The "Human Factor" in Pure and in Applied Mathematics

Systems Everywhere: their Impact on Mathematics Education

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When thinking about the next 20, 30 years of mathematics education one can ask "What will happen?" or "What should happen?" Differentiating the second question one can, e.g., ask "What should happen from the point of view of society (Western, developing countries, . .)?" or "What should happen from the point of view of mathematics?", thus possibly paying tribute to one's own profession. To be candid, I will not refer to these different orientations systematically. As an optimist I think what should be will be, and as a lover of harmony I think that what is good for society is good for mathematics, and vice versa, at least in the long run. So the following considerations are a mixture of answers to very different questions.

Pure and applied mathematics

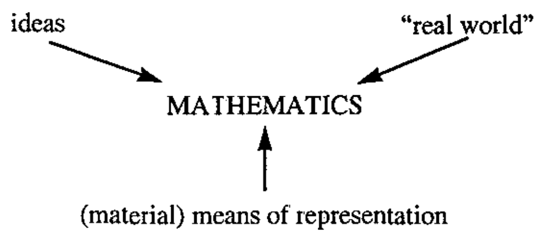
What are the key factors influencing mathematics and mathematics education? (Here is another differentiation which I will not consider in this paper.) The classical answer is that there are *theories* and *applications*, pure and applied mathematics, which correspond to two driving forces: inner coherence, beauty, simplicity, on the one hand and usefulness, problem solving capacity, flexibility on the other. During the last 30 years one has seen the considerable impact of these two factors on mathematics (education). On the one side we have had Bourbakism and the New Maths movement, on the other side we have had the immense diversification of applications, most striking in fields other than physics and technology, namely in biology, the social sciences and economics, in management and business administration, especially in fields dealing with decisions. The growth of statistics at least has had an influence on schools, but also elements of discrete mathematics (e.g. graphs), an important set of tools in these fields, can be found in the curricula.

The role of the means

Pure and applied mathematics are two different answers to the question "What is mathematics?": an eternal body of true statements, and a collection of models determined by some "reality". Their sources of development are ideas, invented or discovered by men, and "mathematical" structures with relations to reality, respectively. (Maybe these sources are interrelated.) But the last 30 years have also shown that the classical answers to the question of the essence and driving forces of mathematics do not tell the whole story. The *computer* has vigorously pointed to the fact that there is another source of development, one that is neither a human idea nor a "real world" situation to be explained and handled, but a *(material) means* of representation and operation. There is no doubt today about the enormous impact of computers on mathematics, in its applications as well as in its theory. In order to evaluate this impact and to find its characteristic features one has to look back in history and to become aware of the fact that mathematics was always dependent upon material means, e.g. heaps of stones for supporting the counting process, calculation boards (abacuses) in various countries, paper and pencil, these being the dominant means used by mathematics in our culture until computers arose. It is evident that important problems of mathematics have become obsolete — e.g. the solving of algebraic equations by radicals — through the influence of the computer, and new ones have been generated — e.g. the theory of complexity of algorithms.

Reductionism and objectivation

Summing up the key factors of development in mathematics considered so far we obtain the following picture:



Questions arise like "Are there further factors?", or "What is the most dominant factor?" etc. I do not consider them directly, but first I point to the fact that each of the mentioned factors can give rise to *reductionism*. There are people who view mathematics exclusively as a system of ideas, discovered or invented by some very intelligent men and women. Others conceive mathematics to be only a set of algorithms, models, and concepts to be applied in solving real world problems. And for some people today mathematicians are simply those guys having to do with computers.

Another characteristic of the factors is that each of them tries to *"objectify"* mathematics, i.e. to make it *independent of people*. Firstly, ideas, at least if they are settled in a Platonic world, have nothing to do with us; they were there before mankind and will remain when our last days are over. Furthermore we are only able to perceive the shadows of those ideas, not the ideas themselves. Secondly, the real world exists independently of our will and our actions and so do the mathematical laws governing it. They are objectively true, independent of our interests, wishes, thinking, and even of our existence. With means the matter is not that simple, since obviously computers are constructed by people. Nevertheless a computer is constructed on the basis of mathematics rooted in the above "objective" factors, and once such a machine has been constructed it exists and functions objectively, independent of us; thus it is a reification of mathematics on a "meta"-level. The same is principally true for the more primitive means of representation.

The "human factor"

What is clearly lost in the process of objectivation, even more if it is accompanied by reductionism, is what I want to call the *human factor*, the *relationship between people and mathematics*. Ideas are not related to the mathematicians who created them, applications are viewed independently of the interests of people, means are often seen without regard to the purposes standing behind and the interests of the people handling them. It is the central thesis of this paper that the human factor, understood as the influence of interests, properties, competences of people, individually and collectively, *will and should be considered more carefully* in the future. It "should be" because this orientation would yield the following effects:

- the unconscious and disorganized working of the human factor in the past would be replaced by a more conscious and prudent handling of our relations with mathematics, thus improving the efficiency of mathematics for social life;
- mathematics itself, this enormous set of concepts, theories, and procedures would become more understandable and transparent; the interrelation between pure and applied mathematics and the material means — the above mentioned factors — would become more clear; new orientations for teaching and research would be developed.

