AMBIGUITY IN UNITS AND THEIR REFERENTS: TEACHING AND LEARNING ABOUT RATIONAL NUMBER OPERATIONS

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I just picked up a box of 4 dozen donuts at the shop.
Do you want one?

This simple pair of sentences provides an example of how English language can contain precise, quantitative information and can simultaneously lack clarity. The word "a" indicates the quantity of "1" box, the number "4" indicates how many dozens are in the box, and the word "donuts" gives a name to the unit that each dozen is counting. And yet, there is confusion as to the referent of the word "one" in the follow-up question. This scenario, involving three different sizes of "units" (the box, the dozen, and the donut) creates an ambiguity for the listener that is common in multiplicative situations.

Utterances with ambiguous referents like those in the above example occur frequently in everyday English. The meaning of words like "one" can be clarified by the speaker by applying emphasis to certain words in the utterance (Brown-Schmidt, Byron, & Tanenhaus, 2005), by gesturing toward the intended unit (Bjuland, Cestari, Borgersen, 2008), by referring by name to the item she had in mind ("I meant one **donut**, not one **dozen**"), or by examining the context in which the utterance was said (was the listener preparing to feed a large crowd at a staff meeting?) (Brennan, 1995).

Ambiguity in the mathematics classroom

Unlike conversations in everyday situations, language used in mathematics classrooms does not always provide the clarity learners need in order to make sense of numbers or arithmetic operations in problems (Barwell, 2005; Pimm, 1987; Rowland, 1999, 2000; Schleppegrell, 2007). Mathematical discussions of multiplicative and rational number situations can be particularly opaque to learners due to the introduction of many types of units (collections or fractional pieces). In an attempt to illustrate mathematical ideas, including identification of referent units, members of learning communities frequently introduce symbolic, diagrammatic, contextual, or other types of representations. These representations can introduce additional ambiguity if everyone in the community does not share a common conceptualization of the referents of the representations (Anghileri, 1995; Moschkovich, 2008).

Goldin (2003) noted this general difficulty in mathematical communication in that whenever representations, including language, are utilized in learning mathematics, "ambiguity is inescapable." And yet the ambiguity of representations is what gives them "power and flexibility." Rowland (2000), too, wrote of the use by learners of unreferenced (and thus ambiguous) pronouns such as "it" to make conjectures about mathematical ideas that were new to them. For example, a child discussing division and its relationship to multiplication claimed "**it** probably always works." Young students also use the pronoun "you" to try out mathematical generalizations, as in the following pupil's query regarding the relationship between the scale factor and the area of a rectangle, "Is **it** that **you** square it – everytime?" (Pimm, 1987).This use of generic pronouns to label patterns can give novices access to higher level concepts than they might normally have if they were required to name the concepts.

Barwell (2005) suggests an additional benefit of ambiguity in the mathematics classroom, encouraging students to "engage in a process of joint meaning-making, trying out the possibilities of words which they have encountered before extending their experience of using these words to think mathematically together." (p. 124). Allowing students in mathematics classrooms to use their own (non-mathematical) language to express ideas may encourage greater participation by learners, while providing an entry into important mathematics (Moschkovich, 2007). Further, vagueness and lack of precision permits learners "to make mathematical assertions with as much precision, accuracy, or as much confidence as they judge is warranted by both the content and the circumstances of their utterances" (Rowland, 2000 p. *vii*).

Thus, awareness of the inherent ambiguities of language and other representational systems may benefit instructors as they orchestrate tasks and discussions in which students build understanding of mathematics concepts and processes. Mathematics educators, as well, may benefit from the study of language and how it is operating in instructional settings where the "students" are themselves future teachers.

Background and purpose

As a mathematics educator, I had been teaching in classes designed to develop deeper mathematical understanding in future elementary teachers when I decided to find out how the pre-service teachers' (PSTs') explanations changed during the course. By video-taping and transcribing episodes, I was able to provide the detachment needed to study aspects of not only the PSTs' thinking but also my own language. Much as Rowland (2000) described his *a posteriori* reflec-

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tion on research activity, I discovered, after transcribing language used in these mathematics classrooms, possible sources of confusion in the learning of rational number operations. My inclination was to "fix" these problems; to make the meanings of diagrams, contexts, and language transparent. It was after comparing two teaching episodes, one my own and the other a colleague's, both dealing in some way with the problem of identifying referent units of rational numbers, that I realized that the vagueness and ambiguity in language and other representations might be harnessed as a resource to encourage participation. These are the two episodes described here.

The purpose of this article is to examine closely teaching situations where ambiguity in language and other representations may hinder or support students' conceptual understanding. Through a fine-grained analysis of dialogue, I focus on potential ambiguities in the language used by both instructors (mathematics educators) and students (PSTs) in attempting to track referent units during operations on rational numbers. Further, I consider how instructors might take advantage of mathematical tasks with embedded confusion and of ambiguous diagrams, inscriptions, and language to encourage classroom debate, discussion, and struggle around important mathematical ideas. Before presenting and dissecting these classroom vignettes, I present a brief summary of some of the relevant issues that arise in multiplicative thinking and the learning of fractions.

Tracking units in rational number multiplication: the convergence of two critical transitions

Children are known to require extra support as they try to make sense of numbers and operations during the transitions (1) from additive to multiplicative situations (Hiebert & Behr, 1988; Greer, 1994; Herel & Confrey, 1994) and (2) from whole number to rational number usages (Graeber & Tanenhaus, 1993; Lamon, 2005). In both of these learning situations, the notion of the "unit" undergoes a major shift. (Behr *et al.,* 1994; Lamon, 1994).

