Extraneous Clues

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Experience refines our perception of facts and trends in whatever we are busy with — or so one supposes. At any rate having re-read my paper on "Problem-solving or Mathematics" [1], and taught the course described therein some more, I am now asking myself an even more basic question: what exactly do we teach when we give our students problems to solve? And my feeling is that it turns out to be hardly mathematics, not even general problem-solving methods, but rather some — not necessarily less valuable — insight into the thoughts of the person who sets the problem. To explain exactly what I mean, let me discuss some examples.

Notations: "general considerations" is denoted by *G, "use of mathematical knowledge" by *M and "reacting to extraneous clues" by *E.

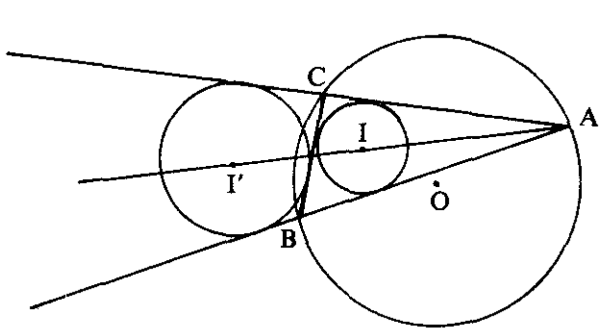
Example I

Consider the problem:

> Construct a triangle when given:
> a) the circumcentre
> b) the incentre
> c) the centre of an excircle.

Solution-protocol:

*G O.K. First consider the triangle and the three circles:

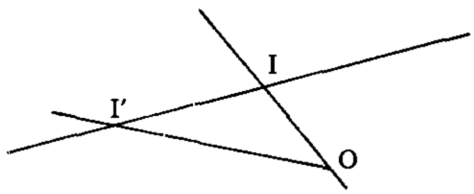


*M We are given O, I, I' only.
Obviously A is on the line II'.

*M Oh-oh, to get A, B and C we need intersections of geometrical loci.

*E Now, since this is an elementary geometry problem, only lines and circles need to be considered.

*M OI, OI' do not seem to help.



*M Circles then. Of course O is the centre of the circumscribed circle.

*M We will have to look for a point on that circle.

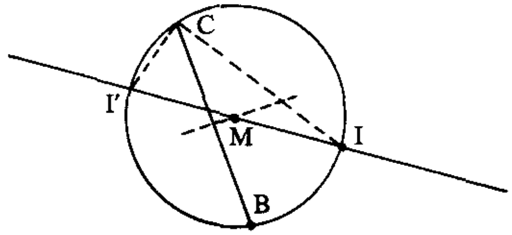
*M Then we will have A and a locus for B and C.

*M We need another locus for B and C.

*E Didn't use I and I' to find circles.

*M How do two points determine a circle?

*M Diameter. Does the circle with diameter II' pass through B and C?



*M Yes. Why? Oh — indeed the right angles between the two bisectors of an angle.

*E Back to looking for one more point on the circumcircle. Then the construction is finished.

*E Any new interesting point appeared recently?

*M Yes. The midpoint M of II', centre of the circle we considered just now.

*E This must be it.

*G How do we prove that?

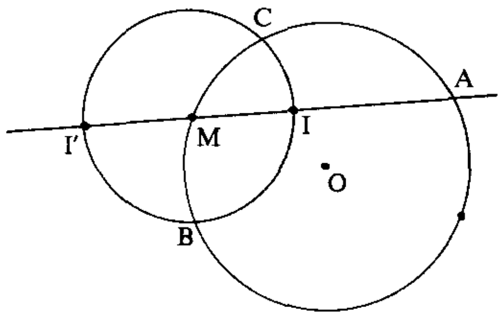
*M Well, M is on the bisector of A and also on the perpendicular bisector of BC, therefore on the circumcircle.

And from here on it is straightforward.

Solution:

> Join II'.
> Let M be the midpoint of II'.

Draw two circles: one with centre M, through I, the other with centre O through M.



Example II
Solve the equation:

$$4x^3\sin a + 6x^2\sin 2a + 4x\sin 3a + \sin 4a = 0$$

Warning: this is a difficult question and the amount of computation may seem overbearing, yet, once the idea of the solution is there the rest is straightforward and develops by itself.

