Communications

A closer examination into elementary students' mathematics discourse: the role of language games

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We examine the ways in which members of a mathematics classroom engage in working towards understanding a mathematical situation, using an episode between a fourth-grade student, a teacher, and a researcher. Our aim is to elucidate the importance of intepreting language usage from the perspective of those involved in constructing understanding. We highlight, using Wittgenstein's concept of language games (1953/2009), how each person comes to use the same words but reaches very different understandings of the mathematical situation.

The episode

The episode under investigation lasted around three minutes and was transcribed from a larger ethnography. At the time of the episode, the students were in the middle of a review for their end-of-year assessments. The exchange took place around a multiplication problem that was a part of a larger conversation on using the box method, a common method associated with drawing a box and multiplying by place values of the multiplier and multiplicand and then adding the products of each.

Researcher Notes: As I sat down next to James, I noted that he was working on three by two-digit multiplication, 684 × 12. I asked him to explain his process "James, can you walk me through what you are working on today?"

1 *James* I am multiplying the bigger number *[pointing to 684]* by the smaller number *[pointing to 12]*.

2 *Researcher* Ok, can you show me what you do?

3 *James* *[James drew a rectangle and then divided the rectangle into six squares.]* The first you got to do, every time for the box method, you draw a box. And then you got to put the big number on the top and the smaller number on the side.

4 *Researcher* Ok, I think I follow you, but can you put the big number on the side and the small number on the top?

5 *James* No, big number always goes on top because it is the bigger number *[He gestured to 684]*

6 *Researcher* Ok, so now that we have this box and you told me where 684 goes, where does 12 go?

7 *James* That's the small one so it goes here *[He gestured to the left side of the drawn rectangle.]*

8 *Researcher* Ok, and we are sure?

9 *James* Oh yeah, yup, that's the way we do it to get the correct answer.

10 *Researcher* And the correct answer is important?

11 *James* Well uh yeah, that's why we do this. Got to get them correct for the test.

12 *Researcher* Ok, I see. So, what now? Is it all set up the way you want?

13 *James* Yeah, we are ready to multiply the numbers and put it in the small boxes. *[He gestured to the boxes he had drawn in the rectangle.]*

14 *Researcher* Ok, walk me through that part then.

15 *James* *[James gestured throughout the multiplication using the box method and carefully placed his solutions into the boxes.]* See, 600 times 10 is 6,000. *[Pointing proudly to the box that held his solution.]*

16 *Researcher* Wow, that's a pretty large number. How did you do that?

17 *James* Yeah my brain is pretty good at those. I can just do them real quick. *[This was his first mention of 'his brain'.]*

18 *Researcher* Can you tell me more about how you do that?

19 *James* Well it's pretty easy. The 600 has two zeros in it and the 10 has one. So, I just count the zeros really fast and that's three of them. And 6 times 1 well that's easy it's 6. Then I add the four zeros to the number and that's 6,000.

20 *Researcher* Interesting way of thinking about that. *[James then completed the rest of his boxes.]* Ok, now the box is complete. Now what?

21 *James* You got to add up this row *[Running his finger across the top row]* and this row *[Running his finger across the bottom row. He added the rows.]*.

22 *Researcher* Ok, so now that you've added the rows, now what?

23 *James* Now I just add these two numbers and we get the answer.

24 *Researcher* Wow, so do you always do this method for multiplication?

25 *James* Yup, I only use the box method. It's the best one for me.

26 *Researcher* What do you mean by 'best one for you?'

27 *James* My brain just works better with that strategy.

Researcher Notes: At this time Ms Baker, the classroom teacher, walked by and James asked if she could look over his work.

28 *Ms Baker* Did you use the box method? You didn't do that last time. *[Two second pause.]* It doesn't mean you are wrong by doing it another way we just need to do it how it works best for us and how we can see it with our math brains.

29 *James* I know I know, 'do it how you see it in your brain.'

Language games

Wittgenstein's (1953/2009) concept of language games refers to language in use and the actions that are associated with how language is woven into interactions and how language creates the opportunity for interactions to occur. Wittgenstein theorized that words, phrases, and sentences have meaning only due to the *rules of the game* that are being played out in interactions; thus, language is contextually oriented. To derive meaning when interacting with specific words or utterances, one must understand the rules to respond or make sense of the situation. A distinction must 'be drawn' to understand the concept of language games—in playing a game, according to Wittgenstein, correct moves are determined while the game is being enacted or played out; thus, the meaning of words are found in their contextual usage (Kitchen, 2017). Take for example, our use of 'be drawn' in the previous sentence; if the phrase were used in another context, one might assume that we are asking the reader to draw a picture. However, we have used it to outline a clear distinction of Wittgenstein's ideas. Thus, to play a language game is to use the rules associated with using particular words and actions.

Language games across positions

For the purposes of this discussion, we use the last six exchanges between the researcher, James, and Ms Baker (Turns 25–29). We focus on the language game being played across the three positions that are made visible through the word 'brain' which is first used in Turn 17 by James. The researcher's subsequent questions were aimed at understanding what the word meant in the context in which James had used it.

Language games between the researcher and James

In this episode, James begins a specific language game. The intent of the researcher was to understand James's rationale for using a specific approach to multiplication. James responds to the researcher's prompts with 'it is the best one for me' (Turn 25). The researcher did not understand this usage of 'best one for me'. In viewing this exchange through the concept of language games, we can indicate that James used a phrase that was known to him in the context but was foreign to the researcher. From the perspective of the researcher, this equates to his not knowing the rules of the game relating to this phrase. It is in this exact moment, that the researcher attempts to understand the rules by responding with "what do you mean?". James continues the game and offers a reference to 'his brain' as a means to communicate his meaning of the phrase "best one for me". It is here, in Turn 27, that this language game is concluded due to the inclusion of the third actor.

Within this game, James is using certain words and phrases in interesting ways. When prompted to elaborate upon his particular usage of phrase, he offers 'his brain' as a response, but in such a way that is outside of his own body, almost as if he and his brain are two separate entities. He does this in a similar manner in Turn 17. The researcher attempts to understand the rules for the use of his 'brain' within the context of the episode, which is amplified by the larger context of the episode because the researcher was prompting James to provide details of the solution processes. However, even though the researcher did not understand the usage at the time, James was responding based upon how he understood such games are played.

Languages games between James and Ms Baker

Ms Baker enters into the exchange and a new language game is played. James prompts Ms Baker to examine his work. Ms Baker outlines the rules for the new game by first asking a question. In the teacher's usage of an immediate proposition after the question, the language game becomes intertwined with a previous exchange. In terms of language games, Ms Baker is using a previous game to outline the rules. Although she asked a question, it did not allow for a response; thus, the rules of using it in this particular game mark it as rhetorical in nature. This lack of opportunity is further characterized by the pause in the conversation. Even though no words are spoken, it becomes part of the language game being played between the student and the teacher. Ms Baker ends this pause by responding to an assumption James might have made about his novel use of the box-method. This is evidenced by the use of "it doesn't mean you are wrong by doing it another way". Ms Baker has not yet looked at James' work, meaning this language game being played is not around his mathematical thinking but instead on his use of a particular method.