Referent units for multiplicative situations

One of the obstacles children face as they are introduced to the operation of multiplication is that a group of objects must be seen as both a single entity (*the group unit*) and as a collection of several objects (*the object units*). Further, multiplication requires the coordination of these composite units, focusing at times on the object unit and at other times on the group unit (Steffe, 1994; see also others in the same volume,). This presents a major difference between multiplicative and additive situations, where only one type of referent unit is required. Also, when two numbers are added, the referent units of the addends as well as their sum are the same (3 apples + 3 apples = 6 apples). But multiplication can be a "referent-transforming operation" (Schwartz, 1988); for example, the product often has referent units that are different from either of the factors (2 baskets × 3 apples per basket = 6 apples). In multiplicative situations such as the donuts scenario mentioned at the outset, the identification of the intended "one unit" could be challenging due to the choice of several possible units.

Referent units for rational numbers

As learners encounter rational numbers, two related areas of difficulty arise regarding the concept of "the unit." First, students must recognize that there is an underlying whole (a fraction is a fraction of some whole) and secondly, they must refer the fraction to that whole using unit labels. With whole numbers, like 3, discrete, countable quantities can be used without explicitly acknowledging the existence of an underlying unit. With the introduction of fractional ($^3/_4$) or decimal (0.8) numbers, attention to the unit is vital: $^3/_4$ of what unit? Or 0.8 of what unit? When the fraction is an operator (*i.e.,* a factor that stretches or shrinks another quantity) the referent unit may be made up of discrete, countable objects (three-fourths ($^3/_4$) of the children or eight-tenths (0.8) of the marbles), which is handled differently than in situations where the fraction has a measurement meaning and the unit is a continuous quantity ($^3/_4$ of a cup or 0.8 of a mile). In either case, however, a *new* unit (fourth or tenth) is created by partitioning the referent whole (Lamon, 2005). Note the variety of units in this scenario:

> The recipe calls for one and three-fourths cups of flour. Would you like to pour one in?

The intended referent unit of "one" is ambiguous (the **one-fourth-cup**, the **cup**, the **three-fourths-cup**, assuming there is a measuring cup that holds three-fourths of a cup, or even the entire $1^3/_4$ cups – one measure of what the recipe requires).

Everyday and classroom language may prevent the understanding of these new, smaller units. For example, querying college students, including pre-service elementary teachers, as to the verbal name of the symbols "$^3/_4$ c." or "0.8 mi." many would say "three out of four cups" or "zero point eight miles" (personal observation). These verbalizations suggest they are reading the surface symbols and are not focused on the meanings of the symbols. Further, the reference to whole numbers (three, four, eight) followed by the plural (cups or miles) suggest discrete (and countable) quantities larger than one. Hence, by introducing a context (fraction of a cup or mile) and the language that is commonly used with that context, the referents of fractions can become more obscure (*i.e.,* there are several possible units to attend to). That 'three-fourths' means three measures each of which is one-fourth of some understood whole needs specific grounding (Behr, Harel, Post, & Lesh, 1994; Lamon, 2005).

Studying mathematics classrooms for pre-service teachers

Just as children experience difficulty with units in fractional and multiplicative situations, adult learners also confront some of these same issues (Ball, 1990; Ma, 1999; Simon & Blume, 1994). In this article, two cases situated in mathematics classrooms for pre-service elementary teachers (PSTs) will be examined. In both of these classroom settings, PSTs grapple with mathematical concepts and ideas with the aim of understanding more deeply the subject they will teach. As I present a detailed description of video transcripts and other artifacts of teaching and learning from these classes, we get a glimpse of the types of language and representations that are utilized by both the instructors (mathematics educators) and the PSTs. In this study, we consider the following questions: What are the struggles these adult students experience as they try to make sense of operations on rational numbers? What are effective ways for the instructors to support these future teachers in their struggles? In particular, what is the role of ambiguity in the choice of language or context and how do these choices impact the opportunity of learners to focus on the referent unit in rational number operations.

Case A: Multiplying decimals with base-10 area diagrams

In the first case examined, PSTs are working on understanding what it means to multiply decimal numbers. The instructor has introduced a rectangular diagram to the whole class to help them visualize the area meaning of multiplication. She is striving to link the partial products seen as smaller rectangles in the diagram to the steps of a multiplication algorithm (Schultz, 1991; Owens & Super, 1993; Bassarear, 2005). Because of the PSTs' prior experience in the course using rectangular base-10 diagrams to visualize multi-digit whole number multiplication, the instructor hoped that the same types of diagrams would clarify decimal fraction multiplication and begin to show why the algorithm for "moving the decimal point" works.

The PSTs in the class were instructed to use one of three ways to think about the multiplication problem 3.4 × 2.6 as the area of a rectangle on base-10 grid paper. Briefly, these ways are: (1) 3.4 length units by 2.6 length units, (2) 2.6 blocks each composed of 3.4 area units, or (3) 3.4 blocks each composed of 2.6 area units. The PSTs' work and conversation (both here and in other observations) illustrate, among other things, that they have difficulty distinguishing

