Looking at a Pizza with a Mathematical Eye

MARION WALTER

There are many ways to generate mathematical problems from a starting point. Here, I would like to describe my thoughts which led me to produce a number of them. I made a conscious effort to make notes and keep track as much as possible while working on the problems.

These problems, many of them new to me, range in level from Kindergarten to college. I realize, though, that if I describe how I came to pose the problems *before* stating them, I would be giving some hints for possible ways of solving them. Consequently, I will start by stating the problems that came first in the chain, so that you the reader have an opportunity to solve them without any hints. After you read the initial problem, see in how many different ways you can solve it before reading on. You decide on the amount of rigor you demand of yourself.

The first problems

Imagine you are in a restaurant and order one large and one small size plain cheese pizza. The waiter brings them out and hears you discussing how to split the pizzas fairly among the three of you. The waiter says:

> Well, since the large pizza is twice the size of the small one and there are three of you, you can just halve the large pizza and you'll each have the same amount.

How can you check that the waiter was right?

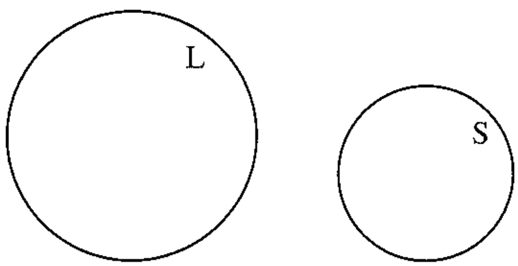


Figure 1: Two pizas: large L and small S

No, you did not bring a tape measure or ruler and you cannot go into the kitchen to weigh them. No need to worry about crusts, these pizza are uniform to the edge. (Of course, if, later, you did consider crusty edges there might be more problems lurking.) So, now before reading on, try to find various ways of showing that L = 2S.

Before describing how I came to this problem, I pause to recollect the 'usual' kinds of pizza problems (partly to discourage you from peeking ahead). Standard pizza problems usually deal with the cost of pizzas or with numbers of available choices.

Here are two generic examples:

> Which is a better deal - one 14" pizza at $13.00 or two 10" pizzas at $7.00 each?

> If there are five different toppings available and the 'special' allows the choice of any three of them, how many different ways can a pizza be ordered?

Now I will describe how I came to the first problem which, in turn, led to quite a number of others. In the past, in classes and during talks, I have asked how one can check quickly whether a given large square L has twice the area of a smaller square S, if measuring the side of each square is not allowed.

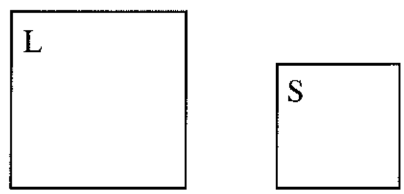


Figure 2: Is the area of the large square L twice that of the smaller one S?

Sometimes, I motivate this question by showing a sequence of nested squares using colored acetate on an overhead and asking which square is twice as large as the smallest one. (See Figure 3a.)

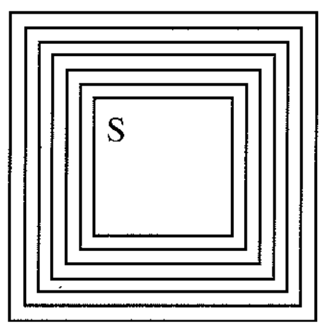


Figure 3a: Using colored acetate: which square is twice as large as square S?

I also customarily show a Frank Stella or Josef Albers painting along with the mathematical problem. (See Figures 3b and 3c overleaf.)

One easy way to check whether square L has twice the area of square S is shown in Figure 4 (overleaf).