But what encourages me to think that what should be will be? There are some indications which I consider in the following, before looking at some impacts of these relatively abstract considerations on mathematics (education). I present three sets of arguments for an increasing role of the human factor:

- a growing interest in philosophy, history and sociology of science;
- new paradigms in didactical and psychological research;
- a growing awareness of the human factor in technology, management and design

Reflections about mathematics

Reflecting about mathematics is becoming a more and more acceptable kind of scientific work among mathematicians. There is an increasing number of books and articles dealing with *history, philosophy and didactics of mathematics,* sometimes in a very personal (e.g. (auto-)biographical) way, and "normal" university mathematicians are engaged in these fields. Some of the books are: Davis/Hersh [1981], Monna]1986], Steen [1978], Halmos [1985], Ulam [1986], Albers/Alexanderson [1985], Campbell/Higgins [1984], Reid [1986] The most prominent journal in the field of reflection about mathematics is "The Mathematical Intelligencer". The reflections in these books and articles are, to a large extent, deliberations about the human factor in mathematics.

This phenomenon of increasing interest in reflection can be viewed against the background of discussion in the *sociology of science*, starting with T.S Kuhn and P. Feyerabend, stressing the influence of the social situation of the (in-) group of the scientific community on the contents and methods of science [cf Lakatos/Musgrave 1974] Work in this field relating to mathematics has been done by Lakatos [1979], MacKenzie [1981] and Mehrtens/Bos/Schneider [1981]. This discussion can also be embedded in the *sociology and history of knowledge*, investigating the relationship between society as a whole and the development of a body of knowledge. Classics in this field with respect to mathematics are the treatise of O. Spengler [1973] on the role of mathematics in ancient Greece and in modern times, and the comparison between "Mathematics in China and the West" at the change from medieval to modern times by J. Needham [1956] These works have been written decades ago, but there is an increasing interest in studying connections between mathematics, science and culture [cf Restivo, 1983]. Not should we forget ethnomathematical studies like the investigations into the geometry of Indians and the mathematics of Papua New Guinea [cf Pinxten *et al*, 1983, and Bishop, 1979].

There is a general philosophical context for all these developments: *constructivism,* especially the belief that knowledge is primarily a social construction by people [cf Watzlawick, 1981] This view has received support from disciplines such as psychology, biology, cybernetics, systems theory, and is being "applied" in fields like psychotherapy and business administration [cf Trappi, 1986; Ulrich/Probst, 1984; Selvini-Palazzoli *et al*, 1981.

Research about individual and social cognition

In recent years new research paradigms in *didactics and the psychology of mathematics* have arisen. These take seriously the individual learner, his/her construction of knowledge, his/her strategies of problem solving, and even his/her strategies of making "errors". The methodology of this research — e.g. in-depth interviews, and observations of "thinking aloud" problem solvers — as well as the interpretation of results give equal weight to the "objective" knowledge being learned and the learners's construction [cf Davis, 1984; Kaput, 1985; and many other articles in the Journal of (Children's) Mathematical Behavior]. In this view errors are not deviations from the correct path of objective truth but simply differences between the subjective and official constructions. This view not only respects the individual but enables the researcher to focus on the *relation* between the individual and knowledge, and sometimes to learn about the hidden assumptions embedded in the latter. So-called stochastic misconceptions for example, are often due to different model-assumptions, and sometimes the "common sense" solution of a problem is more appropriate than the "mathematical" solution [cf Bentz/Borovcnik, 1985].

Learning is not only an individual but also a social enterprise. In recent years some studies have been conducted on the *interaction and communication processes* in learning groups (with or without a teacher), which adopt the principle of giving equal attention to the "human" situation and to the (mathematical) content [cf Bauersfeld, 1980; Voigt, 1984]. The emergence of meaning is a phenomenon influenced by the knowledge presented, by the individual's pre-constructions, *and* by special social interactions, communication routines, the history of the group, etc. A dominant function of such construction of meaning is its role in giving security and stability to the group. This reflects a function of science for society in general.