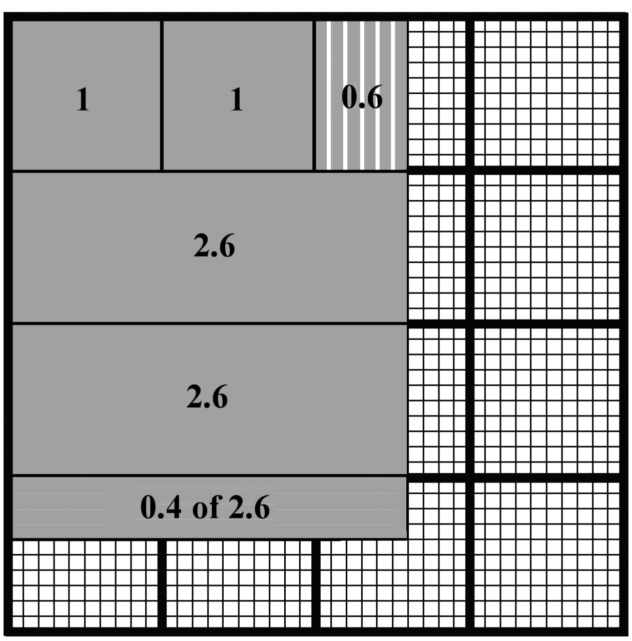

Figure 1. Instructor's rendition of 3.4 2.6 as 3.4 (blocks) of 2.6 (area units)

the units that measure one-dimensional length and width of the rectangle from the units that measure two-dimensional area. (These difficulties are also discussed in Simon & Blume, 1994 and in Reinke, 1997.) To avoid this additional uncertainty in creating the model, as a starting point, the PSTs were encouraged to start with option (2) or (3). Thus, utilizing an "equal measures" meaning of multiplication, the two PSTs presented in this small group discussion with the instructor are thinking about the problem as 3.4 (groups or blocks) each made up of 2.6 area units (see fig. 1 for a completed model using this way of thinking). The reader might consider, before reading on, which units a person needs to attend to in solving this problem. Where might difficulties occur in attending to units? How might ambiguities in identifying units hinder the ability to create the diagram and interpret the product?

Segment A-1: Representing and iterating a 2.6-unit "unit" 3.4 times

The two PSTs begin the process by composing an area of 2.6 area units. This step requires the construction of one area unit and its iteration (copying) to represent the "2" in 2.6. To complete the assembly of 2.6 area units, they must add on six-tenths. At this point, the first ambiguity arises in the nature of the unit: Six tenths of what? In order to decode the number 2.6 and represent each digit appropriately, the PSTs must realize that the "six-tenths" is counting six of the one-tenth pieces of one area unit (and not, for example, six-tenths of the two area units already constructed). Also, the everyday naming of 2.6 as "two-point-six" rather than "two **and** six-tenths" may obscure that six-tenths must be **added on** to the two units.

After the 2.6 area units are successfully constructed, a shift must take place in the view of the unit. Where the PSTs had been considering multiple area *units,* they must now "package" those 2.6 units into a new singular type of unit (the 2.6-units "Unit") and iterate the new Unit three times. [In the transcripts, "T" refers to the instructor/mathematics educator, "S1" and "S2" are the two pre-service elementary teachers. Bold words indicate those pronounced with emphasis throughout the transcript.]

T: So, there's **one** of two and six-tenths [pointing to rectangular region shaded orange]. Can you draw me **two** of those? **Two** of the orange parts. Maybe draw it right underneath it.

St1: Just … do it again [begins drawing 1 square *area* unit … not 2.6 area units; see fig. 2]?

T: What you've got there, now … what she's creating is **two** of …

[student returns to the diagram and begins shading a second 1 by 2.6 rectangle, but stops short after shading one area unit, as in fig. 2].

T: [to St1] No, the whole **thing**.

The instructor anticipates (erroneously as we now see) that the students and she are referring to the same parts of the model as they discuss the diagram. The instructor's

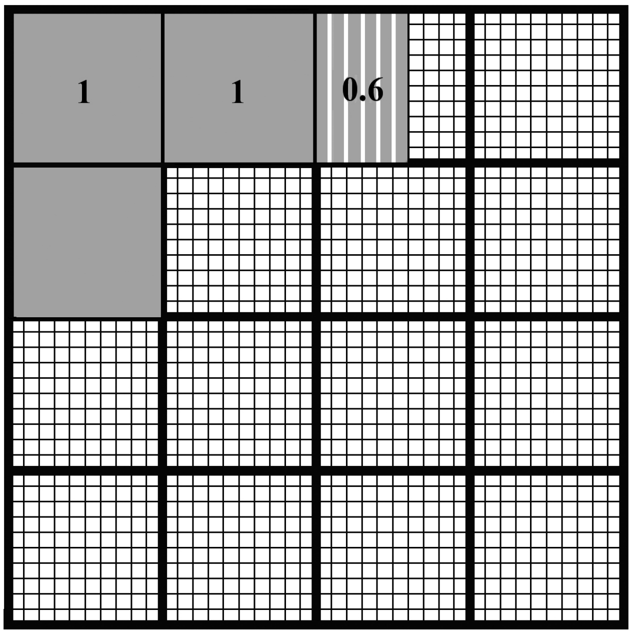

Figure 2. PST's model of 2 2.6

suggestion to, "Maybe draw it right underneath it," implies that she holds a vision of the completed diagram in her mind and wants to help the PSTs to create what she sees. As the diagram unfolds, she assumes that the student has mentally constructed the 2.6 area units into a new kind of "package." Because this unit is clear to the instructor, she speaks vaguely about making a copy of this package. Much like the speaker in the conversation about donuts mentioned at the outset of this article, the instructor uses everyday language, rather than mathematical language, to connect with the students on a common level. A characteristic of everyday speech that is evident in this transcript is the prevalence of vague or unreferenced pronouns. Examine how pronouns are used by the instructor (in italics) to indicate certain parts of the model and how they are used by the PST in response:

T: Can you draw me **two** of *those*? … Maybe draw it right underneath it.

St1: Just … do *it* again?

T: What you've got there, now … what she's creating is **two** of … No, *the whole thing.*

For the instructor, the phrases "two of those," "it," and "the whole thing" refer to an entire 2.6 area-units "Unit." The PST's question, as well as her uncertainty about creating the next part of the area diagram, indicates that she is considering the referent of these utterances to be a single (1 × 1) area unit. The ambiguity of unreferenced pronouns and the changing nature of the unit in decimal multiplication make the instructor's well-intentioned everyday language unclear if not confusing.