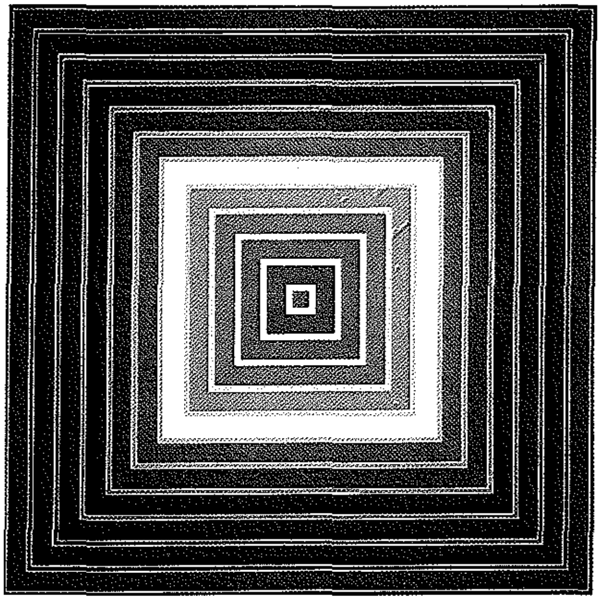


Figure 3b: *Sacramento Mall proposal #4, Frank Stella (1978) ©Frank Stella/ARS (New York)/ SODRAC (Montréal) 2003, Gift of the Collectors Committee, Image ©2003 Board of Trustees, National Gallery of Art, Washington*

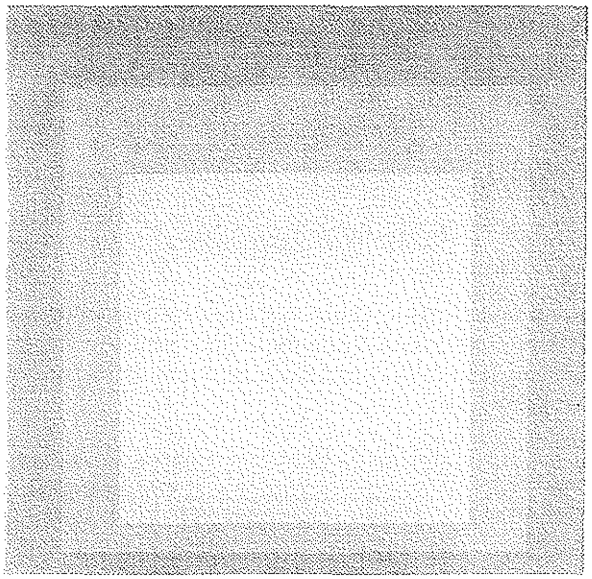


Figure 3c: *Study for Homage to the Square: Departing in Yellow, Josef Albers (1964) ©Estate of Josef Albers/Bild-Kunst (Bonn)/SODRAC (Montréal) 2003; ©Tate, London 2003*

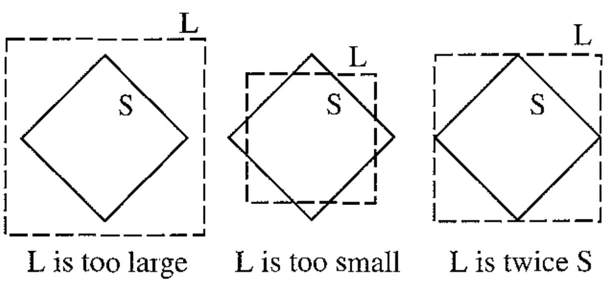


Figure 4:

After establishing that square L has twice the area of square S, I often circumscribe a circle about square S and about square L and rotate either square to obtain squares with parallel sides as shown in Figures 5a-c.

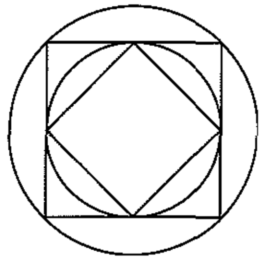


Figure 5a: *Drawing the circumcircles*

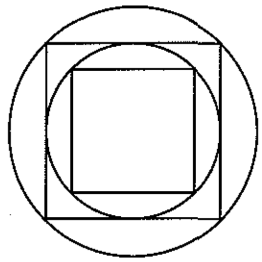


Figure 5b: *Rotating the small square*

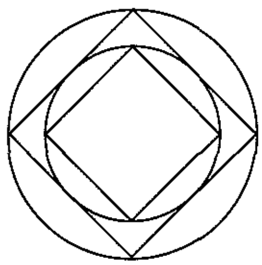


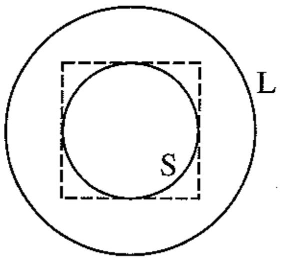
Figure 5c: *Rotating the large square*

I was preparing for a talk - contemplating the above square situation which I love to sneak in - but I did not really want to use this square problem again. I had already sketched Figure 5a and so asked myself how I could perhaps disguise the problem. I heard Polya's rich voice saying both "Think of a related problem" and "*Look* at the problem". [1] So I looked again at Figure 5a and saw it not as a problem dealing with two squares but as a problem dealing with two circles. Aha, here was a disguised problem - how can one tell quickly whether one circle is twice as large as another? But now I had no Stella or Albers painting to motivate or embellish the problem. How could I present this problem?