Technology, management and design

There is a growing awareness that developing, implementing, and using new technologies is essentially a social movement, carried along by social systems and acting back on them. Machines like automobiles and computers are not simply means to be used when we want to, but artifacts which could only be produced and implemented because of a certain readiness in society — more than just the existence of specific knowledge — and which by themselves change society to a high degree. The "interface" between people and computers is not the keyboard and the screen, it lies within us. We are, partially, machines, and machines are, partially, human beings [cf Bamme *et al*, 1983]. This means that the relationship of society to technology is much more complicated and interwoven that was assumed 30 years ago, and we have to cope with this fact. The "user-friendliness" of machines is only a minor aspect of this matter.

The realization that human beings cannot be separated from their constructions — this is another formulation of the main thesis of this paper — has led to new developments in management and business administration. the constructions here can be technologies as mentioned before, but also social systems like a company or the market-system itself. Classical methods of management use mathematical instruments to treat these construction like objects which obey certain rules. Often they are not able to cope with the dynamics and the reflexivity of these constructions, the latter reminding us that managers are parts of the constructions. New paradigms have been developed, like "evolutionary management", which give a place to the strong interrelationship of human factors and systems [cf Malik/Probst, 1984]

Technology and management are brought together. The relevance for mathematics becomes obvious when a computer scientist formulates "a new foundation for design", as T. Winograd does, writing with the management expert F. Flores [Winograd/Flores 1986]. From the preface to their book I take:

> This is a book about the design of computer technology... it is both theoretical and practical; it is concerned with computer technology and with the nature of human existence; with the philosophy of language and with office automation. [Winograd/Flores, 1986, p. xi]

The emphasis on the "human factor" here is obvious. The book starts with a theory of cognition rooted in the philosophy of H.G. Gadamer and M. Heidegger, and in the epistemological ideas of the neurobiologist H.R. Maturana. On the one hand it calls on the humanistic tradition of hermeneutics and phenomenology, stressing the relation of the individual to the (social) context; on the other hand it provides an explanation how systems can be structurally determined but nevertheless not predictable even in generating specific behaviors. This leads to "the rejection of cognition as the manipulation of an objective world" and "to the conclusion that we create our world through language, an observation that has important consequences for design" [Winograd/Flores, p. 11].

Again the aspect of constructivism brings the human factor in. And what is true for the development of computers and their programs must be equally true for mathematics since computers are no more than materializations of what in principle mathematicians always did. The difference is that by materializing formal thinking and acting in computers, and thereby exposing it to the usual economic processes (production, selling, buying), this kind of activity has become much more widely distributed and influential.

Possible impacts

What does the increasing role of the human factor mean for mathematics (education) in concrete terms? In the following I want to give some hints, but it is clear that it is not possible to draw conclusions with compelling force from such a general idea as "the increasing role of the human factor." Nevertheless I will try to say what an emphasis on the human factor could mean for the two classical factors of development: for pure and for applied mathematics.

Open mathematics

I start with the latter by expressing my conviction that a *new kind of applied mathematics is already arising* and that it would be worthwhile to foster this process. the main difference between the old and the new applied mathematics lies in the fact that the old one is based on (and bound to) some objective reality and its (mathematical) structure, whereas the new one is much more aware of the fact that the model-builder plays a genuine role. It is aware that constructing a mathematical model always means interpreting reality on the basis of interests, purposes, paradigms, convictions, etc., and that applying mathematical concepts does not always aim at finding some "absolute" "objective" solutions to a problem, but rather to give a good *description* of the problem in order to enable and foster communication about it. I like to call this kind of orientation "open mathematics" since it does not claim to develop models which give a full, closed, description of some situation but uses mathematical modes of representation to explain the views of people about a situation, or merely to give some structure to it. This has consequences for the kind of mathematical concepts that are used as well as for the kind of use itself. I will try to explain through some examples.

Sociometry

It is not accidental that "open mathematics" in the above is represented more in *the social sciences* than in other fields of applications. One very simple, but nevertheless worthwhile, tool to describe the situation of small groups is the *sociogram*. This is a directed graph the vertices of which stand for the members of the group and an arrow leads from vertex A to vertex B if, for example, A is willing to cooperate with B. When drawing such a graph different arrangements are possible, thus pointing to different aspects of the structure. Various indices can be calculated such as the

$$\text{expansiveness} = \frac{\text{number of arrows}}{N}$$

(N = number of vertices) or the

$$\text{interaction} = \frac{\text{number of arrows}}{N(N-1)}$$

or the

$$\text{index of isolation} = \frac{\text{number of isolated vertices}}{N}$$

etc. [Nehnevajsa, 1973; Proctor/Loomis, 1951]. An interesting question is how to identify "subgroups". Several definitions have been developed which have various pros and cons — see Seidman [1985].