Segment A-2: Making "three two-point-sixes"

After some uncertainty as to whether she should copy the one square area unit a number of times or iterate the entire 2.6-area-unit block, Student 1 is directed to make copies of the 2.6-units "Unit." At this point in the video segment, the PST seems to have constructed in her mind 2.6 area units into this new larger Unit. Now she must shift to thinking about how many total copies (3.4) of this new Unit she needs to make.

T: … **Two** of the two and sixth-tenths. There's two of them … so far, so good? Here's **one** of them [pointing to first 1 × 2.6 rectangle]. Here's **two** of them. [pointing to second 1 × 2.6 rectangle]. Can you draw me another one? 'Cause we need three of them, right?

St1: Why do we need … why do you say we need **three** of them?

T: Well, this is a *two-point-six,* right? You're good with that? Here is another *two-point-six.* Now we've got **three** *two-point-sixes.*

St1: OK, 'cause we're trying to …

T: 'Cause we're trying to eventually get three-point-four of them, but now we have **three** times *two-point-six* … you're okay with that?

In this segment, the instructor begins to focus attention on the larger, composed units by physically gesturing to the relevant region with her finger and naming each composed unit a *"two-point-six"* rather than the more ambiguous "one of them," "another one," or "the whole thing." However, if the PSTs are not completely clear what is indicated in the diagram by a *"two-point-six,"* this gesturing and naming might not help to clarify how to complete the diagram or what it might mean after it is constructed.

Segment A-3: Four-tenths of what?

As the PSTs proceed to construct their area model, they must continually shift their attention between the individual area unit, the 2.6-area-units Unit, and the 3.4 which is counting the number of the larger Units they must make. At this point, the PSTs are facing unit ambiguity again. The "point four" could refer to four-tenths of an area unit, to four-tenths of a 2.6-units Unit, or to four-tenths of all 3 of the 2.6-units Units they have made thus far. In this next segment of transcript, you can hear evidence of reference to four-tenths of each of these possible "units." Note again the role of unreferenced pronouns in contributing to the PSTs' uncertainty as to the unit of focus.

T: So, now we've got **three** … one, two, three of the two-point-sixes. Can you draw me another four-tenths of one of those orange things that you just drew?

St1: Four tenths?

St2: Like you can do it … [taking over the pen from St1].

T: Another four-tenths of this orange thing … but, I would put it up here [pointing to put another "row" on top of the 3 × 2.6 array].

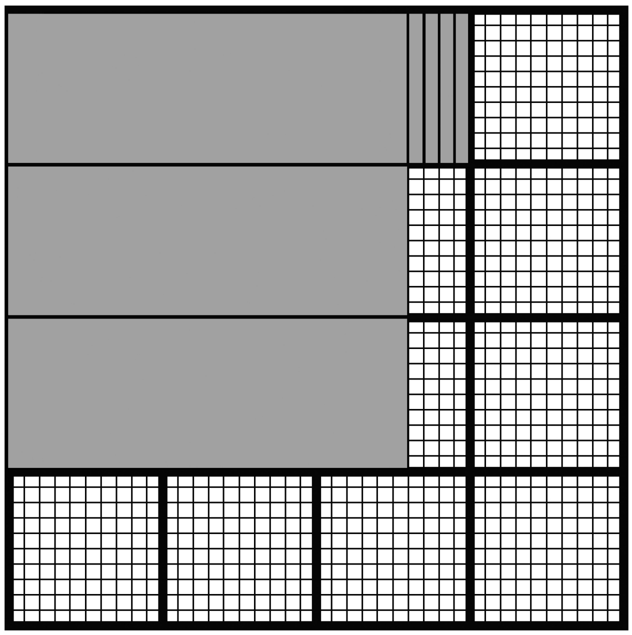

Figure 3. PSTs' interpretation of four-tenths of "an orange thing"

St1: [inaudible] four of …?

T: We've got one of these orange things, two of these orange things, three of these orange things and now, we want four-tenths of one of these orange things.

St1: So, couldn't you … [inaudible] … 'cause it's a whole 'nother thing …

T: … you could put it over here … how would you do it?

St1: One, two, three, four [counting 4 *"longs,"* see fig. 3].

T: No, four-tenths of the **whole** orange.

St2: The whole thing.

St1: Of all the orange [pointing to 3 × 2.6 rectangle that has been made so far]?

T: No, just of **this** orange [pointing to a 1 × 2.6 rectangle]. This is one of them.

St1: Why did you say just to make one of them?

T: Why just one of those? Because if we did four tenths of this whole orange thing, that would be four-tenths of **three** of them. We want four-tenths of just **one** of them.

The answer to "Four-tenths of what?" is not yet clear. Although the PSTs have constructed 3 blocks with 2.6 area units in each block, the referent of the final four-tenths remains somewhat elusive to both students. The instructor's gesturing and "pointing" to parts of the model does not seem

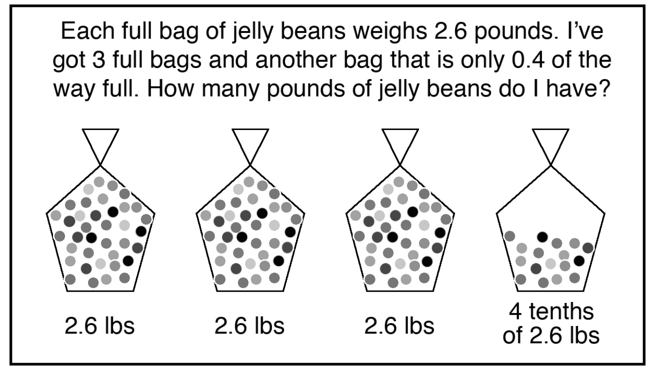

Figure 4. A context for 3.4 2.6

to engage the focus of the PSTs to meaningful referents. To add to the confusion, the instructor's words are again ambiguous ("four-tenths of this orange thing"). In particular, if we examine the use of unreferenced pronouns and other vague terms throughout the transcript *(i.e.,* "one of those things"), we get a glimpse of not only what the problem is, but also what a potential solution might be. The instructor's intended units of focus may be less elusive or ambiguous if they are named, such as in a context. One possible scenario is explored in the following section.