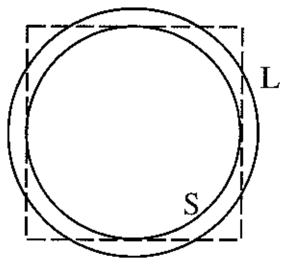
I decided to turn it into a pizza problem. So here was the first problem I started with - how can you tell that pizza L is twice as large as pizza S? I hope you have already solved it in several different ways.

Here are some ways I thought about it and all are related [2]

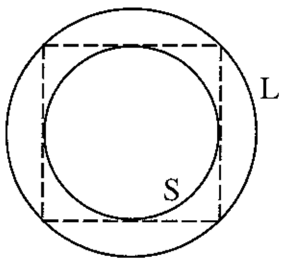
(i) Place one pizza concentrically on top of the other and by eye (or using your fingers) mentally trace the four tangents to the smaller pizza to see if they form a square inscribed in the larger pizza (See Figure 6.)



S too small



S too large



S just right

Figure 6:

(ii) I noticed that the diameter of the small circle is equal to one of the tangent line segments with end-points on the large circle (See Figure 7.)

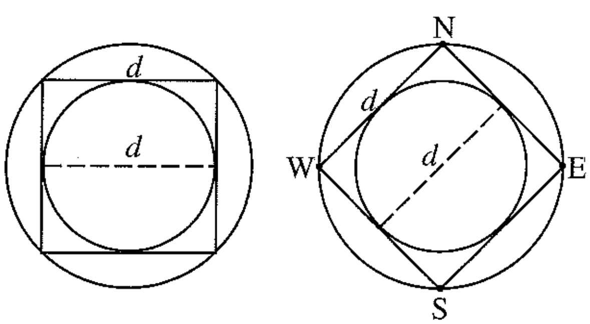


Figure 7:

So allowing the small pizza to be cut in half means we can check to see if the diameter just stretches between two compass points of the large circle (See Figure 8a.)

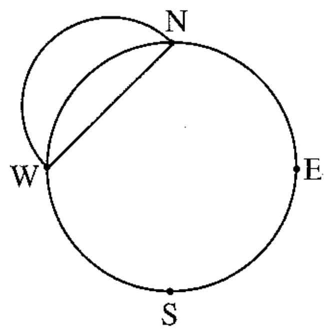


Figure 8a: Using half the small pizza

(iii) If you do not want to cut the small pizza, check the diameter of the small pizza to see if it stretches between the two compass points. (See Figure 8b.)

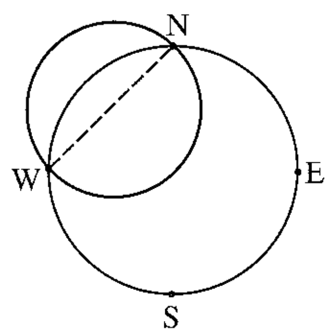


Figure 8b: Using the whole small pizza

Next, I cut out some wax paper circles because they are good for using on overheads. But they are also good for experimenting with, because they are transparent. I cut the smaller one in half and had one half in each hand, so it felt 'natural' to place them as shown in Figure 9 and see if the end-points of the small diameter lay on the end-points of the large diameter

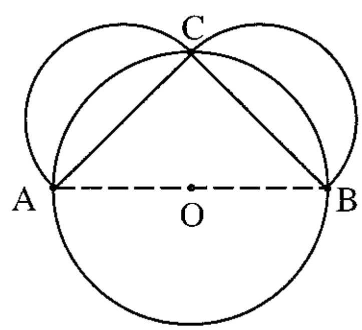


Figure 9: The two half-pizzas form the legs of a right-angled triangle

Since AB seems to pass through the center of the large circle, the angle at C is a right angle and so the generalization of the Pythagorean theorem applies. So we can conclude that the area of the two small semi-circles equals the area of the large semi-circle [3] Placing the two paper circles concentrically made me realize that I could have asked the original question in a different way.

Show that the shaded part is one-half of the large pizza (See Figure 10.)

