k \leq r\right\}$$

The ( ) indicate ordered *r*-tuples

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