Solution-protocol:

*M This is a 3rd degree equation, we could look up the Cardan formulas.
*E Come on! this can't be what the author had in mind.
*M We could also try to guess one solution and then solve a second-degree equation.
*M We should for that purpose first develop the different $\sin(ka)$ in the equation.
*E As problem-posers do, I would rather look for something symmetrical. How did he get this equation — are there no more clues?
*M Hm. There is a 4-6-4 series of coefficients. This leads us to binomes and 4th powers.
*M Yes — we could look at something like

$$[x+\alpha]^4 - [x+\beta]^4 = 0$$

as a new way of writing our equation......

So we try:

$$\alpha - \beta = \sin a$$
$$\alpha^2 - \beta^2 = \sin 2a$$
$$\alpha^3 - \beta^3 = \sin 3a$$
$$\alpha^4 - \beta^4 = \sin 4a$$

and it soon turns out that we get into a contradiction.

*M This doesn't work. We should try something else.
*E No. 4-6-4 is not a random choice of coefficients. We forgot something.
*M Right. One more parameter would do. Oh, I see, we should try

$$\lambda\{[x+\alpha]^4 - [x+\beta]^4\} = 0$$

Now we have

$$\lambda(\alpha - \beta) = \sin a$$
$$\lambda(\alpha^2 - \beta^2) = \sin 2a$$
$$\lambda(\alpha^3 - \beta^3) = \sin 3a$$
$$\lambda(\alpha^4 - \beta^4) = \sin 4a$$

Since $\lambda(\alpha^2-\beta^2) = \lambda(\alpha-\beta)(\alpha+\beta) = \sin 2a = 2\sin a\cos a$ we have $(\alpha+\beta) = 2\cos a$.
Similarly $\lambda(\alpha^3-\beta^3) = \lambda(\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2) = \sin 3a = 3\sin a - 4\sin^3 a$, so that $\alpha^2+\alpha\beta+\beta^2 = 3-4\sin^2 a$, and since $(\alpha+\beta)^2 = \alpha^2+2\alpha\beta+\beta^2 = 4\cos^2 a$ it follows that

$$\alpha\beta = 4\cos^2 a - 3 + 4\sin^2 a = 1$$

It remains to check that $\lambda(\alpha^4-\beta^4) = \sin 4a$ does not lead to a contradiction. (DO IT!).

*M This looks fine. Remains to find α and β.

We compute λ:

$$\lambda(\alpha-\beta) = \sin a \text{ or}$$
$$(\alpha-\beta)^2 = \sin^2\alpha/\lambda^2 = (\alpha+\beta)^2 - 4\alpha\beta =$$
$$4\cos^2 a - 4 = -4\sin^2 a$$
So that $\lambda = i/2$.

Therefore

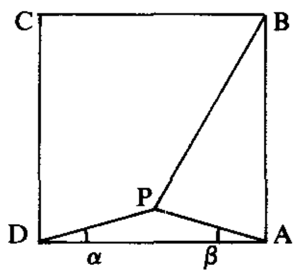
$$\alpha = \cos a - i\sin a$$
$$\beta = \cos a + i\sin a$$

Our equation becomes

$$(x+\cos a - i\sin a)^4 = (x + \cos a + i\sin a)^4$$

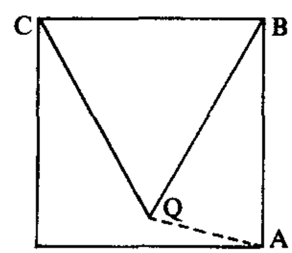
and one easily sees that $x = -\cos a$ is a solution.

Example III
In the square $ABCD$ the angles α and β measure 15 degrees. Prove that $BP = AB$.



Solution-protocol:

*E Nobody starts by drawing angles measuring 15 degrees. So we should look for another way of arriving at this situation — or at the point P.
*M Well, if $BP = AB$ we also have $BP = BC$, therefore we could start with taking a point Q such that $CQ = BQ = BC$.



*M Then $\measuredangle CBQ = 60°$, so that $\measuredangle QBA = 30°$, and since we also have $BQ = BA$ the triangle BQA is an isosceles triangle, and $\measuredangle BQA = 75°$.
O.K. we found our angles of 15°.

*G And by the same token the proof we were looking for.

Discussion

In all three examples the mathematical clues, steps, or associative remarks, outnumber the clues we decided to call extraneous. But these few were the major starting points on the way to a solution.

First we list them in a slightly more sophisticated form:

I —The origin of your problem determines where you should look for its solution.
II —Ask yourself how the person who set the problem arrived at its formulation.
III —Almost always all the necessary information is provided.
IV —Read the question carefully. Quite often the text or some of the givens will hint at the solution.