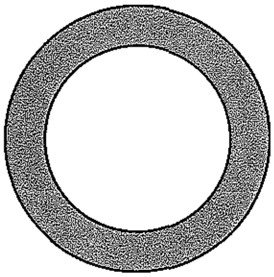


Figure 10: Is the shaded part equal to one-half of the large circle?

If I had asked myself the question this way at the start, I wonder if I would have used a different approach? Now it set me off in another direction, as I realized this is an interesting way of splitting a pizza in half. I could move the inner circle to be tangent to the circumference and show the shaded half of a circle in a crescent way. (See Figure 11a.)

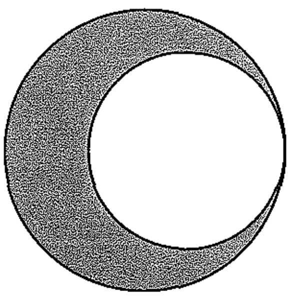


Figure 11a: Another interesting half large pizza

If this is cut in symmetric halves, we get an interestingly shaped quarter large circle (See Figure 11b.)

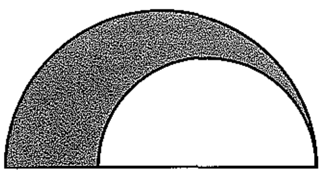


Figure 11b: One quarter large pizza

So now I was on to:

How many *different* ways can a pizza be cut in half?

I have dealt with halves many times, both in two and three dimensions. I have often used photos of Max Bill's half-sphere sculptures. In particular, I have asked participants to cut an apple in half like the sculpture shown here in Figure 12.

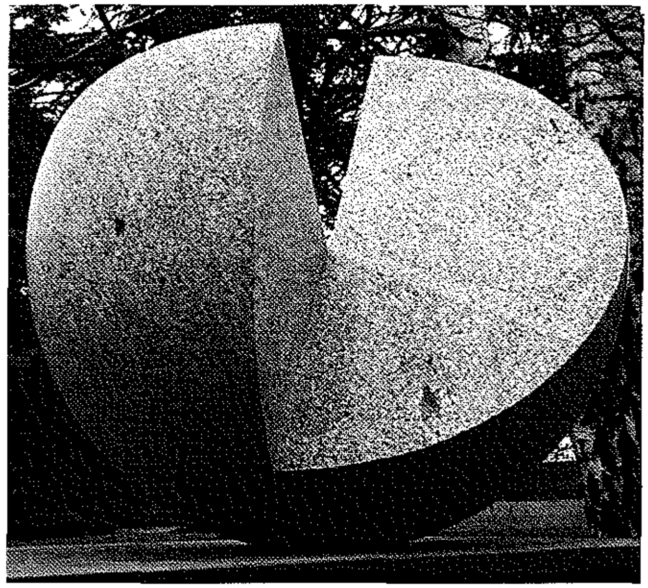


Figure 12: Max Bill, Half-sphere around two axes (1965-6) ©ProLitteris/SODART 2003

So, going back to the question of how many ways can you cut a circle in half, it is possible to generate many more 'obvious' answers besides the ones given in Figure 13.

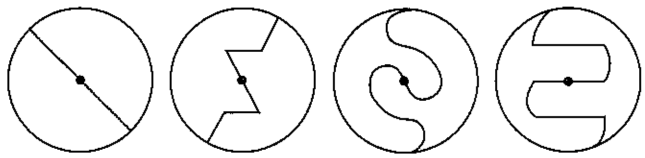


Figure 13: Some usual and less-usual halvings of a circle

Using the What-If-Not? idea [4] on the first image in Figure 13, focusing on the attribute 'the cut is straight', I first asked: "What if the cut does not have to be straight?" One then gets the infinite number of ways that one is used to already from having cut the square in half. So, next, I thought of the attribute 'the two parts are equal in area and congruent' and asked what if they do not have to be congruent? Well, we might get the picture of the concentric ring At *this* point, I did not investigate other shapes. Only while finally typing this article did I think of halving in ways as shown below and opposite in Figures 14a-b

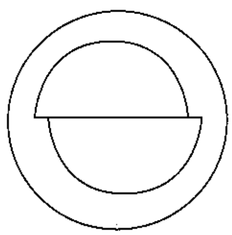


Figure 14a: Recombining two semi-circles of the small circle

Instead, I focused on the attribute 'each half is in one piece': what if it does not have to be in *one* piece? That brought to mind the SMILE [5] poster, showing ways of halving a square (see Figure 15 opposite). And that prompted me to draw Figure 16 (opposite) and ask what if the half is in a pie shape and the remaining half is in two pieces?