Concluding thoughts

It would not be extreme to state that language use plays an instrumental role in the ways and extent to which students come to understand mathematics. While language is at the core of learning mathematics (Sfard, 2015), some researchers note that learning mathematics means to also learn the language of mathematics (Pimm, 1987). Thus, language cannot be overlooked in mathematics education research. Here we have attempted to provide insights into the importance of understanding the language in use from an insider perspective and to offer an opportunity to reflect on mathematics education research. It is important to consider the question, 'How often do we make assumptions of what a student might understand about mathematics based on our own interpretations of the situation or based on the language games we as educators might bring into the classroom?'. This communication highlights how language games can come to be understood in the context of mathematics classroom-based research. While we illuminate some of the rules from an insider perspective, this is just a snapshot. We cannot simply understand the mathematical language needed to learn mathematics, we must begin to understand how students, teachers, and even researchers socially construct the language games around learning mathematics.

References

Kitchen, W.H. (2019) Wittgensteinian pedagogy for mathematics. *Philosophy of Mathematics Education Journal* **35**, 1–20.

Pimm, D. (1987) *Speaking Mathematically: Communication in Mathematics Classrooms.* Routledge & Kegan Paul.

Sfard, A. (2015) Why all this talk about talking classrooms? Theorizing the relation between talking and learning. In Resnick, L.B., Asterhan, C. & Clark, S. (Eds.) *Socializing Intelligence through Academic Talk and Dialogue,* 245–253. American Educational Research Association.

Wittgenstein, L. (2009) *Philosophical Investigations* (rev. 4th ed) (Hacker, P.M.S. & Schulte, J., Eds., Trans.) (Original work published 1953).

La résolution de problèmes mathématiques et l'art de la représentation

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Les recherches en enseignement des mathématiques suggèrent que l'apprentissage de la résolution de problèmes devrait impliquer la modélisation et la représentation visuelle (par exemple, Lesh et Zawojewski, 2007). Nunokawa (1994) suggère que l'utilité des représentations visuelles « lies, we think, in the fact that it can show relationships between elements in the problem clearly » [1] (p. 34). Selon les chercheurs, la pratique de transformation de sa propre représentation mentale d'une situation en une représentation visuelle des relations mathématiques entre les quantités améliore la pensée mathématique des élèves et contribue au développement des capacités de résolution de problèmes. Dans nos cours dédiés à la formation des futurs enseignants, nous mettons en œuvre une approche relationnelle de la résolution de problèmes (Polotskaia et Savard, 2018). Nous présentons à nos étudiants un système de relations quantitatives de base avec leurs représentations visuelles. Afin de développer la maîtrise par les étudiants de cet outil de modélisation spécifique, nous leur proposons des problèmes plus complexes pour les faire représenter visuellement. L'une de ces représentations a déclenché la remarque d'une étudiante: «C'est du Picasso pour moi», pour signifier « c'est difficile à comprendre ». Nous aimerions aborder la question suivante: quel est le lien entre les arts visuels modernes (par exemple le cubisme) et une représentation visuelle d'un problème mathématique? Qu'est-ce qui rend notre compréhension d'une représentation visuelle ou d'un dessin difficile?

Représentations visuelles en mathématiques

Depuis l'Antiquité, les mathématiciens apprécient la visualisation de différentes idées mathématiques et utilisent des représentations visuelles pour étayer leur raisonnement et leurs preuves. Par exemple, Roger Nelsen (1993) fournit dans son livre de nombreuses preuves visuelles développées par des mathématiciens anciens et modernes du monde entier. De nos jours, les mathématiciens utilisent des ordinateurs pour créer des représentations d'objets mathématiques très complexes. Par exemple, la figure 1 présente une structure fractale créée à l'aide d'un logiciel.

On sait également que certains mathématiciens utilisent leur talent artistique pour produire des images de concepts mathématiques très abstraits. Une représentation de la *Déformation de la surface de Riemann d'une fonction algébrique* (Figure 2), créée par le mathématicien Anatoly Timofeevich Fomenko, membre titulaire de l'Académie des sciences de la Russie, professeur de l'université Lomonossov de l'État à Moscou, est un exemple de telles œuvres d'art. Ces représentations visuelles aident certainement à mieux comprendre les idées mathématiques.

Représentations visuelles des relations mathématiques

À l'école élémentaire, les élèves apprennent généralement à représenter les objets un par un pour les compter et à représenter visuellement des nombres en dizaines et en unités. Dans certains pays, les élèves utilisent des schémas simples pour représenter des opérations ou des relations quantitatives élémentaires (par exemple, Venenciano et Dougherty, 2014; Davydov, 1982). Ces schémas aident certainement les élèves à apprendre à résoudre des problèmes écrits présentant des structures mathématiques simples (non complexes). Cette utilisation de base de la modélisation (schématisation) est-elle suffisante pour que les élèves puis-

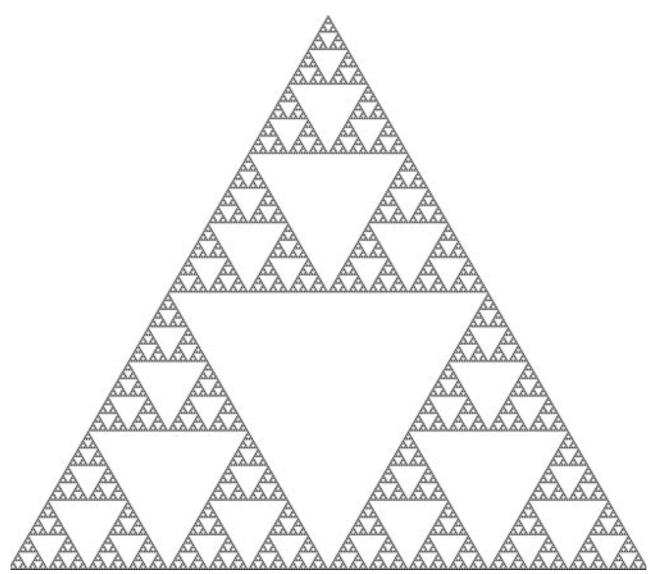

Figure 1. Triangle de Sierpinski, structure fractale nommée d'après Waclaw Sierpinski — célèbre mathématicien polonais.

