Fundamental Questions for Teachers

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I have just been reading an article which presents a discussion of the foundations of mathematics for the benefit of teachers. The discussion is firmly based on an ontological premise: mathematics exists — somewhere, somehow — and one of our problems is to discover its nature. This seems to me a very unhelpful way of looking at the issue, for mathematics remains separate from us, distant, problematic, and to a large extent unknowable. The alternative approach is to treat mathematics as a creation of the human mind, with no existence other than in the minds of people and characterised by some typical human mental activities. I find this view more humane, acceptable and intelligible. Moreover, one of the reasons for trying to understand what mathematics/mathematical activity is about is in order to deal more satisfactorily with, in turn, the justification of mathematics teaching, and the form it should take. The ontological approach, emphasising the objective nature of mathematics, seems to me to give little help here, while looking at the problem in terms of what people do when they act mathematically is much more fruitful. I would like to use my reactions to the article as a vehicle for pursuing this view.

The article is by Alexander Wittenberg and is entitled "An Unusual Course for Future Teachers of Mathematics" [1]. Wittenberg explains that he sees the necessity for teachers to have an understanding of the nature of their subject in order that they can pass on their insight to others. He describes a course which he taught in pursuit of this aim and which was designed to "make future teachers of mathematics think through some of the fundamental ideas and problems in the foundations of mathematics on their own". True to his belief that teaching should not be indoctrination, he says that he did not attempt to provide ready-made answers to these problems, but rather to raise them in the student's mind and make him think about them for himself.

The course consisted of a careful discussion of the single question "What is geometry about?", and, as it developed, four answers to this were elaborated. They are views on the nature of geometry which have evolved historically, and the course recapitulated this phylogeny. Briefly the four views are:

> The study of properties of the figures we draw with chalk or pencil, or which appear as faces of solid bodies or as cross-sections when we cut through them.
>
> The study of ideal figures which exist in our minds in a platonic way, for which physical drawings serve only as rough reminders.
>
> The study of whatever sets of objects can have the terms "points", "lines", and so forth assigned to them in such a way that they satisfy the axioms of geometry.
>
> The study of formal systems of geometry: statements called axioms and formal inferences from them, with no reference to any interpretation of this game.

Of course, Wittenberg gives a lot more detail of these views, of the historical transitions from one to the next, and of the discussion of them that took place in his courses. Indeed his article gives a very clear perspective on a lot of foundational issues.

However, I am left wondering about his initial aim. How many students and teachers I know would have happily followed the philosophical subtleties of this discussion? Indeed, Wittenberg himself says it is by no means an easy course, and could "be taken profitably only by students who have attained a certain degree of intellectual maturity and sophistication". I feel that considerations such as these would make the course inappropriate for most primary school teachers who are not mathematics specialists, most B.Ed. students studying mathematics, and many graduate mathematicians taking a post-graduate teaching certificate. Is this then a suitable way of conveying insight, as Wittenberg wishes, to future teachers of mathematics about the nature of their subject?

Wittenberg says that the course ends on a question mark: he is unable, in the end, to give an answer to the question of what mathematics is. Moreover I feel that there are other questions, related to this, which are bound to be raised in a teacher's mind by such a discussion and I think Wittenberg's approach will also preclude any satisfactory answer to these:

> Why do people hold divergent views on the nature of mathematics?
>
> (Why, for that matter, do people hold divergent views on the significance of mathematics teaching?)
>
> Why should we teach mathematics?
>
> How should we teach mathematics?

However I think that by shifting the perspective of the debate, and altering the fundamental question from which it derives, it would be possible both to bring it within the concerns and outlook of the average teacher, and also to help him towards formulating his own answers to all these questions.

Let me first point out some clear parallels between, on the one hand, the four philosophical bases for geometry that Wittenberg offers and, on the other, four views that might be put forward by a mathematics teacher of what his pupils are doing in his classes. (Wittenberg is clearly using geometry as a vehicle for the consideration of the nature of mathematics as a whole. If I express alternative views in terms of mathematics rather than geometry, it is only because I am trying to deal with the general problem.)