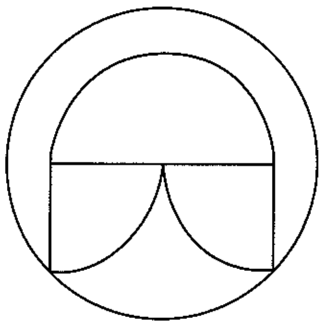


Figure 14b: Recombining two quarter-circles and a semi-circle

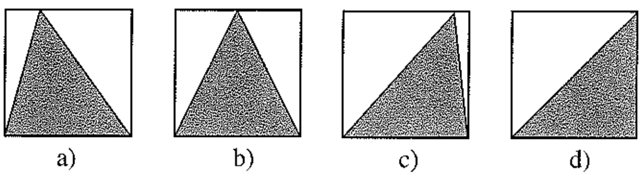


Figure 15: Take half

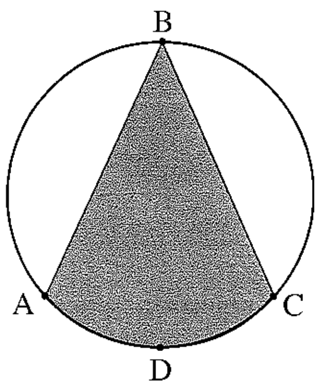


Figure 16: Half of a pizza in a pie shape and the other half in two pieces

How can I determine this slice? I liked this problem as I had not seen it before and so decided to work on it next. I could consider half of the diagram and ask: "Given a semi-circle with BD as diameter, find C so that BC bisects the semi-circle". 'I'll have to use calculus' was my first (but not very long-lasting) thought. But I changed my mind quickly, because I was going to talk to an audience that also involved middle-school teachers. So, I decided to solve the problem without calculus.

I had drawn a slice that has a line of symmetry - BD (Figure 16) analogous to Figure 15b in the square. So the slice would be determined if I knew where A and C were. So here was my next problem: if ABCD is half of the pizza of radius 1 (subject to the usual disclaimer of no loss of generality), what is the central angle AOC, since if I knew the central angle, then A and C would be determined. (See Figure 17.)

The equation I obtained for θ was one whose type I had not seen before. Having no fancy calculator, I solved it by iteration. [6]

It was only later when I looked at the pie shape again and had started to write up this article that I had the two drawings

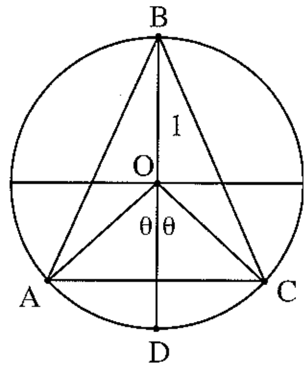


Figure 17:

(Figures 18a and 18b below) lying on my table in front of me and I asked myself the question what if the pie shape had no line of symmetry?

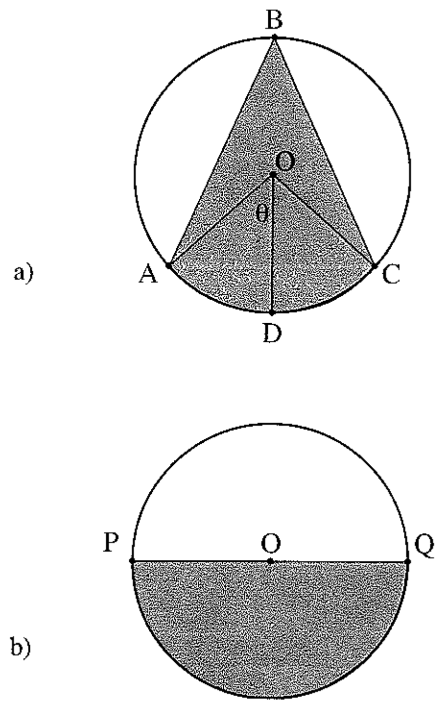


Figure 18: The two extreme (and symmetrical) cases

In other words, what if AB were not equal to CB? I thought about moving B along the circumference of the circle but keeping ABCD equal to one-half the area of the circle. Clearly, AC would have to move too. Because I had the two extreme cases in front of me and I had had a course in film animation – and the book *Geometric Images* (Beeney *et al.*, 1982) also came to mind – I asked myself how the story-board for an animated movie would look if the half-area of the circle that was pie-shaped (Figure 18a) changed to the semi-circle (Figure 18b). I have to confess I made several false starts until I focused on the diameter PQ of the circle and reassured myself of what happened to the line AC as B moved from the highest point down to coinciding with the end-point of the diameter PQ of the circle

