interactions across and within quantified relationships and would form the basis for developing algebraic methods related to skill acquisition. Students could confidently approach routine and non-routine problems, modelling them in multiple ways, with an understanding of why their method is appropriate.

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What is wrong with the teaching of algebra?

ISTVÁN LÉNÁRT

> [I]f we can present evidence of younger, elementary school children engaging with algebra, and using and understanding the syntactic rules of algebra, we have to ask ourselves why so many adolescents face difficulties with algebra. Perhaps [...] the teaching or curriculum to which the students have been exposed has been preventing them from developing mathematical ideas and representations they would otherwise be capable of developing.
>
> It is our belief that, as previously stressed by many, the difficulties middle- and high-school students have with algebra result from their previous experiences with a mathematics curriculum that focuses exclusively on arithmetic procedures and computation rules. (Brizuela and Schliemann, p. 33)

Brizuela and Schliemann, in FLM **24**(2), give a convincing argument, with documentation, in support of their standpoint. Far from casting any doubt on their reasoning, I want to describe a complementary approach to the same problem. Hopefully, the interference of different opinions will lead not to a weakening, but a strengthening of our mutual efforts to reach the truth!

My focus is that adolescents' problems with algebra originate not only in their *previous* experiences, but also in their *simultaneous* middle- and high-school experiences within their algebra curriculum. In other words, the problem lies not only with the preliminary, introductory stage, but also with the actual form of presenting these ideas in the classroom.

What does 'algebra' mean for general education? What does a mathematics teacher want to teach under the name 'algebra' for middle- or high-school students? What is the central message of school algebra to the students?

It would appear, from my reading of the same article, that many educators and researchers apparently believe that the main objective of teaching algebra is to reach a well-understood and precisely performed handling of equations with constants denoted by numbers, and unknown quantities denoted by letters.

Another method of teaching the basics of abstract algebra lies in the algebraic formalisation of geometric transformations, that is, through the concepts of transformation groups. However, my personal experience is that, for the majority of high-school students (and also for many college and university students), this way to abstract algebra is not a viable one. For a beginner, it is hard, often impossible, to accept that a transformation of infinitely many different geometric shapes into infinitely many other geometric shapes might be considered as a single element of an algebraic set.

In my own case, I had problems in my university years understanding the first step of vector algebra, namely, all vectors with the same length and direction constitute one and the same element in a certain set of algebraic objects. I thought this theory stupid if it could not tell two vectors apart that I found clearly distinguishable from each other!

So, what does algebra mean for me? If I agree – as I do – that algebra is among the most powerful thinking tools of modern mathematics, what is my main (but not by any means the only) aim in teaching abstract algebraic concepts to an average high-school student (or first-year college student or university student)?

I think that, under the name of 'algebraic concepts', we should not teach one fixed algebraic system that takes its origin from sets of numbers and the operations amongst them. Instead, we should focus on the ways and means of creating a given world of algebra, changing to another world if necessary. The educational task is to teach about the art of creation in algebra, or, in a broader sense, in any branch of mathematics.

This method of teaching has a message even for a future *non*-mathematician who does not care much about the algebraic language of equations or the associative property of group theory.

Our goal can perhaps best be achieved by alternative models, given that the ancient tradition of introducing algebraic concepts via the language of arithmetic has given problems. I think that abstract algebraic concepts are best introduced by alternative, non-arithmetical micro-worlds.

Such a micro-world should be close enough to the topics usually labelled as 'algebra', but, at the same time, distant enough from students' former algebraic experiences to be rewarding enough to make the hard abstraction to algebra. Probably, one of the reasons why adolescents do not recognise abstract algebraic concepts is that these abstract concepts are not really striking enough for them, compared with the well-known (to them) properties of numbers. Students may feel that abstract algebra only speaks about the properties of numbers under funny pseudonyms.

Still another requirement, perhaps the hardest of all, is that the alternative micro-world should not be more complex, more difficult to grasp, than the traditional worlds that are usually offered in the curriculum.

If the teacher displays an alternative system that is correct in the scientific sense of the word, but much more intricate than the familiar ones, then the student is likely to say: "Well, only mad scientists make such crazy systems!"

I do not enter here into the difficult problem concerned with whether it is at all necessary to teach abstract algebraic concepts in middle- and high-school levels of general education. What I do claim is that once you decide to teach abstract algebra to younger students, then the micro-world shown below, based on simple facts of spherical geometry, is a suitable method. College and university students also get benefits from the study of this micro-world, in abstract algebra and in other branches of mathematics.

The traditional interpretation implies that the ultimate task of the teaching of algebra is developing abstraction from sets of numbers and operations, mainly addition, multiplication and their inverses, to the abstract laws of group and field theory. However, no mathematical theory, including the extraordinarily developed group theory, contains the final truth in mathematics. Likewise, there is no one exclusive system in algebra that is worthy of being taught as the only possible way of thinking algebraically.