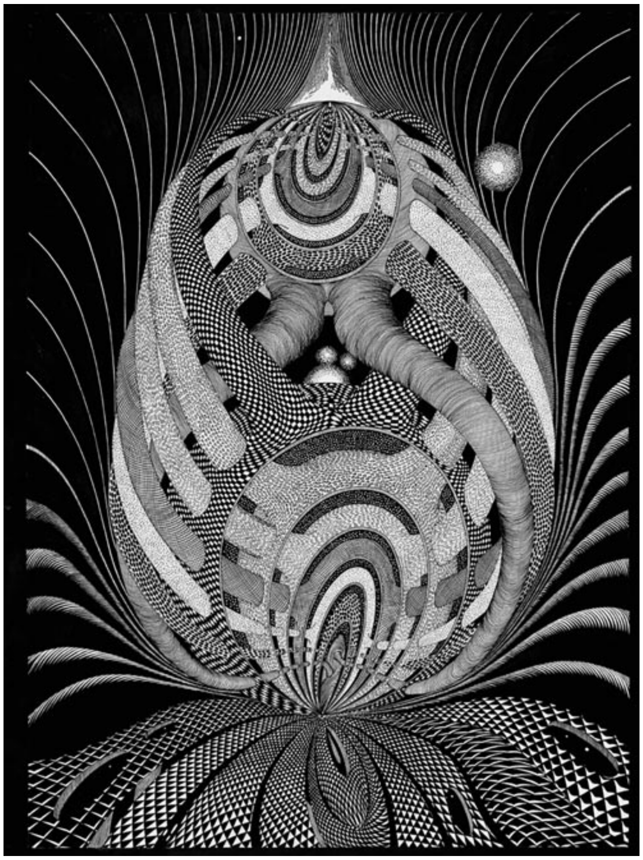

Figure 2. Fomenko, A.T., 1983. Déformation de la surface de Riemann d'une fonction algébrique. [2]

sent résoudre des problèmes plus complexes? Qu'est-ce qu'un problème complexe? Analysons l'exemple qui a déclenché la remarque de notre étudiante mentionnant Picasso.

> Problème: Je dois emballer des pains dans des boîtes. Si je mets 3 pains dans chaque boîte, j'aurais besoin de 8 boîtes de plus que si je les mets 5 par boîte. Combien de pains ai-je?

Nous avons représenté ce problème sur la figure 3.

Le rectangle de gauche représente le cas où les pains sont placés par trois, et celui de droite lorsque les mêmes pains sont placés par cinq. La partie supérieure du rectangle de gauche représente 8 boîtes de plus que dans le cas de droite. La partie droite, ombrée, du rectangle de droit représente les 'deux pains de plus' dans toutes les boîtes de 5. Il est facile de remarquer que la partie blanche du rectangle gauche et la partie blanche du rectangle droit représentent le même nombre de pains. En même temps, les rectangles gauche et droit représentent le même nombre total de pains. Par conséquent, les deux rectangles ombrés représentent le même nombre de pains. Ainsi, nous pouvons proposer une solution arithmétique à ce problème.

> 5 – 3 = 2 (pains plus dans chaque boîte de cinq que dans une boîte de trois)
>
> 8 × 3 = 24 (pains dans les 8 boîtes de trois = tous les pains supplémentaires dans les boîtes de cinq)
>
> 24 ÷ 2 = 12 (boîtes de cinq)
>
> 12 × 5 = 60 (total des pains)

Afin d'évaluer la complexité de ce problème, nous proposons de voir le nombre de relations mathématiques à analyser et modéliser, ainsi que leurs natures.

> *Si je mets 3 pains dans chaque boîte* — ce texte décrit une relation multiplicative R1 (Figure 4): 3 × NB3 = T (NB3 — nombre de boîtes de trois, T — un total de pains).

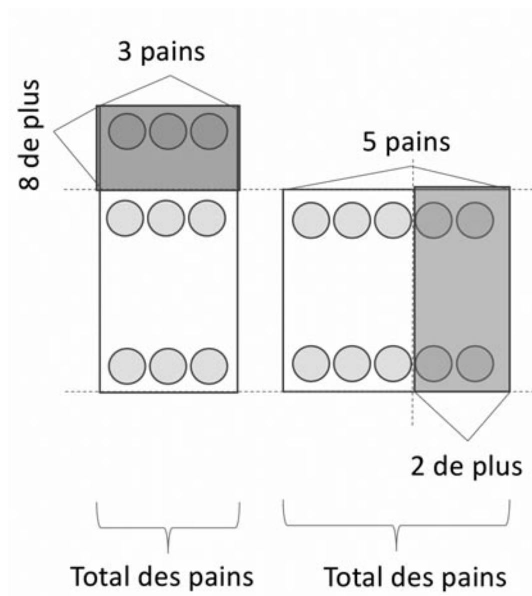

Figure 3. Représentation du problème des pains.

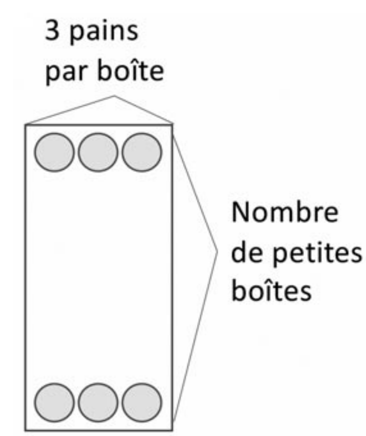

Figure 4. Représentation de la relation R1.

J'aurais besoin de 8 cases de plus que — ce texte décrit une relation additive R2 (Figure 5): NB3 – NB5 = 8 (NB5 — nombre de boîtes de cinq).

Si je les mets 5 par boîte — ce texte décrit une relation multiplicative R3 (Figure 6) : 5 × NB5 = T

Deux autres relations sont implicites.

R4: T = T, car dans deux cas, le même nombre de pains a été utilisé.

R5 (Figure 7) : 5 – 3 = 2, par conséquent, nous mettons 2 pains de plus dans chaque boîte de cinq (comparativement à la boîte de trois).

Les relations R1, R2 et R3 impliquent chacune deux quantités inconnues. Par conséquent, elles ne peuvent pas être traitées par un raisonnement arithmétique immédiatement, et notre compréhension du problème ne peut pas être simplifiée progressivement par des calculs intermédiaires. Nous avons utilisé ici une approche de modélisation visuelle, qui permet de visualiser et de découvrir *toutes les relations mathématiques* impliquées (Figure 3).

De cette façon, nous pouvons voir ou trouver de nouvelles relations et finalement produire une solution arithmétique.

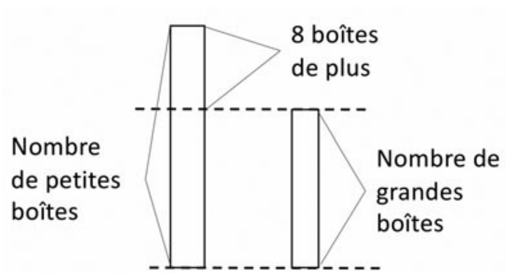

Figure 5. Représentation de la relation R2.

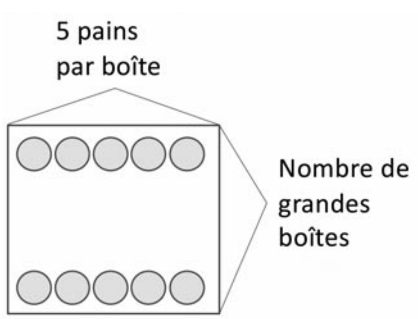

Figure 6. Représentation de la relation R3.

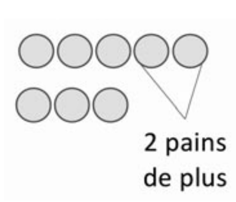

Figure 7. Représentation de la relation R5.

Les relations suivantes peuvent *être trouvées :*

R8 : Le total des pains dans de petites boîtes est composé de deux parties : les pains dans les 8 petites boîtes et le reste.