I also thought about the theorem which states that for fixed A and C, the angle ABC remains constant as B moves on its arc of the circle I chided myself that over all the years I never once asked myself what might be able to be said about the area ABCD for fixed A and C as B moves along the arc of the circle other than that the area goes to the area of ACD. (See Figure 19.)

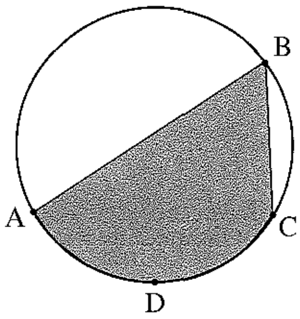


Figure 19: The pie shape has area one-half of the circle, but AB≠BC

But, for now, I was interested in moving B so that the slice ABCD remains a constant area (one-half that of the circle). If the area is to remain constant, AC will have to move too

My third pizza problem

> How can one determine a pie 'wedge' that has area one-half of the circle but is not symmetrical? (See Figure 19.)

Though the wedge ABCD was no longer symmetric, I could still determine half the central angle, since OA and OC are still symmetric with respect to the perpendicular bisector OD of AC. As in the symmetric case, I calculated the area of the wedge by considering the area of the triangle ABC plus the area of the sector AOC minus the area of the triangle AOC - and then set this wedge area equal to $\pi/2$, that is half the area of the circle.

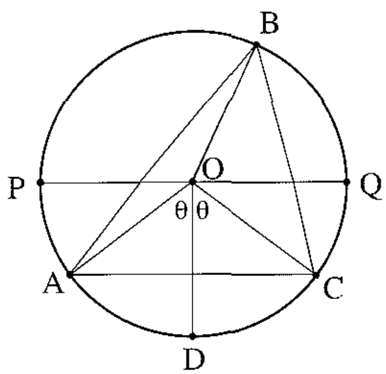


Figure 20: OD is still symmetrically placed with respect to AC

I chose positions of B for which OB made nice angles with the diameter PQ, namely ones for which I knew the sine and cosine (60°, 45° and 30°) - see Figures 21a-e. I leave it to the reader to complete the calculations

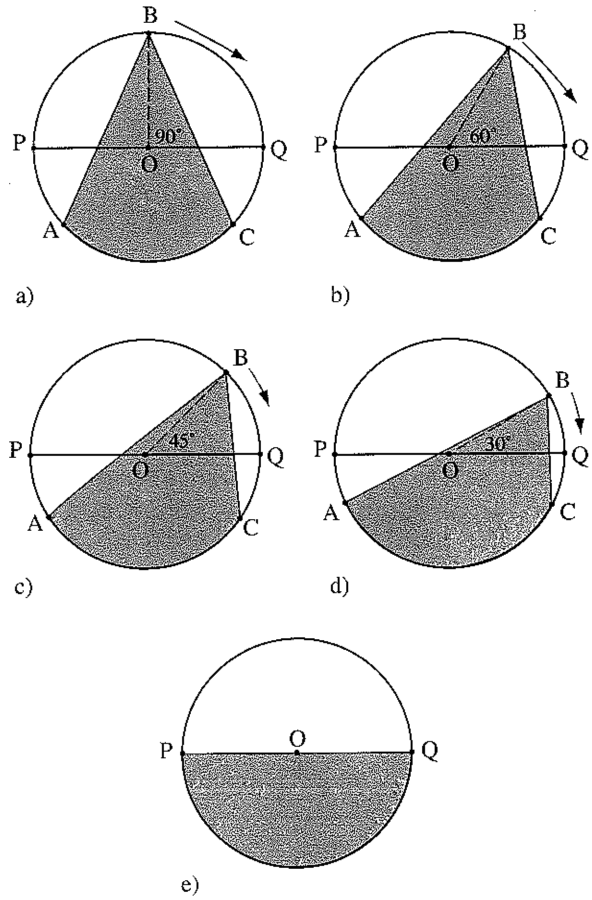


Figure 21: Sketches of pie shapes each of area one-half the circle

Taking problem three further

But I was not done with the pizza problem, which by now I was hooked on I had some paper circles left over - all of the same size - and started to move them about and happened to overlap two of them (See Figures 22a-d.)