Consider the sphere and the great circles on it. Among the coordinate lines of the geographic coordinate system, the longitudes and the equator are great circles, but the other latitudes are small circles (see Figure 1).

For our purposes we only need to state that we can define *perpendicularity* between two spherical great circles. Two great circles are perpendicular to each other (see Figure 2) if they divide the spherical surface into four congruent regions (as with an apple or a water melon divided into four congruent pieces).

Given two different straight lines on the plane, how many perpendicular straight lines do they have in common? None, if they are intersecting; infinitely many, if they are parallel.

These simple facts of Euclidean plane geometry need discussion – even with college students. In the case of intersecting lines, students often come up with concentric circles whose centres are on the point of intersection. It takes time to clear up that the circles are perpendicular indeed to both of the two straight lines, but here we only seek common perpendiculars that are straight lines. In the case of parallel lines, students say that there are infinitely many common perpendiculars – and they draw lines parallel with the two original parallel lines! They immediately correct themselves, saying, "Oh, I confused parallels with perpendiculars", but this mistake is surprisingly common.

Given two different great circles on the sphere. How many perpendicular great circles do they have in common? Only one, the equator of their two opposite points of intersection. (Any two different longitudes on the globe have only one perpendicular great circle in common, namely the equator.)

Although spherical geometry is much less familiar for most of the students than that on the plane, this statement usually causes fewer problems than the corresponding case on the plane. This may be partly because of the close resemblance to the geographic system; and partly because the spherical case is more uniform and more elegant than the planar case.

Now the question of questions, 'What mathematical concept does this finding remind you of?' We have found a set, a basket of mathematical objects, namely, great circles. Then we took out a first great circle from the basket; then again we took a second great circle from this same basket; we pronounced a magic called 'Draw a common perpendicular', and we had a single resulting great circle from the same set. What was this magic?

It is pretty hard to formulate this question; it is even harder to understand it. When the students do, they begin a brainstorming of geometric ideas, such as: 'It is like parallels, because the two great circles are both perpendicular to a third line, just as with parallel straight lines on the plane.' 'It is about finding the pole points, because the two longitudes meet each other in the north and south poles.' The boundaries between different branches of mathematics are so strongly fixed in their minds that it takes quite an amount of time and effort from the teacher to draw their attention from geometric terms to operations of arithmetic. Very often, the older the student, the harder the shift!

Finally, they come to realize that we have arrived at a binary operation in the set of great circles, just as addition or

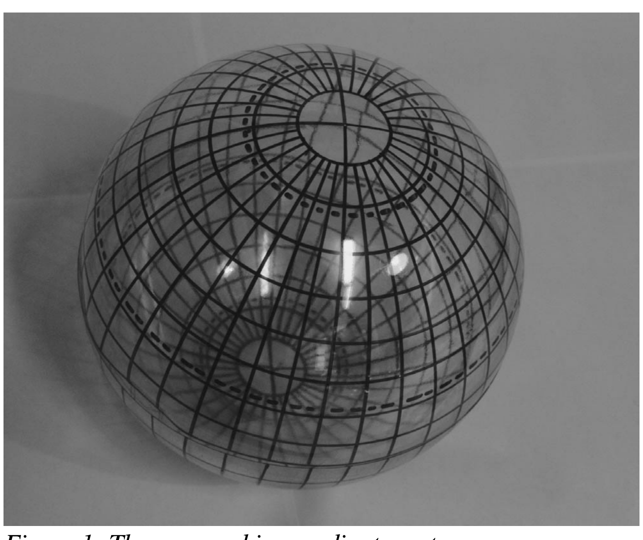

Figure 1: The geographic coordinate system.

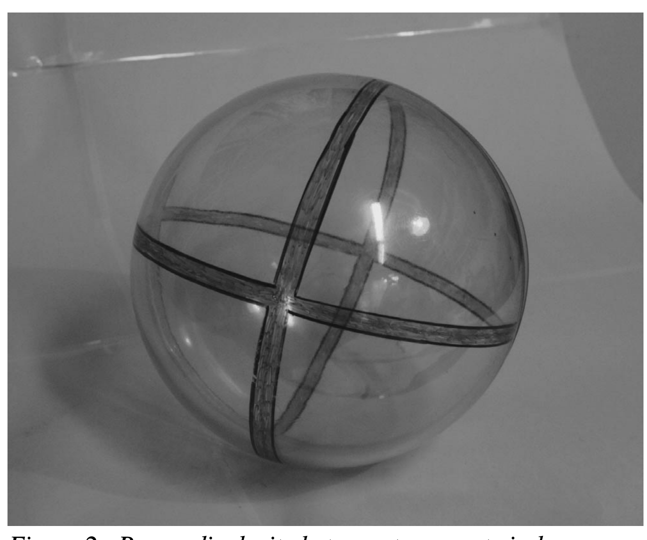

Figure 2: Perpendicularity between two great circles.