R9 : Le total des pains dans de grandes boîtes est composé de deux parties : tous les groupes de 3 pains et toutes les groupes de 2 pains.

R10 : « Le reste des pains dans les petites boîtes » est le même que « tous les triplets de pains » qui se trouvent dans les grandes boîtes car ces rectangles ont les mêmes dimensions : 3 et « le nombre de grandes boîtes ».

Une autre approche à utiliser serait la modélisation algébrique. Les personnes familières avec l'algèbre diraient qu'il est facile de composer l'équation suivante $5x = 3(x + 8)$. Cependant, celui qui compose cette équation devrait être capable d'analyser les mêmes relations mathématiques (mais cette fois, la personne qui résout devrait le faire dans sa tête). Ainsi, de nombreux élèves auront du mal à composer une telle équation pour le problème.

Difficultés avec la complexité

Revenons sur la métaphore lancée par notre étudiante : « C'est du Picasso pour moi ». La remarque de cette étudiante suggère que l'obstacle cognitif qu'elle a rencontré était de même nature que certains spectateurs rencontrent face à la peinture abstraite. Selon la typologie des dissonances cognitives de Weltzl-Fairchild (Weltzl-Fairchild et Emond, 2000), l'analogie exprimée par l'étudiante pour manifester son incompréhension ferait principalement référence à deux types de dissonance liés à une expérience esthétique difficile : une dissonance entre les attentes du spectateur et l'expérience de l'œuvre (un conflit entre l'objet artistique et les notions de beauté et/ou de communication) ainsi qu'une dissonance perçue dans l'objet artistique lui-même (un conflit entre certaines parties de l'objet artistique ainsi qu'un conflit entre le message symbolique et les moyens d'expression). Ainsi, nous verrons comment les éléments d'information donnés à la fois par le schéma du problème algébrique et la composition fragmentée du peintre espagnol compliquent leur analyse et leur interprétation respectives. Il s'agira donc d'étudier les similitudes entre le processus de modélisation par traitement visuel proposé ci-dessus et le processus d'appréciation esthétique d'une œuvre appartenant au cubisme analytique, par exemple l'œuvre Le Poète (1911) de Pablo Picasso [3].

Apparemment, le sens et la beauté de l'art abstrait ne nous viennent pas directement des formes et des objets eux-mêmes. À première vue, *Le Poète* est composé de prismes rectangulaires, de courbes et de lignes droites. Cependant, en constatant l'interaction entre les contrastes de lumière dans la composition ainsi que les enchevêtrements de lignes et les différents angles qu'ils créent, on peut voir émaner une certaine harmonie et un rythme qui traduit le regard tout à fait singulier que porte l'artiste sur le monde. À l'instar de Sinclair (2002), nous suggérons que « le personnel et le subjectif » jouent un rôle analogue dans l'expérience esthétique de l'art et dans la résolution de problèmes mathématiques, déterminant largement la qualité de cette expérience. Ainsi, pour un observateur non préparé, les relations multiples (entre plusieurs données inconnues) présentes dans une œuvre d'art ou dans un problème mathématique rendent l'interaction entre ces éléments difficile à saisir.

Csíkszentmihályi et Robinson (1990) précisent que la dimension cognitive de l'expérience esthétique en art se présente sous deux formes: fermée et ouverte. Ils affirment que « Certain individuals, for example, employed intellect in the service of achieving a kind of closure, while others used cognitive means to open up works to more varied interpretations » [4] (p. 42). L'œuvre de Picasso invite l'observateur à imaginer, à proposer une interprétation qui peut aller bien au-delà de ce qui est directement observable. Dans notre exemple mathématique, l'étudiante essayait probablement d'interpréter le problème et sa modélisation visuelle strictement sur la base de son expérience scolaire, où les nombres sont souvent représentés visuellement sous forme de dizaines et d'unités. Ainsi, sa relation au problème peut être interprétée comme étant « fermée ». Le rôle du professeur d'université sera alors d'aider l'étudiante à « s'ouvrir », à considérer de nouveaux points de vue, à développer de nouvelles façons de penser et de modéliser, et ainsi préparer la future enseignante à mieux répondre aux besoins variés de ses élèves. Cela nous ramène à Picasso et au souhait que nos étudiants adoptent son interprétation de l'apprentissage: « J'essaie toujours de faire ce que je ne sais pas faire, c'est ainsi que j'espère apprendre à le faire » [5].

Conclusion

Surprises par une remarque d'une de nos étudiantes comparant une représentation mathématique à une peinture abstraite, nous avons tenté de comprendre les liens entre la complexité d'un problème mathématique et celle de l'art abstrait. Dans les deux cas, l'observateur-résolveur doit analyser et reconstruire pour lui-même plusieurs éléments connus et inconnus (objets visibles et implicites) et de multiples relations entre les éléments. Le but de ce processus est de créer une représentation mentale holistique harmonieuse et logique de la situation: les mathématiques derrière la description du mot ou la représentation graphique ou les idées de l'artiste derrière les lignes, les formes et les couleurs.

Dans sa récente publication, Alshwaikh (2018) plaide pour une meilleure utilisation des diagrammes pour soutenir l'apprentissage des mathématiques par les élèves. Il écrit:

> mathematics is often presented in classrooms and textbooks as abstract, symbolic and devoid of human agency, a view which affects students' access to mathematics. If we understand what is communicated in diagrams—whether they tell a story and include human agency or whether they are conceptual and 'timeless'—we can better design textbooks and better understand what they communicate to learners. [6] (p. 13)

Nous pouvons ajouter que la capacité à embrasser la complexité du monde est cruciale pour les mathématiques ainsi que pour l'expérience esthétique de l'œuvre d'art. Malheureusement, le développement d'une telle capacité est régulièrement ignoré par les programmes scolaires. Peut-être que l'utilisation de dessins simples, tels que les schémas dont nous avons discutés, peut aider les futurs enseignants à apprécier combien il est difficile de développer l'aisance avec des problèmes complexes.

Notes

[1] « réside, pensons-nous, dans le fait qu'elles peuvent montrer clairement les relations entre les éléments du problème » (notre traduction).

[2] En ligne sur http://virtualmathmuseum.org/mathart/ArtGalleryAnatoly/Anatolyindex.html

[3] Pablo Picasso, *Le Poète,* 1911, huile sur toile de lin, 131,2 x 89.5 cm, Collection Peggy Guggenheim (Fondation Solomon R. Guggenheim, New York), Musée Guggenheim de Venise. Voir https://www.e-venise.com/musees_venise/guggenheim/pablo-picasso-le-poete-peggy-guggenheim-venise.html

[4] « certains individus, par exemple, employaient l'intellect au service d'une seule interprétation, tandis que d'autres utilisaient des moyens cognitifs pour ouvrir des œuvres à des interprétations plus variées» (notre traduction).

[5] Tiré de https://citations.ouest-france.fr/citations-pablo-picasso-658.html.

[6] « les mathématiques sont souvent présentées dans les salles de classe et dans les manuels comme abstraites, symboliques et dépourvues d'agent humain, une vision qui affecte l'accès des élèves aux mathématiques. Si nous comprenons ce qui est communiqué dans les diagrammes — s'ils racontent une histoire et incluent l'action humaine ou s'ils sont conceptuels et «intemporels» — nous pouvons mieux concevoir des manuels et mieux comprendre ce qu'ils communiquent aux apprenants. » (notre traduction).