It is obvious that sociograms don't give a unique, precise, picture of the group situation: there are different arrangements for drawing the graph, different indices for the same aspect, maybe different definitions and degrees of willingness to cooperate among the group members, etc. Therefore this kind of "mathematization" is sometimes rejected. But certainly the very process of discussing different ways of representation can contribute to an illumination of the situation of the group, especially if this discussion takes place among the group members themselves! Another argument which is also sometimes used against this method is the fact that group situations can change very quickly — perhaps just after they have become obvious by drawing a sociogram. But, as I see it, this is exactly what the model is being constructed for! The structure is described in order to be overcome, that means, in order for it not to be true any longer provided there is a motive for change in the group. In other words, the sociogram is a tool for fostering group dynamics.

It is interesting that for the founder of sociometry, J.L. Moreno, the main purpose of the precise analysis of the structure of groups was its potentiality for causing "revolutions", first in small groups and later in whole societies. It is "the practical principle of microsociology to foster small, but sound, sociometric revolutions which express the hope for a transformation to a society that is worth living." [Moreno, 1974, p. xiv, translation by R.F.).

Economic inequality

A second area where the tendency towards "open mathematics" can be seen is *mathematical economics*. One of the important problems in economics is to estimate economic inequality, e.g. in a nation. Several measures for inequality have been developed by economists. To give an idea I shall present two of them. For the first let $y_1, y_2, \ldots, y_n$ be the incomes of the persons in a community and μ the arithmetic mean of these incomes. Then

$$C = \frac{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(y_i-\mu)^2}}{\mu}$$

is called the "πcoefficient of variation" of this income distribution [see Sen, 1973, p. 27]. For the second measure we need a function which associates to each income the utility to be derived from this income. Such a function U ("utility function") is generally assumed to be increasing and concave so that with increasing income less and less utility can be drawn from one unit of money. Now the following measure of inequality can be established:

$$D = 1 - \frac{\sum_{i=1}^{n} U(y_i)}{n \cdot U(\mu)}$$

This is called Dalton's measure [cf Sen, 1973, p. 37]. If U is *strongly* concave then $D = 0$ if and only if all y_i are equal ($i = 1,2,\ldots,n$). For C the corresponding property is clear immediately. For both measures it is feasible: the larger the inequality, the larger the corresponding measure.

But the measures are different. Ordering nations according to the "amount" of their inequality of income distribution can yield different orders of succession for different measures. Furthermore the measures have different properties which may be considered more or less desirable. For example it is interesting to note how a measure changes when an income transfer is made from the rich to the poor in a population of middle or of high income. (Of course, in either case the measure should become smaller.) Should an inequality measure take into consideration the absolute income level; should it remain invariant when all incomes are changed proportionally or additively? A basic difference is seen between "descriptive" measures, such as the coefficient of variation, and "normative" measures, such as Dalton's, where a certain "norm" is introduced by the utility function. The economist A. Sen pleads for a simultaneous use of different measures and argues that the individual measures should not be interpreted too strongly. He writes:

> I have tried to argue in favour of weakening the inequality measures in more than one sense. First of all, the mixture of partly descriptive and partly normative considerations weakens the purity of an inequality index. A purely descriptive measure lacks motivation, while a purely normative mea-

sure seems to miss important features of the concept of inequality ... Second, even as normative indicators, the inequality measures are best viewed as "non-compulsive" judgements recommending something but not with absolutely compelling force. This has implications in terms of the treatment of inequality rankings as prima facie arguments and permitting situation-specific considerations to be brought into the evaluation if such supplementation is needed [Sen, 1973, S. 75].

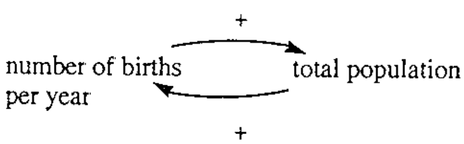
System dynamics

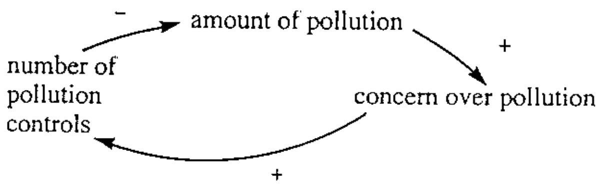
The third example comes from *applied systems analysis*, the attempt to mathematically describe large systems with the aid of computers. In recent years a fundamental change in the modelling of such systems has taken place. Instead of the use of classical mathematical methods — taken e.g from operations research — which provided "solutions", now so-called "interactive decision-support-systems" have been primarily developed. The task of these systems is not to predict the future but rather to accomplish the following functions:

- research
- communication
- legitimation
- education [Meadows, 1986, p. 83]

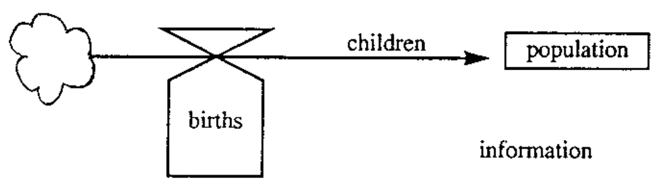
Only the first of these functions is mainly related to the "matter", the exploration of the situation under consideration. The other three tasks explicitly have to do with the "human factor". ("Legitimation" here means the validation of the status of the constructor of the model with respect to the group concerned with the problem)