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Geometry as the study of pencil drawn figures	Learning from concrete apparatus, diagrams and models, and from real life situations about the nature of the world
Geometry as the study of ideal platonic figures	Learning to abstract and generalise Learning to construct and manipulate mental models, i e to have their own idealisations (concepts, images) for numbers, relationships, etc
Geometry as the study of the necessary consequences of its axioms	Learning to be logical, to think carefully, to be consistent, to prove
Geometry as the manipulation of meaningless symbols according to arbitrary rules	Learning to manipulate the mathematical symbols in the correct way ("get the sums right"!)

These four views of what might actually be going on in a mathematics classroom correspond remarkably closely to the philosophical views of the nature of mathematics Also, I think that they would be fairly readily recognised by a teacher, so that a discussion of mathematics could profitably start from this naive point and there would be a realistic hope that we all knew what we were talking about One might later refer to the similar views on the nature of the subject which have been discussed by mathematicians and philosophers, but this would be an interesting side-light on the discussion, not its genesis.

However the gain is not solely in intelligibility In his course, Wittenberg does not in the end come up with an answer to the question "What is mathematics?". That is not very surprising, for great minds have struggled with it for hundreds of years and failed to find a satisfactory solution. I think that the question is the wrong one It assumes that there is a thing called mathematics which has some sort of objective existence, and thus a nature which can be defined But there is no such thing: we are deceived into thinking that there is by our inveterate habit of using nouns. We find "mathematics" a useful word, to fill a place in the curriculum, or to describe what it is we do But in fact there is only the doing (And here we have a double curse, because there is no common verb for doing mathematics.) Anyone who believes that mathematics exists "out there" is welcome to attempt to define it, but his failure to do so should lead him to question his original belief.

The parallels above between sophisticated and naive views of mathematics are not in fact surprising It is a human proclivity to do mathematics, and thence to hold ideas on the nature of this activity whether at an advanced or an elementary level Indeed these views all give characteristics of human mathematical activity After all, it is people who do mathematics, whether they are school-children or David Hilbert, and the fundamental forms that their activity can take are dictated by the nature of the human mind.

Abandoning the impossible question "What is mathematics?", then, the classroom approach is posing a different one: "What (mathematical) is going on in a mathematics classroom?". This is a version of "What do mathematicians do?", and it is one that should be relatively accessible to teachers Its great virtue however is that it is a question to which it *is* possible to give answers Moreover, the answers need not conflict with each other nor exclude each other: mathematics pupils do many different things, and they can all be mathematical. Doing mathematics includes looking at parts of the real world and trying to organise and structure them It includes classifying, abstracting and generalising, and creating mental models which seem to approximate to what we observe but to possess greater generality than the occasional experience. It involves a lot of wondering "What if . ?" and working out the consequences of hypotheses, checking for inconsistencies, proving results in such a way as to convince others. It very much depends upon using symbols and diagrams, to communicate with ourselves as much as with others, as an aid to the convenient handling of complex ideas

So treating mathematics as a human activity rather than an ontological problem has the distinct advantage that we can feel we are dealing with an answerable question It is an approach, moreover, that can help in the resolution of other questions. If mathematics were an entity that existed, it would be strange that clever people should be firmly convinced of divergent views as to what its nature is. Given that it is a collection of activities of the human mind, however, the question "Why do people hold divergent views of the nature of mathematics?" does not seem so difficult. There are many different types of mathematical activity, and those who wish to emphasise one will reify it into a defining characteristic of the subject This raises in its turn the interesting question of why individuals will place the emphasis where they do Answers to this can presumably be discussed either at the individual psychological level, or from a cultural and sociological perspective [2]

(Similar answers can be sought to the fascinating question why people hold their varying views on the significance of mathematics teaching I cannot help wondering if those who advocate the careful drilling of learners in the manipulation of symbols, whether of arithmetic, or algebra, or analysis, have a view of education in which its main purpose it to produce a society of conformers Of if those who think in terms of helping learners to develop their own mental models are trying to produce individualists as a subversive strategy)