Références

Alshwaikh, J. (2018) Diagrams as communication in mathematics discourse: a social semiotic account. *For the Learning of Mathematics* **38**(2), 9–13.

Csíkszentmihályi, M. et Robinson, R.E. (1990) *The Art of Seeing: An Interpretation of the Aesthetic Encounter.* The Paul Getty Museum.

Davydov, V.V. (1982) Psychological characteristics of the formation of elementary mathematical operations in children. Dans Carpenter, T.P., Moser, J.M. et Romberg, T.A. (dir.) *Addition and Subtraction: A Cognitive Perspective,* 225–238. Lawrence Erlbaum Associates.

Lesh, R. et Zawojewski, J. (2007) Problem solving and modeling. Dans Lester, F.K.J. (dir.) *Second Handbook of Research on Mathematics Teaching and Learning,* Vol. 1, 763–787. IAP.

Nelsen, R. (1993) *Proofs without Words. Exercises in Visual Thinking.* The Mathematical Association of America.

Nunokawa, K. (1994) Improving diagrams gradually: one approach to using diagrams in problem solving. *For the Learning of Mathematics* **14**(1), 34–38.

Polotskaia, E. et Savard, A. (2018) Using the relational paradigm: effects on pupils' reasoning in solving additive word problems. *Research in Mathematics Education* **20**(1), 70–90.

Sinclair, N. (2002) The kissing triangles: the aesthetics of mathematical discovery. *The International Journal of Computers for Mathematical Learning* **7**(1), 45–63.

Venenciano, L. et Dougherty, B.J. (2014) Addressing priorities for elementary school mathematics. *For the Learning of Mathematics* **34**(1), 18–24.

Weltzl-Fairchild, A. et Emond, A.M. (2000) Oh, that's beautiful! or consonance and the adult museum visitor. Dans Allard, M. et Lefebvre, B. (dir.) *Museè, culture et eḋucation / Museum, culture and education,* 143–152. Editions MultiMondes.

The grounding of mathematical concepts through fictive motion, gesture and the motor system

OMID KHATIN-ZADEH, BABAK YAZDANI-FAZLABADI, ZAHRA ESKANDARI

We argue, on well-established theoretical grounds, that body-based approaches to mathematics task design are needed.

In our description of concepts, sometimes we use a spatially dynamic situation to describe a static concept. Talmy (1996) refers to this phenomenon as fictive motion, a cognitive mechanism through which a static entity is conceptualized in terms of a dynamic concept. The sentence *the road runs through the desert* is an example of such sentences. Fictive motion sentences are a special type of metaphor. The findings of some behavioral studies have suggested that understanding fictive motion sentences involves spatial processing [1]. Matlock (2010) notes that the understanding of a fictive motion sentence seems to be based on a simulated motion along a trajector or a linear extension of the trajector. The strong version of embodiment theories argues that the understanding of a metaphor in which a concept is described in terms of a body movement involves the activation of the motor system (Gallese & Lakoff, 2005). Therefore, a strong embodied approach holds that processing the metaphor *grasp an idea* may activate those motor areas that control the actual action of grasping. If this is the case, a similar thing can happen for the processing of fictive motion statements. Within an enactivist or embodied approach to mathematics education, some attempts have been made to use real body movements and fictive motions to teach mathematical concepts. For example, Carotenuto, Mellone, and Spadea (2021) discuss three cases of using body movements in elementary geometry education. They show how body movements can help preschool children acquire a better understanding of straight line, jagged line, and curved line.

Fictive motion and mathematical concepts

Fictive motions underlie many mathematical concepts (Lakoff & Núñez, 2000). The metaphor *numbers are locations in space* is a mathematical metaphor that is based on fictive motion (Alibali & Nathan, 2012). The metaphor *f(x) never goes beyond 1* is one realization of this mathematical conceptual metaphor. When a mathematical idea is expressed through a conceptual metaphor, it is usually accompanied by gestures (Alibali & Nathan, 2012). In such cases, gestures present a visual description of mathematical concepts. Limit is one of the mathematical concepts that can be metaphorically understood through fictive motions. When it is said that

$$\lim_{x \to c} f(x) = L$$

it means that the difference between $f(x)$ and L can become smaller than any small value if the difference between x and c is small enough. In other words, when x approaches c, $f(x)$ approaches L and it can become unlimitedly close to L. In the former description, the concept of limit is understood as an abstract operation among numbers. In the latter description, this concept is understood as a fictive motion: as x approaches c, $f(x)$ approaches L. Although the two descriptions of the concept are essentially the same, the latter description is much more understandable than the former one, as it gives a visual representation of the concept. When this visual representation is accompanied by gestures, the process of understanding is facilitated, as the concept may be grounded through sensory and motor systems.

In a graphical representation of a function, the algebraic form of a function can be understood through a visual tool, fictive motion, and accompanying gestures. When the function $y = f(x)$ is represented in a Cartesian plane, we can easily understand the nature of relationship between x and y. The graphical representation of this function can be described through gestures. The gestures can demonstrate the trace of a moving object along the contour of the graphical representation. In this case, the function is understood through a fictive motion along the contour of the graphical representation. Therefore, the algebraic form of function, which is highly abstract, can be grounded through the visual and motor systems. Here, two media, visual representation and accompanying gestures, have been used to ground an abstract concept.

Continuity is another fundamental mathematical concept that can be understood through fictive motion. In conceptualizing continuity in terms of fictive motion, continuity is understood as an uninterrupted motion of an object along a course that has a source, a path, and a goal (Lakoff & Núñez, 2000). This way of understanding continuity is consistent with the way through which continuity is understood outside the realm of mathematics, which draws on our shared intuitions of space and motion (Núñez & Lakoff, 1998). Understanding a function, limit of a function, and continuity of a function through fictive motion allows us to conceptualize that function as something that has motion and direction (Lakoff & Núñez, 2000). The effectiveness of using real motions to acquire the concept of continuity has also been demonstrated. In their study, Carotenuto *et al.* (2021) observed how preschool children could acquire the concept of continuity of a line through real motion and gesture. Dynamic resources that are employed to conceptualize mathematical ideas and concepts are not limited to spoken language, written language, symbols, and graphs (Marghetis & Núñez, 2013). Núñez (2005) suggests that the employment of co-speech gestures, as a dynamic resource, emphasizes the fact that fictive motion has a cognitive reality.

Features of fictive motion and accompanying gestures

When we understand a concept in terms of a fictive motion, we focus our attention on a moving object along the course of movement. In this way, our attention is distributed across all parts of that concept. In fact, it can be said that two mechanisms are employed to understand that concept. Through the first mechanism, various parts of the concept are understood one by one as the object moves from source to goal. Here, the comprehender's attention is gradually distributed across all parts of the concept. This mechanism is a bottom-up process. Understanding a function through its graphical representation is a good example. As the object or focus of attention moves along the contour of graphical representation of the function, comprehenders can realize where the function is ascending, where it is descending, where local maximum and local minimum points are, where the function is concave on its domain, where it is convex on its domain, what is the slope of tangent line in each point of graphical representation, and many other features. As the focus of attention of comprehenders move, they can realize how the behavior of the function changes across various parts of the domain. Through the second mechanism, the whole concept is understood globally. Here, the comprehenders' attention is on the whole concept as they perceive it in its total form. This is a top-down process. When a concept is understood through a fictive motion, these two mechanisms can be simultaneously employed to process that concept.