There is an elementary method for analyzing systems, due to J. Forrester, which I sketch briefly because I would like to suggest it be considered for school curricula [cf Roberts *et al*, 1983]. In applying the method one first tries to find "causal loops", i e positive or negative influences of one magnitude on another. For example:





The second step is to draw a flow diagram, distinguishing between "levels" and "rates":



Here "population" is a level and "births" is a rate. Now a system of difference equations can be established:

$$\text{POP}(t) = \text{POP}(t - \Delta t) + \Delta t * \text{BIRTHS}$$

$$\text{BIRTHS}(t, t + \Delta t) = \text{GF} * \text{POP}(t)$$

(GF = "growth" factor)

This system of equations (usually much bigger than in this example) is the basis of a computer simulation.

As before one can make the objection that this method is inaccurate: it depends on too many assumptions about the influences, the model will not give a true picture of reality, etc. But its advantage is its *flexibility*. It gives the possibility of modeling even very complex systems without a coherent theory about the total system, only about the connections between some parts of the magnitudes involved. The simulation can be conducted and compared with experience. If new information is gained, if new aspects or influences arise, or if assumptions about the connection between two variables are altered, the model can easily be modified.

Material and visual means

Coming back to the general idea of "open mathematics" as a way of stressing the "human factor" in applied mathematics I will finally describe *two features* of this orientation. The first is the view that mathematics offers means of representation which are (literally) *material and visual*. I have already mentioned the (hi)story of calculation stones, abacuses, written symbols associated with paper and pencil techniques, and finally the computer. What I want to stress is the *visual-representational character* of these means. A revaluation of this aspect is taking place. There exists a "Visual Mathematics Library" [Abraham/Shaw, 1984], conferences are held about "the geometry of diagrams" and B. Fusaro speaks about "diagrammatic mathematics" [cf Albers/Rodi/Watkins, 1985, p. 289]. The computer can be viewed as a "new representational window" [Kaput, 1986] or as a "dynamic blackboard" [Hawley, 1986]. It is important to note that even the symbolic representation is visual! A mathematician writes:

> Before finishing let me stress again that "user friendly" outputs like colour graphics, movies, are likely to be more important in experimental mathematics than rows and rows of numbers. Also symbolic calculation and formula manipulation is likely to grow in relative importance, again because symbolic formulae are better suited to human pattern recognizing abilities than numbers. [Hazewinkel, 1984, S. 29]

Mathematicians know how important a good notation is, how important the interplay between different modes of representation can be. Nevertheless this competence is not developed systematically today. I conjecture that the competence to check alternative modes of representation with respect to their appropriateness and to invent new ones will become more important than the traditional competence of skillfully handling single modes of representation because the latter will be entirely taken over by machines.

One advantage of a view of mathematics as a set of material means is the establishing of a *distance* between structures in the "real" world and their mathematical models, putting a greater emphasis on the *constructive role of people*. The influence of purposes and interests on these means can then be discussed much more easily.

"Local" tools

The second feature of "open mathematics" I want to discuss is the *flexibility and theoretical poverty of the tools*. They need no grand theories about "reality", as seems to be the case with mathematical physics; they are adaptable to different, sometimes changing, structures. The relation between mathematical models and reality is determined by data and by theory. Conventional applied mathematics tries to obtain "strong" conclusions from few data and needs strong theories and a stable image of reality. Open mathematics fits better with situations having an abundance of data and a lack of theoretical assumptions.

This is reflected in the development of some branches of *statistics* over the last twenty years. Inferential statistics usually needs a lot of theoretical assumptions, for example about the gathering of the data (e g. stochastic independence). Descriptive statistics, especially in the growing field of exploratory data analysis [Tukey, 1977], is applicable to raw data of very different origins. It can be used to visualize sets of data in various ways, to explore their structures, to obtain hypotheses — which can then be tested by inferential statistics — to convince others about some views on the data, etc.

Another field which provides flexible tools in the spirit of "open mathematics" is *discrete mathematics*. Discrete methods are of increasing importance because of the influence of the computer [cf Ralston, 1985], but what is of special relevance in the present context is that discrete mathematics offers very broadly applicable tools, such as graphs, matrices, or difference equations.

> To give an example: the concept of the "chromatic number" of a graph can be useful when dealing with the following problems: a time table for making meetings of committees, scheduling examinations, providing TV stations with transmitting frequencies, arranging trips for garbage trucks, making a service plan for a truck pool, etc [cf Roberts, 1986].