Multiplying decimals in context

In an effort to address meanings more fully and to reduce the ambiguities, the instructor might introduce a common situation where 3.4 × 2.6 could be used. The thought is that by clarifying language and eliminating obstacles to understanding, learners might make more of the intended connections between procedures and meanings. Figure 4 shows one potential context.

As students connect with personally meaningful referent units and representations, how might the discussion proceed more productively? To envision this scenario using the "bags of jelly beans" context, compare the following real transcript to an imagined transcript that specifies referents and names decimals with place value meanings.

Real Transcript
T: Right, 'cause one of those would be that and there'd be another point four of one of those up here and we'd be done. But, we've got three of them. You're comfortable with this being three times two point six?
S: It just seems that it should be like that.

Transcript in Context
T: Right, 'cause one bag would be 2.6 pounds and there'd be another four tenths of one bag up here and we'd be done. But, we've got three bags. You're comfortable with the whole diagram being three bags each with two and six tenths pounds?

Beyond the clarity in instructor language, consider how the PSTs in this segment might track the different types of units as they are assembled, copied, sectioned, and unpacked. For 3.4×2.6, they might think about making three and four-tenths bags, with each full bag containing 2.6 pounds. One full bag is assembled (2.6 pounds) and two more copies are made to represent the weight of three total bags ($2.6 + 2.6 + 2.6$). The difficulty of "four-tenths of what?" could become clearer when students can ask the question in a context: "How many pounds are in a bag that is four-tenths of the way full?" By placing the computation into a context that is perhaps more familiar to the learners than rectangular area, informal mathematical language in an everyday context rather than vague and imprecise language might be used by participants to give meaning to the different types of units.

So, contextualization is one strategy the Case A instructor could explore to clarify the language of the lesson and thus direct the PSTs' attention to the appropriate unit as it changes during multiplication. Examination of Case B suggests a very different approach.

Case B: Comparing fraction stories and operations [2]

In Case A, a context and avoidance of unreferenced noun phrases were proposed as promising mechanisms to "clean up" instruction by clarifying for the students the meanings of numbers and operations. Interestingly, this approach presents a move in two paradoxical directions. In one sense, ambiguity is reduced, which is characteristic of the mathematics "register" (Pimm, 1987). But as 'everyday', contextual language is brought in, more ambiguity can be introduced (*i.e.,* now there are new units: the bag, the pound, and even the jelly bean).

An alternative approach to instruction around rational number operations is to place the ambiguity of the unit onto center stage and allow the students themselves to sort out the referents. That appears to be the aim of the instructor of a second mathematics class for pre-service teachers where participants are expected to justify their mathematical thinking. In Case B the instructor intentionally introduces a fraction operations task where, from experience, she knows the students will find the unit unclear. When students begin to see that there is a choice of possible referent units, discussion ensues and participants present arguments for or against the use of a particular unit. Within these discussions, speakers employ their *own* context, language, and diagrams to clarify their reasoning and to convince their classmates of the validity of their idea or strategy. In this scene, the PSTs were asked to write story problems to match the arithmetic expression $3\frac{1}{2} - \frac{2}{3}$. Although the task does not specify multiple meanings for the unit, experience with this problem has shown that two different interpretations of the expression usually appear (Rathouz & Rubenstein, forthcoming). In this class, one group has written the following on the board.

> John has to run $3\frac{1}{2}$ miles. He started running but then took a break after … $\frac{2}{3}$ of a mile? OR … $\frac{2}{3}$ of the way? How much further must he run?

The whole class is now wondering which of the two wordings correctly matches the computation $3\frac{1}{2} - \frac{2}{3}$. Before reading on, the reader might consider the nature of the unit(s) in each of these two possible story problems. How does the interpretation of units change as you reason about which problem is consistent with $3\frac{1}{2} - \frac{2}{3}$? How might you use verbal communication or other representations to convince someone else of your reasoning?

Segment B-1. "Two-thirds of a mile" or "two-thirds of the way?"

The instructor for this course frames the problem to engage the entire class. Then, the students begin to use a variety of tools, such as meanings of fractions and operations, to clarify their thinking.

> T: So this group actually … they weren't sure what to do, so they said, "John has to run $3\frac{1}{2}$ miles. He took a break after …" They couldn't decide if they should say 'two-thirds of the way' or 'two-thirds of a mile' … "How much further must he run?"

> S1: It depends on whether you want to take it out of the whole or out of a single mile.

> S2: In a way, you're looking at all three and a half miles and you're saying, like, here's the three and a half miles and you're saying here we are on the number line and you're trying to figure out how much further you have to go to get to three and a half.

In order to help clarify the two different situations and their corresponding referent units, another student shares his explanation and diagram (see fig. 5).

> S3: [presenting his ideas at the front of the class] The amounts you're measuring are such … like … so very different that it's so clear to see what you're measuring. You're measuring two-thirds of the way. The way John is running is three and a half miles so two-thirds of the way would be a little bit over two miles 'til three and a half. Two-thirds of a mile is … two-thirds of a mile would be like right here, so about two-thirds … right here. These are two, clear, different amounts.

When the PSTs experience uncertainty about which unit the two-thirds refers to, they address each other with questions to try to understand better the claims that the speaker is making. The speaker then refers to his own diagram to justify his reasoning using language clarified on demand of the participants.