The gestures that accompany fictive motion sentences can be helpful tools in several respects. Goldin-Meadow and Wagner (2005) suggest that in the same way that writing a mathematics problem and representing it in the form of written symbols can reduce the amount of required cognitive resources, gestures can help us externalize our thoughts and thus have the potential to facilitate the process of learning. Staats (2018) argues that gesture is one of the tools that can be employed to shift attention from familiar (known) to unfamiliar (less-known) concepts and emphasize the logical structure of arguments. Furthermore, it has been argued that gestures can activate and maintain mental images (Wesp Hesse, Keutmann &Wheaton, 2001). This function of gestures can play an important role in describing concepts in terms of fictive motions. In fact, gestures help comprehenders keep a mental image of the concept in mind during the period of processing. This reduces the load of cognitive processing as some parts of the load are managed by gestures or by the mediation of gestures. Another feature of gestures that could play an important role in cognitive processing is the format of gestures. While information is encoded linearly in words of speech, gestures can simultaneously encode several pieces of information (Goldin-Meadow & Wagner, 2005). This is particularly the case with the encoding and the processing of spatial information, which may be difficult to encode in words.

Visual representation and fictive motion

The final point that must be noted here is that the visual representation of any concept may be understood at least partly through a fictive motion. When we create a mental image of something, we scan various parts of the image in our minds. This scanning is similar to the motion of a point along the contour of that image. The trace of the movement indicates the fictive motion of an object along the contour of that image. This could happen even for very complex images. Although scanning, as well as understanding in terms of a fictive motion, can be used to partly understand a concept, it still has a role in the processing of that concept. Here, the nature of the concept and the complexity of the visual representation or image can be an important factor. If the image is very complex, some parts of it or the global structure of that concept may be understood in terms of a fictive motion. If the image of a concept is complex and includes a large number of parts, it may not be an effective way to conduct a scanning process on the whole parts of the image. In such cases, the best way is to conduct a scanning process on the global structure of the image. In this way, the motor system can contribute to the grounding of the global structure of that concept. In other words, that concept may partly be grounded through the motor system. Therefore, it is suggested that the nature of visual representation of a concept is one factor in the degree of motor strength of concepts. In other words, it could play at least a partial role in determining the extent to which a concept is grounded through the motor system.

Conclusion and implications

The sensorimotor system is one of the cognitive resources that can be employed to ground and understand mathematical concepts. Therefore, using tasks that involve the activation of the sensorimotor system can have a significant contribution in the process of mathematics teaching and learning. Task-based mathematics teaching with a focus on embodied learning and physical movement can be a framework for designing future mathematics courses. Although many mathematical concepts are described in terms of motions and body actions, there is a lack of systemic use of body-based tasks to teach mathematics in current courses. Although there are cases of body-based mathematics instruction, it seems that there is a need for more attempts to make such approaches more widely known and to make them more systematic. The techniques used in the study conducted by Carotenuto *et al.* (2021) is just one case of these attempts. Such techniques need to be systematically incorporated into broad body-based approaches to mathematics education. Our educational system needs to reconsider the methods and techniques that are used in mathematics teaching and learning. It seems that body-based tasks, those tasks that involve the employment or the activation of the sensorimotor system as a contributor to cognitive processing, have a lot of already-unemployed-potentials.

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Note

[1] For a review, see Matlock (2010).

References

Alibali, M.W. & Nathan, M.J. (2012) Embodiment in mathematics teaching and learning: evidence from learners' and teachers' gestures. *Journal of the Learning Sciences* **21**(2), 247–286.

Carotenuto, G., Mellone, M. & Spadea, M. (2021) Moving in early geometry education. *For the Learning of Mathematics* **41**(1), 30–36.

Gallese, G. & Lakoff, G. (2005) The brain's concepts: the role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology* **22**(3-4), 455–479.

Goldin-Meadow, S. & Wagner, S.M. (2005) How our hands help us learn. *Trends in Cognitive Sciences* **9**(5), 234–241.

Lakoff, G. & Núñez, R.E. (2000) *Where Mathematics Comes from: How the Embodied Mind Brings Mathematics into Being.* Basic Books.

Marghetis, T. & Núñez, R.E. (2013) The motion behind the symbols: a vital role for dynamism in the conceptualization of limits and continuity in expert mathematics. *Topics in Cognitive Science* **5**(2), 299–316.

Matlock, T. (2010) Abstract motion is no longer abstract. *Language and Cognition* **2**(2), 243–260.

Núñez, R.E. (2005) Do *real* numbers really move? Language, thought, and gesture: the embodied cognitive foundations of mathematics. In Hersh, R. (Ed.) 1*8 Unconventional Essays on the Nature of Mathematics.* Springer.

Núñez, R.E. & Lakoff, G. (1998) What did Weierstrass really define? The cognitive structure of natural and δ-ε continuity. *Mathematical Cognition* **4**(2), 85–101.

Staats, S. (2018) Coloring conjectures with sound, silence, syntax and gesture: a multimodal poetic analysis. *For the Learning of Mathematics* **38**(2), 2–8.

Talmy, L. (1996) Fictive motion in language and "ception". In Bloom, P., Peterson, M.A., Nadel, L. & Garrett, M.F. (Eds.) *Language and Space,* 211–276. MIT Press.

Wesp, R., Hesse, J., Keutmann, D. & Wheaton, K. (2001) Gestures maintain spatial imagery. *American Journal of Psychology* **114**(4), 591–600.

On two definitions of 'function'

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Does the diagram in Figure 1 represent a function?

Imagine two secondary students, Bob and Penny, having the following conversation:

Bob That's not a function. It doesn't map *d* to anything.

Penny What would make it not be a function is if it mapped *d* to two things!

Bob A function has to be defined for everything in its domain.

Penny And it is. *d* is not a member of the domain, as you said. *d* isn't mapped to anything.

Bob That means it's not a function. If it were a function, everything in its domain would be mapped to something.

Penny What do you mean 'were' a function? It is a function, and its domain is $\{a,b,c\}$. Everything in that set is mapped to something.

Bob But X is given as the domain.

Penny X is not the domain. As we have already established, *d* isn't mapped to anything.

Bob and Penny are operating with different definitions of 'function'. There has been abundant work on the historical evolution of the function concept and the comparing and contrasting of operational and structural definitions. Our work does not address these issues of historical versus modern or structural versus operational. Instead, we consider how, precisely, 'function' is defined. The purpose of this short communication is to highlight three central points:

1. In modern mathematics, there are at least *two* different formalized definitions of 'function'.
2. These definitions are *not consistent* in that they yield different answers to important mathematical questions that a student might encounter—which is surprising given the centrality of the function concept.
3. Educators should be aware of both definitions and not be dogmatic about the 'right' one.