Sometimes the lack of theory in discrete mathematics is used as an argument against it, especially when discussing the question whether it will replace other fields of mathematics in the curriculum. For example S. Mac Lane writes:

> Calculus is a coherent and deep intellectual discipline with manifold roots in science. Discrete mathematics [...] is a grab bag of all sorts of things, from graph theory to number theory — some deep, some disconnected, some dismal. [Mac Lane, 1985]

As I have pointed out above it is exactly this "grab-bag character" which I find makes discrete mathematics useful for open mathematics. Too coherent and strong a theory in general reduces the applicability of the tools.

> I want to illustrate this last aspect by an example from my own learning history. I first learnt matrices as modes of representation for linear mappings. Matrices and linear mappings were tightly connected, almost synonymous. Operations with matrices, such as addition, multiplication, transposition or determinants arose in direct connection with linear mappings. All was rather clear to me. The first problem I had was when using matrices for describing quadrics. Where is the linear mapping? I asked. The problem was set forth when matrices were used to describe Markoff processes or the adjacency relations of a graph. I always asked myself: where is the linear mapping? It took much time until I realized that only some attributes of matrices are used in the respective context, "not all" of them. And "not all" of the aspects of matrices are reflected in linear mappings — at least not "naturally" in the sense of fostering the process of cognition. The separation of matrices from their "semantic" background (= linear mappings) as proper modes of representation, as well as from a comprehensive theory of matrices, renders freedom for new "local", "partial", applications.

Pure mathematics — a beautiful social construction

What about the relevance of the human factor for *pure mathematics*? Let me first say that I view pure mathematics as a collective human (re-)construction, highly independent of the individuals involved, but not of their society as a whole. I want to illustrate this view by two quotations. The first is taken from a lecture, "Mathematik — Kunst und Wissenschaft" (Mathematics: Art and Science), by A Borel:

> . . I will agree with the thesis that we tend to endow all that belongs to a culture, in the sense that we share it with other people and communicate about it, with existence. Something becomes objective (as opposed to "subjective"), as soon as we are convinced that it exists in the minds of other people in just the same way as in ours, and that we can jointly think about it and discuss it. Since mathematical language is so precise, it is ideally appropriate for the definition of concepts about which such a consensus exists. [Borel, 1982, p 27; translation by R.F.]

The second quotation is from a paper by the philosopher P Heintel. It puts mathematics into the context of a process of development of mankind

> Mathematics is the human being's highest achievement of abstraction from his inner (sensual) and outer determination by Nature: no prearrangement may claim validity, all is resolved in its being-for-itself quality. The only validity lies in the arrangement which the intellect gives to itself and which is jointly confirmed. Senses and activities are removed from their interweaving with environment, life, and everyday banality. In "looking inside", man finds himself in the empire of thoughts and arrangements that are determinable by himself; the manifold illusions and allurements of reality are left outside. He is not concerned with reality, faced with this world of ideas; it becomes an outside, an object, without any right of determination. Hierarchies, priorities, norms lose their influence. Between Nature

> and themselves people have established an *intermediate world of mind,* which has definitively rejected the claims of the one side, and which has liberated people equally definitively on the other side to choose their arbitrary option. (Heintel 1979, p 34; translation by R F.)

It is not the arbitrariness of the individual, but that of all who have collectively constructed that body of knowledge we call mathematics.

> Through mathematics the absolute revolution of mankind against Nature has been prepared; one is no more interested in its allusions, one wants to be independent and to stay with oneself. (Heintel 1979, p. 35; translation by R F)

A collectively constructed "intermediate world of mind ... to stay with oneself" — this is the picture of pure mathematics sketched by these quotations. Two motifs in the construction of such a world have been mentioned: revolution against the dependency on Nature and the wish to have something in common. I add a third motif: *aesthetics*. It is, I think, the most efficient one for the individual mathematician. Simultaneously it is a connection between the individual and the (scientific) community.

In what does the *beauty* of mathematics consist? I think it has to do with the fact that sometimes a large system of concepts, statements and procedures can be reduced to some principles, not only in the sense that they can be logically derived from the principles, but also that the principles establish connections that let us understand the whole architecture. Often a certain "closedness" corresponds to beauty. Then the principles are, metaphorically speaking, the borderstones of the field from which we can (re-)construct the whole system.