> S4: How did you get that one?

> S3: Get what one?

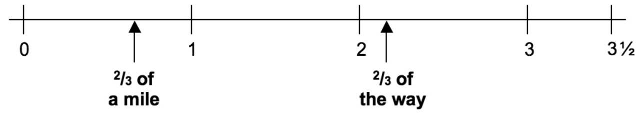

Figure 5. Student 3's representation of two interpretations

S4: Two-thirds of the way.

S3: Two-thirds of the way. Because if it's "of the way", I interpret that "of" … as like "the three and a half miles."

S4: Then you find it for one mile first and then …

S3: Well, I just like estimated like two-thirds of the way … I knew that two-thirds of three is two, so two-thirds of three and a half is gonna be a little bit more than two.

S5: Just to answer her question … you could've put like two extra marks in there between zero and the first mile and that would've broken your first mile into thirds and you could've seen two-thirds.

S3: You're saying I could've put a notch right here so it would be like 1, 2, 3 marks, and this would be two-thirds of a mile, so none of this takes into account. … Just one mile.

S6: Right, so, are you saying that three and a half is what we're trying to find?

S3: I interpret "two-thirds of the way" as two-thirds of three and a half miles. 'Cause "of the way" I think of like the total amount.

S6: Three and a half is your whole.

Although the students have not yet returned to the original issue of which story matches the subtraction arithmetic, they are beginning to clarify that the two stories require attention to (at least) two different units. The question throughout this discussion really is "What are we finding two-thirds of?" What is the referent unit of two-thirds in each of the story problems? Student 3's number line provides a common focal point to engage the rest of the class in discussion. He clearly shows both types of unit ("one mile" and "the whole way") and describes his reasoning about where 'two-thirds of a mile' and 'two-thirds of the way' would be on the diagram. Additional "units" (*i.e.,* thirds of a mile) might be brought out in student 6's suggestion to, "… put, like, two extra marks in there between zero and the first mile and that would've broken your first mile into thirds …". Because this story problem has a choice in wording and the diagram has several potential units to attend to, the need for clarifying the referents of units is highlighted.

Segment B-2. PSTs sort out the units

As the discourse continues, the PSTs begin to justify that one of the wordings involves subtraction alone and the other uses both multiplication and subtraction. Interestingly, the participants argue their points most convincingly when they draw attention to the variety and potential confusion of different units.

T: Somebody present the other argument for why it might be two-thirds of a mile.

S7: Two-thirds of a mile is keeping the same units, which is miles, and if it's two-thirds of three and a half miles, your unit's like three and a half miles total so it's …

[Many students speaking …]

T: Wait, hold on.

S8: It's keeping … the same units as … two-thirds of a mile … I don't know how to say it …

S9: No, you're right … keep going.

T: So, wait, you're saying three and a half miles minus two-thirds of a mile, you're keeping the units the same?

S4: But "two-thirds of the way" you could say "two-thirds of the mil**es**"

S8: But two-thirds of the miles is two-thirds of three and a half …

Student 4 brings up further complications at the end of this segment because the two-thirds is a scalar multiplier in the "of the way" story. However, the other students seem to be clear that it is the **structure** of the unit that is relevant in deciding the referent of two-thirds in subtraction or multiplication, not just the **name** of the unit. Through further discussion and debate in this classroom, progress is made.

Contrasting the cases

At this point, it might be helpful to draw some comparisons between the two cases presented, the language and representations used in each classroom, and the role of ambiguity in the learning of rational number operations. The PSTs in Case B, like those in case A, experience confusion at times during their discussion. Also similar to the first case, the language used by the participants is closer to everyday speech than mathematical discourse.

The sources of ambiguity of the referent of the fractions in Case A and Case B present a possible difference in how confusions are handled by the PSTs in each class. In Case A, the referent unit of the "3.4" (a block of 2.6 area units) was elusive to the PSTs and the imprecise language seems to have suggested several possible referents. In Case B, the ambiguity of the referent for two-thirds (one mile or three and a half miles) is introduced by the two story contexts and number line representations that suggest different "answers" that, thus, must have different symbolic expressions ($3^1/_2 - {}^2/_3$ or $3^1/_2 - ({}^2/_3 \times 3^1/_2)$). Another possible contrast in the two cases is evident in the resolution of the "problems." In Case A, the units did not hold meaning to the PST participants in the conversation. This lack of connection to a known context made it difficult, if not impossible, to discuss or to reason about what to do with the objects in the diagram. I hypothesized that introducing and referring to a more familiar context would be helpful to the learners. In Case B, the context of a $3^1/_2$-mile run and perhaps the imprecision of the presenting PST's original explanation helped to foster mathematical discourse in an everyday context. Through discussion, the PSTs were able to resolve the ambiguity in referent unit by using meanings of the operations to justify why only one story matches the given expression.

Considerations for teaching mathematics

What can we learn about the role of ambiguity in mathematics classrooms from the study of artifacts from these two environments? As I examined my own and my colleague's classroom videos and transcripts, I became aware of both mathematical and language issues that may impact student learning. The transitions from whole number to rational number thinking and from unit-preserving operations (*e.g.,* addition) to unit-transforming operations (*e.g.,* multiplication) appear to be non-trivial. As these adult learners move through multiplicative situations involving rational numbers, the identifying, naming, and representing of the referent unit is noted as particularly challenging due to the changing and often ambiguous nature of the unit.