The Ordered Pairs definition

Halmos (1960) defines a function as *any set of ordered pairs, f,* that is *univalent*: "if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$" (p. 30). In this case, the *domain* of f is the set of all x such that there exists some y where (x, y) is a member of f. This definition has appeared in secondary textbooks, introductory proof textbooks, domain-specific advanced texts, as well as in education literature.

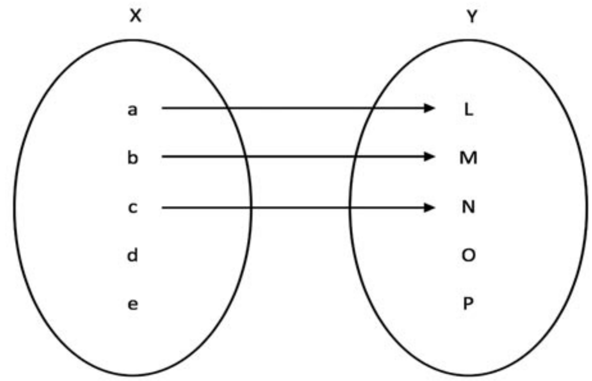


Figure 1. A function?

The Bourbaki Triple definition

Bourbaki (1968) defines a function as a triple (F, A, B), where A and B are sets and F is a univalent and total subset of $A \times B$. That is, for all x in A (*total*), there exists a unique y in B (*univalent*) such that (x, y) is a member of F. The set A is the *domain* of the function, B is the *codomain*, and F is the *graph*. This definition is sometimes given in secondary textbooks, introductory proof textbooks, and in education literature.

An informal summary of the difference is that an Ordered Pairs function is determined by its graph, but a Bourbaki Triple function also includes two additional components: domain and codomain.

Nonequivalence

It is well-known that the same mathematical concept can be defined in different ways. For instance, the real numbers can be defined as Dedekind cuts or equivalence classes of Cauchy sequences, and there is an isomorphism between the reals defined in these two different ways. The definitions are equivalent in that they entail the same answers to nontrivial mathematical questions, and results about the real numbers can be proven using either definition.

This is *not* the case with the two different definitions of 'function' given above. The two different definitions of 'function' provide different answers even to common questions in school mathematics! This is more like the case of trapezoids—having two common, non-isomorphic definitions (inclusive and exclusive [1]).

What is a function?

When functions are named, the reader's underlying definition of 'function' can influence *what*, exactly, is being named. Returning to Figure 1, mathematicians adopting the Bourbaki Triple definition would likely interpret the diagram to depict the triple (F, X, Y) with $F = \{(a, L), (b, M), (c, N)\}$. Under this interpretation, the diagram does not depict a function, because F is not total on X. In contrast, mathematicians adopting the Ordered Pairs definition would focus on the question of whether $\{(a, L), (b, M), (c, N)\}$ is a function. Under this interpretation, the answer is 'yes' because the set is univalent.

People who use the Bourbaki Triple definition and people who use the Ordered Pairs definition will also interpret function notation differently. Use of the notation '$f: X \rightarrow Y$' does *not* uniquely indicate either definition of 'function'. At first glance, it might appear that such notation suggests a Bourbaki Triple by explicitly naming three parts. Indeed, for someone who adopts the Bourbaki Triple definition, the notation '$f: X \to Y$' denotes the triple (graph(f), X, Y). For someone who adopts the Ordered Pairs definition, though, the function f being referred to is just a set of ordered pairs; X and Y are not parts of the function itself. The notation instead means that f is a function (a univalent set of ordered pairs) whose domain is X (consisting of whatever the x-values are in the set of ordered pairs) and whose range is a subset of Y.

Consider g and h defined below. Are g and h the same function?

$g : R \to R$, where $g(x) = x^2$

$h : R \to [0, \infty)$, where $h(x) = x^2$

According to the Ordered Pairs definition, g and h are identical as they have (*are*) the same set of ordered pairs. According to the Bourbaki Triple definition, the answer is 'no' because the codomains (and hence the triples themselves) differ. Eccles (1997), who adopted the Bourbaki Triple definition in his textbook, noted that g and h *should* be different because they have different properties: h is surjective while g is not. Questions like this are not only of interest to quibbling logicians. Literature in *FLM* has identified that it is important that students accurately determine when functions are identical (Markovits, Eylon & Bruckheimer, 1986; Mirin & Zazkis, 2020). Students also encounter questions surrounding function identity in introductory proof courses (see, *e.g.* Rosen, 2019). And, as we will see, the differing answers to this question will have further consequences for how we interpret invertibility.

Invertibility

Consider the following statement:

A function f is invertible if and only if f is injective.

Would you agree that this statement is true? Or, do you believe that an invertible function needs to also be surjective? This is an important question—and there appears to be no consensus amongst textbooks and educators [2].

These contrasting viewpoints about invertibility reflect differing definitions of 'function'. That is, the definition of function adopted fundamentally changes the properties required for a function to be invertible—further evidence that the two definitions of 'function' are different and that these differences are non-trivial. Using the Ordered Pairs definition, the inverse of a function f is defined as: $f^{-1} = \{(y, x) \mid (x, y) \text{ in } f\}$. The inverse f^{-1} only needs to satisfy univalence to be a function, so f is invertible if and only if f is injective. However, for $f = (F, A, B)$ as a Bourbaki Triple, its inverse is defined to be: $f^{-1} = (F^{-1}, B, A)$ where $F^{-1} = \{(b,a) \mid (a,b) \text{ in } F\}$. Notice that f will only be invertible if it is injective *and* surjective (see Zazkis & Marmur, 2018, for a more thorough explanation). This has important implications, for example, whether $f(x) = e^x$ (which is not surjective on R) is invertible.

Ambiguity about which definition is used

In addition to there not being consensus on the properties for invertibility, there is often ambiguity as to which definition of 'function' is being used in textbooks. For instance, in his widely-used calculus textbook, Stewart (2003) writes: "A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B" (p. 11). We can either view A and B as part of the meaning of function (suggesting alignment with the Bourbaki Triple definition) or view the statement as an existence statement where f is a function if and only if there exist sets A and B that fit the definition (suggesting alignment with the Ordered Pairs definition). Since two sets are explicitly stated in Stewart's definition, a reader might think the first interpretation is more plausible. However, he treats a function as equivalent to its graph (p. 12) and claims all injective functions are invertible (p. 59).

Discussion

In this short communication, we have pointed out there are two different definitions of 'function'. These definitions are not equivalent; they lead to different answers to basic questions in school mathematics. This itself is not unheard of (although it is somewhat unusual—most commonly adopted, but different definitions in mathematics are equivalent). What is surprising is that ambiguous usage within a single text suggests that mathematicians and mathematics educators are unaware of these competing definitions, or that they do not recognize their non-equivalence. It is also surprising this issue does not appear to be more well-known, since function is such a prominent concept across mathematics.

We make three modest suggestions for mathematics educators. First, we should be aware of our own underlying definition of 'function' and which definition we intend to communicate to students. The field needs to be clear how function is being defined, just as we are in specifying whether we are using the inclusive or exclusive definition for trapezoid (in fact, because of the ubiquity of the function concept, the need for clarity is even higher).