Treating mathematics as pre-eminently an activity of the human mind makes it possible to give reasonable answers also to the question "Why teach mathematics?" By contrast, if mathematics is just a form of knowledge, or the set of all statements of the form "If p then q", or the contents of Bourbaki, it is difficult to justify inflicting it upon children. But if mathematics is what people do, then people should be introduced to doing mathematics. And mathematics is indeed what people do, in everyday life, not just in the classroom or study. Making sense of the real world, constructing mental models, following logical arguments, and manipulating symbols are everyday human activities. Everyone does them, and one might reasonably hope that with some help and guidance they could do them better. The fact that most people presently get labelled as mathematical failures but nevertheless act mathematically suggests there should be a lot of scope for working with the grain of natural mental activity. One would be teaching mathematics because it is a human achievement which gives pleasure to people.

Then *how* should we teach mathematics? If one sees mathematics as a body of knowledge (about the world, or platonic ideals, or formal systems) then one will be led to a view of teaching in which knowledge is passed (poured?) from the knower to the ignorant. What is to be passed can be prescribed in a syllabus and tested in an exam. The most convincing argument against the value of this view, in my opinion, is that by and large people do not pass exams, and even those that do seem in a year or two to have been (cognitively) quite unaffected by their learning experience. If, however, one approaches mathematics as a set of human activities, one will try to introduce these activities to children. The aim will be to help them to experience them and enjoy them. The measure of success will be the extent to which learners subsequently think and act mathematically.

To be fair to Wittenberg, his penultimate paragraph recognises that mathematics is not "a defined entity in some logician's or philosopher's textbook", but an aspect of our experience, a living reality, and this is his resolution of the apparent conflict between different views of (the epistemology of) the subject. It seems to me that it would have been much better to start at this point, not to finish there.

Notes

[1] A. Wittenberg. An unusual course for future teachers of mathematics. *American Mathematical Monthly, 70*. (1963a)

[2] David Bloor for instance ("Hamilton and Peacock on the essence of algebra", in: H. Bos, I. Schneider and H. Mehrtens (eds.) *Nineteenth century mathematics in context*. Basel and Boston: Birkhäuser. forthcoming) argues that Sir William Hamilton's opposition to the Cambridge School's development in the early 19th century of formal algebra, unrelated to number, was connected with his hierarchical view of society — you can't do just what you like with symbols, you must stick to the rules.

MINUTIAE

. . yet I cannot see any reason to suppose [Witt] gave a meaning to the quantity with its [decimal] separator inserted. I apprehend that if asked what his 123|456 was, he would have answered: It *gives* $123\frac{456}{1000}$ not it *is* $123\frac{456}{1000}$. This is a wire-drawn distinction: but what mathematician is there who does not know the great difference which so slight a change of idea has often led to? The person who first distinctly saw that the answer −7 always implies that the problem requires 7 things of the kind diametrically opposed to those which were assumed in the reasoning, made a great step in algebra. But some other stepped over his head who first proposed to let −7 *stand for* 7 such diametrically opposed things.

Augustus de Morgan *Arithmetic books from the invention of printing to the present time* (p. xxiv)

Communications

Educational mathematics

LARRY COPES

Even while issue number 3 is waiting in The Pile, I am reading number 2. I am reacting first to the Editorial, which makes the claim that writing about mathematics education should (but, frequently enough, doesn't) relate either to mathematics or to student learning. In the process of making that claim, it states

> Nevertheless, if it is appropriate, as it surely is, to demand of teaching that it works — i.e., that students learn mathematics — then it is equally fair and appropriate to demand of mathematics education that it works too — i.e. that mathematics teaching improves.

From this it appears that our aim as mathematics teachers is for students to learn mathematics. I believe that Bill Higginson is reiterating this apparently obvious goal on the next page when he writes

> The aim of a mathematics educator is to optimize, from both intellectual and emotional viewpoints, the mathematics learning experience of the student.