From one perspective of instruction, consistent with Barwell's (2005) *formal model,* the role of the instructor is to present ideas explicitly so that ambiguity is avoided. Under this model, instructors' attention to conceptualizations of mathematical ideas through clear language is meant to guide discussion and use of representations so that students make appropriate connections and "discover" the mathematics that the tasks intended. From an alternative perspective to mathematics learning, termed the *discursive model* (Barwell, 2005), a variety of meanings arise as students and teachers discuss ideas using their own language. As we note in the two cases presented in this article, trying to balance these two instructional models can lead to tensions for mathematics teachers. Adler (1997, 1998) termed these issues the "dilemma of mediation" and the "dilemma of transparency." To what extent should instructors validate and encourage informal and often incomplete language that learners use or attempt to move students to produce the abstract and formal language and representations of mathematics?

In Case A, the instructor chose to introduce a model for multiplying decimals that she knew was robust for fractional quantities. She has a particular image of the area model in her mind that she is trying to convey to the PSTs. This case seems more consistent with the Formal Model of instruction. Because the area representation was not familiar to the learners, the instructor attempted to activate the attention of the PSTs to various units by utilizing informal language. In particular, unreferenced pronouns and phrases (such as "it" or "one of those"), introduced to make the mathematical language more "friendly" may have obscured rather than clarified the intended meaning of the symbols. Even the common everyday naming of the symbols used by both the instructor and the PSTs ("three-point-four" for 3.4) rather than language that preserved the meanings ("three and four tenths") may have caused confusion. Further, although the rectangular area model is an appropriate and potentially useful representation of decimal multiplication, every participant in the classroom discussion might not have "attended to" the same features that the model shows. For example, the grid lines in base-10 paper create units of many different sizes, both 1-dimensional and 2-dimensional, so perhaps unnecessary unit ambiguity is introduced with this model.

Case A guides us to the conjecture that without precision and clarity in the language and other representations used to refer to the quantities in the problem, students may miss opportunities to connect the mathematical symbols and representations to their meanings. Also use of a more familiar context in a decimal or fraction multiplication problem may have strengthened the connection to relevant units. However, when placing multiplication computations into familiar contexts, caution must be taken so that the cognitive challenge of the task is not removed. In the jelly bean bag example, separation of the whole number of bags (3) from the fractional part of a bag (.4) may promote the use of more primitive meanings of multiplication such as repeated addition (2.6 + 2.6 + 2.6). When multiplication is "reduced" to repeated addition, there is perhaps less of a need to continually shift focus from one type of unit to another. Also, while the new "jelly beans" diagram helps to make clear what each number is counting or measuring, it is perhaps not as helpful as the area representation in solving the problem numerically. With the jelly bean context to ground their thinking, however, students might return to the area model with new insight as to what each part of the model represents.

With **reference** to the **names** of the units, rather than vague or unreferenced pronouns, the conversation might clarify rather than obscure both the problem and its solution. In doing so, connections might be formed among (1) the symbolic representation (3.4 × 2.6), (2) the contextual representation (If each full bag of jelly beans weighs 2.6 pounds, how much would 3.4 bags weigh?), and (3) the diagrammatic representation (the area model where each square unit represents a pound, each block of 2.6 square units represents a bag, a rectangle of 3.4 of these blocks represents all of the bags, and the area of the entire rectangle represents the weight of the 3.4 bags). To strengthen these connections, instructors might pose the following types of questions: What does the "3.4" mean in the story? Where is one bag in the area model? What does this part (point to 1 area unit) mean in the story? Where are the "two and six-tenths" in the area model?

In Case B, ambiguity in the unit is integral to the task at hand and is chosen intentionally as a device to generate useful discussion by the PSTs around the problem. Asking students to generate story problems for this particular fraction subtraction problem nearly always provokes this confusion and struggle. Hiebert and Grouws (2007) suggested that struggling with important mathematics helps to facilitate conceptual understanding. Having this group of pre-service teachers grapple with the idea of the unit (two-thirds of what?) also seems to generate the use of more precise communication and representation tools to help resolve the ambiguities. In this second classroom, the PSTs connected verbal and diagrammatic representations to symbolic expressions in order to clarify their own thinking. Some of the participants recognized that there was more than one type of "unit" in this problem and they made progress in clarifying this idea for each other. As the adult learners interact on and with their own "terms," the collective understanding of the group appears to deepen. Importantly, the PSTs' own reasoning, context, and diagrams were the source of this progress. Descriptions and analyses of classrooms where the instructors embrace students' multiple (even incorrect) interpretations of mathematical symbols, language, and representations as opportunities for learning provide a vision of instruction where students are allowed to grapple with important concepts (Moschkovich, 2008).

Final thoughts

Teachers of mathematics at all levels might consider their instructional goals as they weigh the advantages and disadvantages of introducing representations, contexts, and language of various types, both vague and precise. In both of the cases examined here, one goal was to support PSTs in their understanding of operations on fractional quantities. Because the referent unit in these situations changes throughout the problem-solving process and because representational systems, including language, can be interpreted in multiple ways, ambiguities of many types arise. If teachers of mathematics are conscious of the issues around fractions and their often unclear referent units, especially in multiplication, deliberate decisions can be made regarding whether to clarify language, to introduce contexts, or to embrace imprecision and vagueness to catalyze discourse and understanding among students.

Notes

[1] While this article has been authored by one colleague, several others have been involved in the instructional design, implementation, revision, and thinking related to this work. In particular, I am indebted to Theresa Grant and her students for allowing the use of her videotape in Case B, to my own students for allowing me to transcribe their work in Case A, and to Rheta Rubenstein for countless discussions regarding manuscript writing and editing. Parts of this work were funded by the NSF DUE #0310829.

[2] Portions of the classroom transcript from this section have been presented elsewhere (Rathouz & Rubenstein, 2010) and were analyzed with a different focus than that presented here.

References

Adler, J. (1997) 'A participatory-inquiry approach and the mediation of mathematical knowledge in a multilingual classroom', *Educational Studies in Mathematics* **33**, 235-258.