> One of the areas of mathematics usually considered beautiful is the *theory of functions of a complex variable*. Here the fundamental theorem of algebra expresses closure. Another beautiful result is Cauchy's integral formula. It allows one to determine the value of a (holomorphic) function within an open, connected subset of *C*, given the values on the boundary. It mirrors the general idea of exploitation of the inner structure of a system, given the situation at the borders. Another example of beauty is the system of different interpretations of the normal distribution. Firstly we have the classical result by DeMoivre und Laplace, which shows the normal distribution as the limit of the distributions of sums of independent random variables, secondly there is the fact of "invariance" of this distribution with respect to convolution. Another very nice interpretation is due to Maxwell. Considering the dispersion of projectiles on a target, decomposing the deviations into two orthogonal directions with (stochastically) independent deviations, one gets the functional equation
>
> $$f(\sqrt{x^2+y^2}) = f(x) \cdot f(y)$$
>
> for the diversity function of the distribution of the linear deviations (using Pythagoras' theorem). The solution is the density function of the normal distribution (cf Schiffer/Bowden 1984, p 34).

When emphasizing the role of mathematics as a collective construction I do not reject ontologies of mathematics. My view is, e.g., compatible with Platonism; that means this construction can also be viewed as a re-construction. This is in analogy with modern theories of learning: students construct knowledge, though, of course, it existed before the students existed. What I am interested in is the question of the role of mathematics for people; why do we choose exactly this platonic idea in a given (e.g. historical) situation? One could say it is not the essence of (pure) mathematics I am dealing with, but its functional aspect. From another point of view one can of course say this functional aspect is the essence of mathematics, if we apply the mathematical way of treating ontology onto mathematics itself. But I am not interested in this differentiation now.

Emphasizing the human factor in pure mathematics

Let's put aside these considerations about the essence and the beauty of pure mathematics and come back to the question of the relevance of an increasing role for the human factor. In the described understanding of pure mathematics the human factor is already very decisive. What can be new in future developments?

First, the above sketched perception of pure mathematics is not common sense. It is much more widely believed that the roots of pure mathematics lie exclusively outside people, in eternal laws, Platonic ideas, etc. Usually it is not accepted that mathematics is, like the arts, also governed by the collective will. For the individual, of course, there is not much freedom, but the different developments of mathematics in different cultures indicate that there is an impress from the people's ways of thinking, acting and living onto mathematics. And if this were not the case, we could make it so — says my fantasy of collective omnipotence. So the first option with respect to stressing the human factor in pure mathematics is to *become aware of its role of collective construction,* to appreciate mathematics as the basis of a very universal understanding, which has more to do with ourselves than with things outside, at least compared with the *status quo*. We, as teachers, should stick honestly to this fabulous world, not selling it through the false arguments of necessity (of eternal laws) or of applicability, but with those of cultural heritage. The fairy tales of mathematics are a way — alongside other cultural achievements like literature or the arts — of helping to establish more coherence, more common understanding among people, maybe at the expense of some connections with so-called reality. I state this despite criticisms coming from those loving (only) applied mathematics, from the humanists, who would refuse to admit the described potentiality of mathematics, and, thirdly, from pure mathematicians, who, I think, don't like their theorems being designated as fairy tales.

"*The essence of mathematics lies in its freedom*", G. Cantor said. Mathematics could therefore serve as a basis for collective fantasy, as belletristic literature does. One can use it to create plans, speculations, etc. A difference from literature is the fact that pure mathematical fantasy is not bound to a specific context. It is just the value of mathematics that it enables us, within the domain of pure concepts and their symbolic representations, to think about their relations, necessities — and their liberties and alternatives. The latter are sometimes not seen when the concepts are interwoven too tightly with some "reality". But independence from context does not mean that there is no context, but rather the seeing of many possible contexts. This is another element of the fantasy in pure mathematics.

> To illustrate this idea one can think of the process of "exactifying" mathematical concepts. Sometimes it is believed that this process is governed by the formal logic of the linguistic expressions. On the contrary, one has to have an idea of alternative interpretations of the formulations outside the context in which the concept arose. So exactifying has much to do with fantasy.

Mathematics as a source of freedom and fantasy has not yet been made accessible to our educational system, not to mention in our social life. Mathematics as a system of rigorous rules, possibly a source of interesting puzzles, dominates. Hopefully mathematics will develop like writing which was also originally used to fix laws but has become a general method of communication in art, science, business, and everyday life.

An aspect of pure mathematics as a basis for collective fantasy which is related to the independence of context is the concentration on *relations* as opposed to elements, fostering thinking in terms of relations instead of thinking in terms of objects (substances). Combined with the idea of large, complex systems, this way of thinking seems to be of great importance in our present day ecological and social life. It is not the elements of the respective systems, it is their functional relations which constitute their identity and which are responsible for any positive or negative effects.

With these last considerations I have turned back slightly to applied mathematics, at least as an argument for pure mathematics. This is not by accident, it corresponds to a "dialectical" element as expressed in the following quotation:

> One can approach a subject matter in different ways, through specific experience and concrete competence or through high abstractness and "metaphysical unconcern (carelessness)". [Otte 1987, p. 45; translation by R.F.]