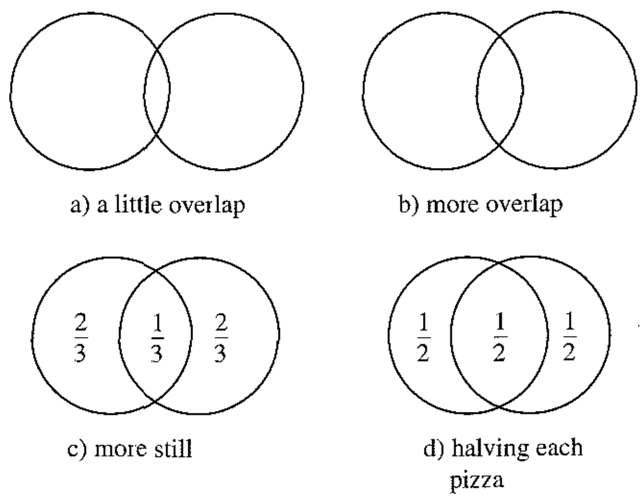


Figure 22:

I asked myself what is the area of the overlap? I then asked myself whether I could place them, so that one can split the two pizzas among three people.

Well, if the areas were shown as indicated in Figure 22c, two people would get two-thirds of a pizza in one piece and the remaining person would get two pieces each of area one-third. I leave it to the reader to do the calculation to obtain this situation

Instead, I continued to move the circles (Figure 22d) and realized if each region had area one-half of the circle, it would be easy to split the two pizzas among four people in a more interesting way than cutting each pizza down the middle!

How far should two circles overlap, or how far apart should the two centers of congruent circles be, so that each of the regions shown has area one-half the area of the circle? To draw a decent diagram, I first had to ask myself where the centers of the circles would have to fall. Convince yourself that they must fall in the middle region. So the problem boils down to asking how far should the centers be apart

The pay-off for my doing the calculation was to realize it fitted neatly into my growing set of pizza problems.

Pizza problem four

> How can you divide two pizzas of the same size among four people - not by cutting each pizza in half with a straight line (or a fancy one like the lines in Figure 13), but by overlapping them and cutting around the overlap?

To do an easy related calculation, look at a different case - the one for which each circle goes through the center of the other It is easy because of the sixty-degree angles What is the area of each region? Again, a film sequence came to mind, one that shows the areas of the regions as the circles move over each other. What is the area of the overlapped region as a function of the distance between the centers?

At this point, I was sorting various papers I had collected over the years that dealt with the Pythagorean theorem and I came across one that I had xeroxed because of its title, but had not actually read yet (McNeill, 1999). Now, I noticed the circle diagrams on the second page and read it at once! Among other things, it discusses how to find the area of a circle whose area is equal to the difference in the areas of two given circles. McNeill expresses the drawing tangent methods I had used for the special case much more thoroughly and beautifully. He also discusses how to do it using the generalized Pythagorean theorem I had been concentrating on the sum of two areas equalling the larger area, but of course one can think of it in terms of difference - except that I did not until I read this article!

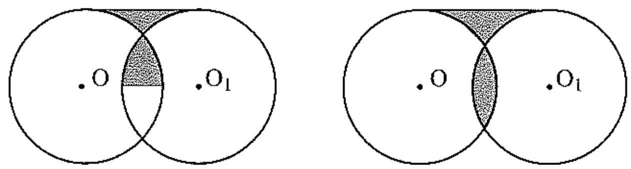
So, here is yet another pizza problem.

Pizza problem five

> Find the diameter of a pizza that is equal in area to the difference of two given pizzas.

There are two other articles I should mention I came across one (Penner, 1978) in another folder which is not totally unrelated, and it contained the following problem which it attributes to Krutetskii (1976, p. 309)

> Find the distance OO_1 if the radius of each circle $r = 1$ and the shaded areas are equal. (See Figure 23a)



a) diagram as given in the article

b) incorrect diagram I drew

Figure 23:

I am including it here because it was the only problem I discussed at the ATM conference that I did not make up myself and, not looking at my notes, drew incorrectly and thereby turned a nice aha easy problem into a nasty one on which people struggled because I had said there was a nice easy way to solve it!