Let us now look at each one of those heuristic principles in turn:

I "The origin of your problem determines where you should look for its solution."

This is a rather self-evident and innocuous remark. Indeed we all will look for a quadratic equation if a problem appears in the relevant chapter and so on. But the moment we can apply it, any real research-work is ruled out.

(This applies of course only to problems posed in an educational setting).

I often call this "the principle of the student" which in its most general formulation says:

> Any question you are asked has an answer that you can arrive at with the means you have been taught, so all you have to do is quickly recapitulate all different relevant techniques and eliminate those that don't work.

II "Ask yourself how the person who set the problem arrived at its formulation."

This is a valuable device that we somehow use but do not teach in general. It should not be confused with the typical student's remark that "I try to give the examiner the answer he is expecting". It means that you use shortcuts in "the principle of the student"; indeed you ask yourself what mathematical devices did the person who set the question want you to use and how was the question then built (Example III typifies this approach.)

Again, for research-work this is of no use whatsoever.

III "Almost always all the necessary information is provided."

This just emphasizes that problems we pose for our students almost never are research problems — which might have no solution or need knowledge we don't have yet.

IV "Read the question carefully. Quite often the text or some of the givens will hint at the solution."

Here again a useful rule that should be interpreted differently from the saying "read the question a few times carefully before starting to work". It is not understanding the question that we are after, but looking at unintentional hints, at the form of coefficients for example (as in ex. II), or at the terms used a.s.o.

Should I mention once more that this does not apply to real research-work?

Coda

The reader probably feels, at this point, that all this is maybe true, but what can we do about it — if we care at all to change something.

My own tendencies are:

1) to intersperse series of routine-problems with questions which have — in increasing order of didactic difficulty —
- — answers which are not integers.
- — answers which do not match primitive intuitions.
- — questions with insufficient data
- — questions which have as answer: there is no solution.

2) to rephrase the statements of the problem so as to minimize the number of extraneous clues and make it look more like a research-problem.
For instance, for Example I, say:

> "Obviously 3 points can suffice to determine a triangle — for instance the 3 vertices. Can you find other simple examples? What about the midpoints of the sides? Can you find examples where there is no solution? What can you say about the 3 points circumcentre, incentre, centre of an excircle?"

3) to give many open-ended questions, like:

> "$\rho = 1 + \cos\epsilon$ is the polar equation of a cardioid. Study the curve."

This is a real research problem, with the added advantage that we know some of the possible answers. It will also induce the student to go and look at the literature — which is all to the good.

More about that in [2] and in another forthcoming article.

Appendix

Obviously no solution-protocol, or, for that matter method, should be considered as unique or as optimal. So here is what happened with Example II on another occasion:

Solve the equation:

$$4x^3 \sin a + 6x^2 \sin 2a + 4x \sin 3a + \sin 4a = 0$$

*M This is a 3rd degree equation, we could look up the Cardan formulas.

*E Come on! This can't be what the author had in mind.

*M We could also try to guess one solution and then solve a second-degree equation

*M We should for that purpose first develop the different $\sin(ka)$ in the equation

The equation becomes:

$$4x^3\sin a + 12x^2\sin a\cos a + 4x(3\sin a - 4\sin^3 a) + 4\sin a\cos a(\cos^2 a - \sin^2 a) = 0$$

Or after dividing by $\sin a$

$$4x^3 + 12x^2\cos a + 4x(3 - 4\sin^2 a) + 4\cos a(\cos^2 - \sin^2 a) = 0$$

*M The solution we should guess must depend on a.

*E And it must be a very simple trigonometric expression.

*M So let us first homogenize the "3".

So we get:

$$4x^3 + 12x^2\cos a + 12x\cos^2 a - 4x\sin^2 a + 4\cos^3 a - 4\cos a\sin^2 a = 0$$

Comparing terms it is easy to see that $x = -\cos a$ is a solution

And then by reducing the degree of the equation we can find the other two solutions

References

[1] A. Evyatar, Problem-solving or Mathematics. *For the Learning of Math* 4, 2 (1984)

[2] A Evyatar, Math. for the gifted — a miniresearch approach, to appear in *Gifted Child Quarterly*

I have more than an impression — it amounts to a certainty — that algebra is made repellent by the unwillingness or inability of teachers to explain why. There is no sense of history behind the teaching, so the feeling is given that the whole system dropped down ready-made from the skies, to be used only by the born jugglers. This is what paralyzes — with few exceptions — the infant, the adolescent, or the adult who is not a juggler himself.

Jacques Barzun