Let me recall the line of arguments about the human factor in pure mathematics:

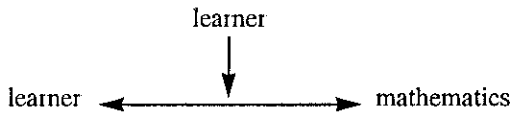
- pure mathematics is a collectively constructed domain of universal understanding, thereby potentially strengthening the coherence of people;
- it can serve as a base for collective fantasy, independent of contexts and therefore open to different ones, stressing relations and large systems.

I do not offer concrete examples as I have done in the case of "open mathematics". One reason is that it is still more important not to lose contact with the spirit of the general idea, which is to emphasize the role of the human factor. But there is still another point, which I want to explain in the next, final, section.

Reflexive meta-learning

Stressing the role of the human factor in mathematics means focussing on the relationship person-mathematics. It means asking the question: "What does mathematics mean for society? for you? for me?" The problem is that there are *no honest general answers* to this question. The above considerations are, in part, hints to answers to that question, but not solutions. It would not be in the spirit of this paper to propose that knowledge about the relationship people-mathematics should be developed — e.g. by the philosophy, history, psychology of mathematics — and then that this knowledge should be taught to students. Stressing the human factor means a re-organization of the educational system on the large and on the small scale.

The central element of this re-organization must be what I call the *reflexivity* of the learning process. Learning takes place on at least two levels: firstly, we gain some knowledge, competence, etc. We thereby establish some relationship between ourselves and the body of knowledge we want to learn. Now, on a second level, we can focus precisely on this relationship, e.g. by observing our learning process or discussing the meaning of the knowledge for us. The following picture visualizes this situation:



In a different terminology, the subject matter of the second level is *meta-knowledge* with respect to the knowledge of the first level. Of course there are different levels of the object-meta-structure *within* each body of knowledge, but the "highest" meta-level is always that which concentrates on the relationship between the subject matter and oneself.

Starting from the keyword "meta-knowledge" I want to present two quotations as arguments for stressing the human factor in connection with mathematical competence. The first is from an article by A. Schoenfeld about mathematical problem solving:

> 1. Metacognitive skills and a "mathematical epistemology" are essential components of competent mathematical performance.
> 2. Most students do not develop very many metacognitive skills or a mathematical epistemology to any degree, largely because mathematical instruction focuses almost exclusively on mastery of facts and procedures rather than "understanding"; these are basic causes of students' mathematical difficulties.
> 3. It is possible, although difficult, to develop such skills in students. [Schoenfeld 1985 b, p. 361]

The second quotation uses another keyword: "style" (in doing mathematics). This word also stands for what I would call the relationship between people and mathematics and is used here by R.W. Hamming, who has worked more than 30 years as an applied mathematician at different research institutes in the USA:

> We now turn to some of my opinions. First among them is: If you believe that there is a difference between training and education, and if you believe that training is what can be told in words, then education is what is not teachable by words alone. In short, education is related to the word "style".

When it is a matter of education we can say how we would do something, we can criticize how the student does it, but we cannot give explicit detailed rules — we can when it is training Since most of us like to believe that we do more than mere training we are forced to face this matter of style. .

At Los Alamos I met first class scientists, and I could not avoid noticing the difference between them and me Fermi, Teller, Bethe, Oppenheimer, Feynmann, and others were obviously doing important things and I was merely a stooge for them to be used to carry out their ideas I became envious, to say the least, and began to study the differences At Bell Telephone Laboratories I had an excellent opportunity to continue the study of greatness. I also started to study great men in the past, what they did, how they did it, and why they did it. It has been a 40 year study. The difference is mainly style! ...

Along the way I also came to believe that the amount of mathematics now known, as well as the amount now applicable, are both so great that we simply cannot prepare the students for the future by trying to tell them all the past applications. We must struggle to get back to fundamentals and hope that the students will be able to develop the applications that they will need to create the future (which necessarily we cannot know or even conjecture accurately about) Hence teaching the details, except as illustrations of the underlying methods, is futile. It is not a decent preparation for the future The methods of mathematics are more important than the results [Hamming 1986, p A 37, A 38 und A 40].

As I have already stated above, we cannot learn about the relationship humans-mathematics, i.e. about style, in such an objective way as we can about "objectivated" knowledge — unless we want to lose exactly the human factor That means the necessity of an *alternative style of teaching* A style, which, corresponding to meta-knowledge, allows for *meta-communication*, that means communication not only about the subject matter, but also the situation of learning and its relation with the subject matter. For this kind of communication the teacher cannot be the expert who knows the truth So the organization must involve elements of "social learning" (such as group discussion).

The question of the human factor cannot be solved by experts, it can only be treated in a cooperative "democratic" way. It must be possible to learn from the learners. One consequence should be that people learn more about themselves: by looking in the mirror called mathematics

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