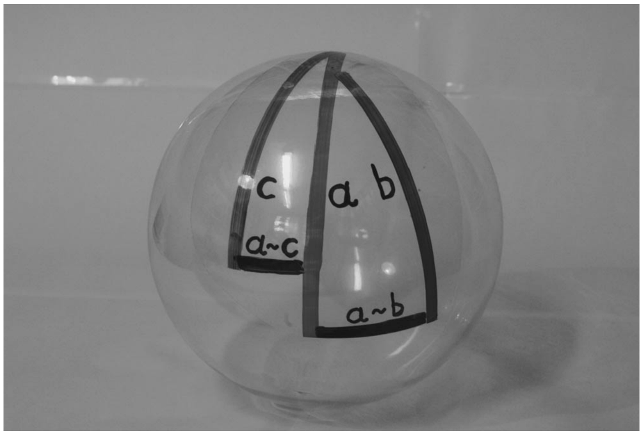

Figure 3: a ~ b and a ~ c

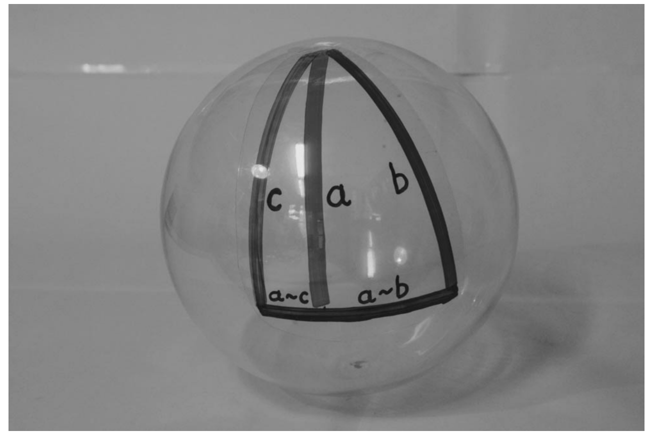

Figure 4: a ~ b = a ~ c

multiplication is a binary operation in the set of natural or real numbers. We have chosen two elements in a given order from the set of great circles of a sphere, and find that these two elements determine a great circle, that is, an element of this same set.

I believe that even this starting idea in itself is worth doing. The students who understand it – who could make the mental shift of connecting concepts that they know from arithmetic with another concept of a definite geometric character – will catch the abstract meaning of algebraic operations.

The next step would be to give a name to this operation: a ~ b = c, and compare the properties of this new operation with the well-known properties of addition and multiplication. For example, what about the commutative property? Does a ~ b = b ~ a? [1]

And what about a + b = a + c implying that b = c? a ~ b = a ~ c does not mean that b = c. However, if a ~ b = a ~ c, then the three great circles go through the same pole points, that is, they are *concurrent*. If great circle a and great circle b have a common perpendicular; and great circle a and great circle c have the same common perpendicular as a and b, then great circle b and great circle c must also have the same common perpendicular (see Figures 3 and 4).

These are but the very first steps of a theory that is rich and rewarding scientifically just as much as educationally. Here I only wanted to show the essence of the whole. This is an alternative system in the field of algebra between traditional addition-multiplication versus the operation `~`, in the same sense as the geometry of the plane and the geometry of the sphere constitute an alternative system in the field of geometry.

Note

[1] There is a sequence of activities in Hungarian and English linked to this introduction to algebra, a summary of which has been translated into Polish by Anna Rybak. Contact István Lénárt for further details.

These notes and references follow on from page 45 of the article "Talking through a method', Speiser, Walter and Lewis, p. 40 (ed.)

Notes

[1] We have described in some detail (Speiser and Walter, 2000) how these students built their number system, in the context of a sharp debate about epistemological as well as mathematical issues.

[2] The first two authors led the sessions. Lewis was Speiser's research assistant. An early version of the analysis appears in Lewis's unpublished Master's dissertation, Brigham Young University, 1997. Contact tarallewis@ yahoo.com.

[3] More precisely, the number, say, of beans in a jar does not depend on how that number may be written. In our students' terms (Speiser and Walter, 2000) that number has a "gut reality" that might be written down in different ways. Similarly for sums and products of such quantities.

[4] For the story of how the five students arrived at this system see Speiser and Walter, 2000, pp. 37-71.

[5] As a theoretical construct, the idea of multiple contexts emerged slowly in the course of our analysis, as we built detailed annotations to the transcript.

[6] In Dewey's (1896, p. 106) terms, the group looks for a stimulus, in this case a sense that the needed reasoning and explanations can be built.

[7] Discoveries as stimuli, opening the way for posing further questions, that is, perceived needs (Dewey, 1896, pp. 106-107).

[8] The following information about the editors of this work has been lightly edited to conform to FLM style but otherwise appeared on the reverse of the title page of the volume: '*The early works of John Dewey* was the result of a co-operative research project at Southern Illinois University. The editorial board consisted of G. Axtelle, J. Boydston, J. Burnett, S. M. Eames, L. Hahn, W. Leys, W. McKenzie and F. Villemain. F. Bowers was the Consulting Textual Editor and J. Boydston the Textual Editor of Early Works.'

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