Last, but not least, I had more or less finished this article when John Sharp, who was at my ATM talk, alerted me to an article I might be interested in, by e-mailing me this message:

> I found a paper in *Mathematics Magazine* from the early 1980s It was about a goat in a circular field which was tethered to the fence so that it could eat half the grass. What is the length of the rope? It gave a history of the problem back to 1890s in various forms

I replied that:

> I think I saw that problem about the goat. But it did not grab me

He replied:

> You have missed the point. It is EXACTLY your problem with a different story.

So I list it here (Fraser, 1982) because it is a delightfully written paper with many historical references.

Looking back

I started by looking at the old square problem. Though I had no idea then where it would lead me, I am surprised how far from the square problem it has taken me, even without asking related questions in 3-D

So, what methods did I use to create these problems? I had an open mind to free associate [7] and let previous problems suggest new ones by first asking, "How can I disguise it?" I asked many questions such as: is there more than one way? Can I move the shapes around? What-If-Not? Paper circles also helped.

I do not think that this pizza problem has yet been milked thoroughly, so I hope readers will continue work on it.

Notes

[1] I had the good fortune to have had a course with Professor Polya over forty years ago at a Stanford University NSF Summer Institute and I can still hear his voice saying, "*Look* at the problem"

[2] I last used this problem at the UK *Association of Teachers of Mathematics* (ATM) meeting in April 2001 and there several people used the paper semi-circles in ways different from mine and some used quarter-circles. I had planned to write up these methods here together with their authors' names, but since I do not have sufficiently accurate records of who did what I decided not to.

[3] This method can reinforce the fact that the Pythagorean theorem (and its converse) is a special case in which the three figures on the sides of the right-angled triangle are squares. It is only necessary that the three shapes on the three sides of a right-angled triangle be similar. I could have now started all over again and asked how one can tell quickly that a gingerbread cookie is twice the size of a smaller one that has the same shape. It was even longer before I realized I was back to a problem I first posed in the 1960s which my colleague and I used in class. See Brown and Walter (1990, p. 107 ff).

[4] For a full description of the What-If-Not? technique and others, see Brown and Walter (1990, 1993)

[5] 'Take half' was a short video made by London-based SMILE mathematics during the 1980s and was part of a collection of short computer programs called *The First 31*. For more information contact: info@smilemathematics.co.uk.

[6] I calculated the area of the wedge by considering the area of the triangle ABC plus the area of the sector AOC minus the area of the triangle AOC and then set this area equal to $\pi/2$, namely half the area of the circle. I did ask graduate student Kay Byler to check my solution to the equation I got:

$$\sin(\theta) + \theta\,\pi/180 - \pi/2 = 0$$

using her TX calculator. Though it took only about ten seconds, I would like to thank her!

[7] Free associating, which I probably do too much all the time and annoys some of my friends, can be very useful!

Acknowledgements

I am extremely grateful to David Wagner for his detailed and painstaking work (both graphic and mathematical) in creating the figures for this piece. For the motivation for writing this article, see the introduction to my earlier article 'Looking at a painting with a mathematical eye' (Walter, 2001)

References

Beeney, R. *et al.* (1982) *Geometric Images*, Derby, Derbyshire, Association of Teachers of Mathematics.

Brown, S. and Walter, M. (1990, 2nd edn) *The Art of Problem Posing*, Hillsdale, NJ, Lawrence Erlbaum.

Brown, S. and Walter, M. (eds) (1993) *Problem Posing: Reflections and Applications*, Hillsdale, NJ, Lawrence Erlbaum.

Fraser, M. (1982) 'A tale of two goats', *Mathematics Magazine* **55**(4), 221-227

Krutetskii, V. (1976) *The Psychology of Mathematical Abilities in Schoolchildren*, Chicago, IL, University of Chicago Press

McNeill, C. (1999) 'Playing with Pythagoras and geometric arithmetic', *The Mathematical Gazette* **498**, 446-452

Penner, S. (1978) 'Oldies but goldies', in Problems and Solutions, *New York State Mathematics Teachers' Journal* **28**(2), 78-79.

Walter, M. (2001) 'Looking at a painting with a mathematical eye', *For the Learning of Mathematics* **21**(2), 26-30