Second, educators should be sensitive to these different conceptions of functions when evaluating students. For the question about Figure 1 that began this communication, it would be problematic to assess a student's answer as wrong—say on a standardized test—without specifying the definition of function being used. It would be similarly problematic to declare a student incorrect for suggesting that non-surjective functions can be invertible, just as it would be problematic to negatively evaluate a mathematics educator for failing to critique such a student in their research reports. If mathematicians cannot agree on what the definition of function is and what properties functions and invertible functions have, we cannot blame students for not sharing our conceptions.

Third, the existence of differing definitions creates an opportunity for educators to assimilate students more deeply into critical aspects of mathematics and mathematical practice. Discussing defining as a practice in mathematics, and the genuine choices we have in determining those definitions, can acquaint students with the humanistic nature of mathematics. Pointing out the stipulative nature of definitions in mathematics—that objects that fulfill the defining features are examples, even when we do not want them to be—invites students to engage more critically (looking for counterexamples) and can familiarize students with the ontological and epistemological nature of mathematics. In sum, differing definitions, like the ones discussed for function here, are opportunities. We encourage educators to be aware, and to take advantage of them.

Acknowledgements

The topic of invertibility was informed by thoughtful conversations with Ofer Marmur and Rina Zazkis. We thank them for helping us clarify these issues.

Notes

[1] The inclusive definition allows for more than one set of parallel sides (parallelograms are trapezoids); the exclusive, exactly one set of parallel sides (parallelograms are not trapezoids).

[2] Stewart (2003) asserts all injective functions are invertible; Rosen (2019) claims that surjectivity is also necessary. In educational literature, Even (1990) operates under the assumption that all injective functions are invertible, whereas Zazkis & Marmur (2018) require surjectivity.

References

Bourbaki, N. (1968) *Elements of Mathematics: Theory of Sets.* Springer.

Eccles, P.J. (1997) *An Introduction to Mathematical Reasoning: Numbers, Sets, and Functions*. Cambridge University Press.

Even, R. (1990) Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics* **21**(6), 521–544.

Halmos, P.R. (1960) *Naive Set Theory.* D. Van Nostrand.

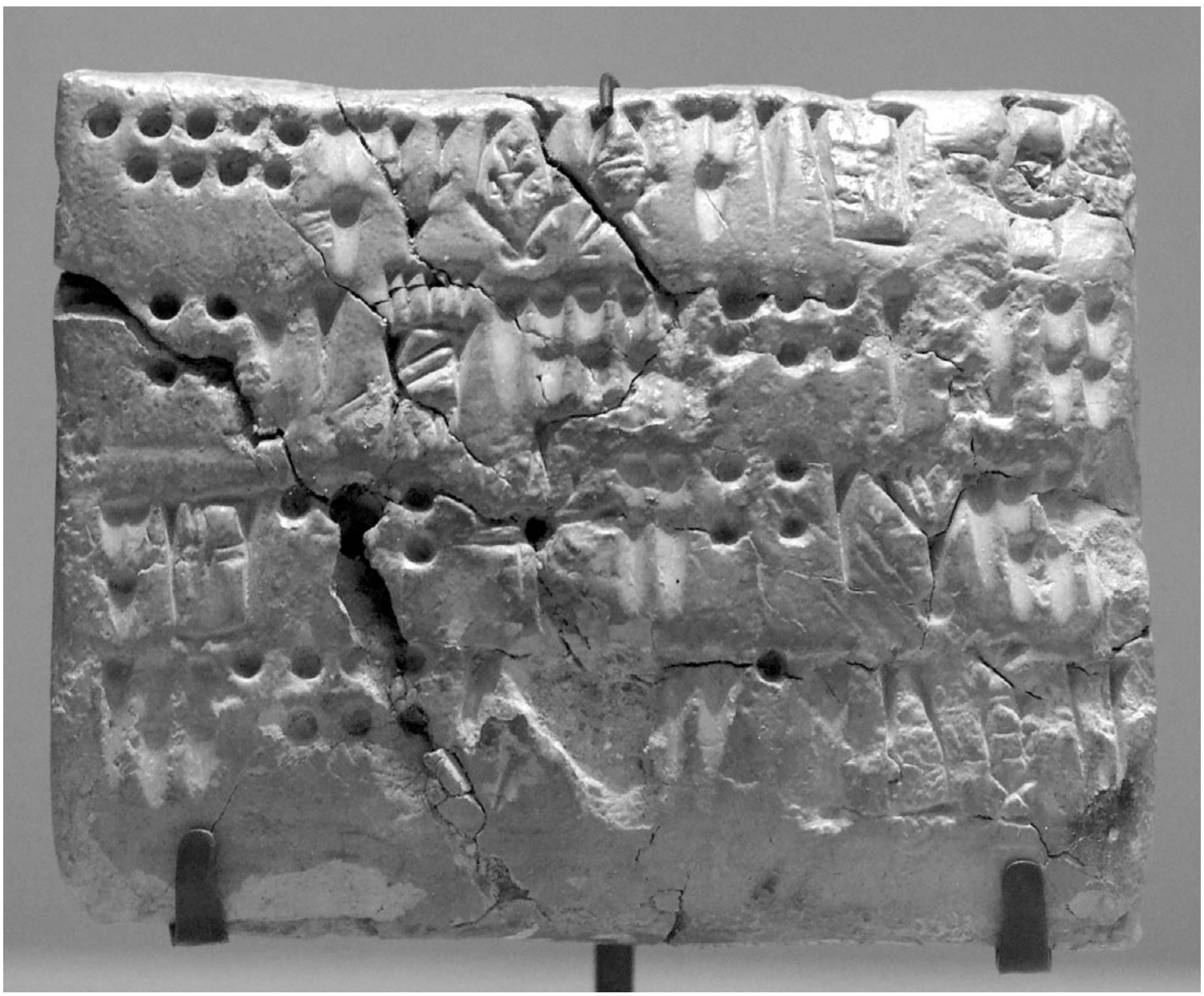
Markovits, Z., Eylon, B.-S. & Bruckheimer, M. (1986) Functions today and yesterday. *For the Learning of Mathematics* **6**(2), 18–24.

Mirin, A.C. & Zazkis, D. (2020). Function sameness: bringing coherence to implicit differentiation. *For the Learning of Mathematics* **40**(3), 14–19.

Rosen, K.H. (2019) *Discrete Mathematics and Its Applications* (8th ed.). McGraw-Hill.

Stewart, J. (2003) *Essential Calculus: Early Transcendentals* (5th ed.). Thompson, Brooks/Cole.

Zazkis R. & Marmur O. (2018) Groups to the rescue: responding to situations of contingency. In Wasserman, N. (Ed.) *Connecting Abstract Algebra to Secondary Mathematics, for Secondary Mathematics Teachers,* 363–381. Springer.



Accounting tablet, Uruk period (3200 BC to 2700 BC), showing indentations in the same forms as earlier clay tokens (see p. 41). Adapted from a photo by Marie-Lan Nguyen, CC BY 2.5.