Adler, J. (1998) 'A language of teaching dilemmas: unlocking the complex multilingual secondary mathematics classroom', *For the Learning of Mathematics* **18**(1), 24-33.

Anghileri, J. (1995) 'Language, arithmetic, and the negotiation of meaning', *For the Learning of Mathematics* **15**, 10-14.

Ball, D. L. (1990) 'The mathematical understandings that prospective teachers bring to teacher education', *Elementary School Journal* **90**, 449-466.

Barwell, R. (2005) 'Ambiguity in the mathematics classroom', *Language and Education* **19**(2), 118-126.

Bassarear, T. (2005) *Mathematics for elementary school teachers, 3rd edition.* Boston, MA, Houghton Mifflin.

Behr, M., Harel, G., Post, T. and Lesh, R. (1994) 'Units of quantity: a conceptual basis common to additive and multiplicative structures', in Harel, G. and Confrey, J. (eds.), *The development of multiplicative reasoning in the learning of mathematics,* Albany, NY, SUNY Press, pp. 123-176.

Bjuland, R., Cestari, M. L. and Borgersen, H. E. (2008) 'The interplay between gesture and discourse as mediating devices in collaborative mathematical reasoning: a multimodal approach', *Mathematical Thinking and Learning* **10**, 271-292.

Brennan, S. E. (1995) 'Centering attention in discourse', *Language and Cognitive Processes* **10**, 137-167.

Brown-Schmidt, S., Byron, D. K. and Tanenhaus, M. K. (2005) 'Beyond salience: interpretation of personal and demonstrative pronouns', *Journal of Memory and Language,* **53**, 292-313.

Goldin, G. (2003) 'Representation in school mathematics: a unifying research perspective', in Kilpatrick, J., Martin, W. G. and Schifter, D. (eds.) *A research companion to Principles and Standards for School Mathematics,* Reston, VA, NCTM, pp. 275-285.

Graeber, A. O. and Tanenhaus, E. (1993) 'Multiplication and division: from whole numbers to rational numbers', in Owens, D. T. (ed.), *Research ideas for the classroom; middle grades mathematics,* New York, NY, Macmillan, pp. 99-117.

Greer, B. (1994) 'Extending the meaning of multiplication and division', in Harel, G. and Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics,* Albany, NY, SUNY Press, pp. 61-85.

Harel, G. and Confrey, J. (eds.) (1994) *The development of multiplicative reasoning in the learning of mathematics,* Albany, NY, SUNY Press.

Hiebert, J. and Behr, M. (eds.) (1988) *Number concepts and operations in the middle grades,* Reston, VA, Erlbaum.

Hiebert, J. and Grouws, D. (2007) 'The effects of classroom mathematics teaching on students' learning', in Lester, F. K. (ed.), *Second handbook of research on mathematics teaching and learning,* Charlotte, NC, Information Age Publishing, pp. 371-404.

Lamon, S. (1994) 'Ratio and proportion: cognitive foundations in unitizing and norming', in Harel, G. and Confrey, J. (eds.), *The development of multiplicative reasoning in the learning of mathematics,* Albany, NY, SUNY Press, pp. 89-120.

Lamon, S. J. (2005) *Teaching fractions and ratios for understanding: essential content knowledge and instructional strategies for teachers,* Mahwah, NJ, Erlbaum.

Ma, L. (1999) *Knowing and teaching elementary mathematics,* Mahwah, NJ, Erlbaum.

Moschkovich, J. N. (2008) '"I went by twos, he went by one": multiple interpretations of inscriptions as resources for mathematical discussions', *The Journal of the Learning Sciences* **17**, 551-587.

Owens, D. T. and Super, D. B. (1993) 'Teaching and learning decimal fractions', in Owens, D. T. (ed.) *Research ideas for the classroom: middle grades mathematics,* New York, NY, Macmillan, pp. 137-158.

Pimm, D. (1987) *Speaking mathematically,* London, UK, Routledge & Kegan Paul.

Rathouz, M. and Rubenstein, R. (2010) 'Supporting preservice teachers' learning: a fraction operations task and its orchestration', *AMTE Monograph VI: Scholarly practices and inquiry in the preparation of mathematics teachers,* San Diego, CA, Association of Mathematics Teacher Educators.

Reinke, K. S. (1997) 'Area and perimeter: preservice teachers' confusion', *School Science and Mathematics* **97**, 75-77.

Rowland, T. (1999) 'Pronouns in mathematical talk: power, vagueness and generalisation', *For the Learning of Mathematics* **19**, 19-26.

Rowland, T. (2000) 'The pragmatics of mathematics education: vagueness in mathematical discourse', London, UK, Falmer.

Schleppegrell, M. J. (2007) 'The linguistic challenges of mathematics teaching and learning: a research review', *Reading & Writing Quarterly* **23**, 139-159.

Schultz, J. E. (1991) 'Area models - spanning the mathematics of grades 3-9', *The Arithmetic Teacher* **39**, 42-46.

Schwartz, J. L. (1988) 'Intensive quantity and referent transforming arithmetic operations', in Hiebert, J. and Behr, M. (eds.), *Number concepts and operations in the middle grades,* Hillsdale, NJ, Erlbaum; Reston, VA, NCTM, pp. 41-52.

Simon, M. A. and Blume, G. W. (1994) 'Building and understanding multiplicative relationships: a study of prospective elementary school teachers', *Journal for Research in Mathematics Education* **25**, 472-494.

Steffe, L. (1994) 'Children's multiplying schemes', in Harel, G. and Confrey, J. (eds.), *The development of multiplicative reasoning in the learning of mathematics,* Albany, NY, SUNY Press, pp. 3-39.