Communications

Juxtapositions

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The other day I had one of those experiences that reminds me why I like FLM so much. I was travelling, and had brought with me photocopies of the FLM articles I needed for my work. To save weight, I had made my copies on two sides of the paper and my photocopier is quicker when I copy two facing pages (even-odd) onto the two sides of a sheet. This meant that when I was reading the conclusion of Fischbein's wonderful *Intuition and proof* in FLM **3**(2), I turned the page to find the beginning of Buerk's equally wonderful *An experience with some able women who avoid mathematics*.

The paragraph I was reading discusses

> the need for a complementary intuitive acceptance of the absolute predictive capacity of a statement which has been formally proved

and the final, interrupted, line is

> It may be supposed that the child – who admits the correctness of [...].

From this vision of absolute correctness, I turned to read:

> On the eighth day, God created mathematics. He took stainless steel, and he rolled it out thin, and he made it a fence forty cubits high and infinite cubits long.

I know the current editor takes great care with the order of articles in each issue, and so it was not hard for me to imagine David Wheeler, faced with Fischbein's article a third of a page too long, deciding that interrupting him at that moment, to present the viewpoint of the students caught on the other side of that absolute correctness, would be a good idea. And it is.

This is an experience I would never have had with an electronic version of FLM (though it would be easier to carry). It makes me cherish the arrival of the envelope. I wonder: How has the physicality of the journal affected the lives of other readers?

Thoughts on learning advanced mathematics

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The research in the field of the teaching and learning of undergraduate mathematics seems to have been prompted by a widely held belief that students often fail to develop the understandings of advanced topics that faculty members desire. Several authors have postulated competing ideas to explain students' persistent failures:

- blaming poor teaching for students' failure – Leron and Dubinsky, 1995; National Science Foundation (NSF), 1992; Mathematical Sciences Education Board (MSEB), 1991
- students lack of effort – Wu, 1999.

Yet, all these authors also believe that advanced mathematics courses, and especially abstract algebra, are *fundamentally different* to students' previous mathematics courses. Dubinsky *et al.*'s (1994) formulation may be the most succinct:

> abstract algebra is the first course for students in which they must go beyond learning "imitative behavior patterns" for mimicing [*sic*] the solution of a large number of variations on a small number of themes. (p. 268)

In an abstract algebra course, students should be constructing an understanding of abstract mathematical ideas based upon a variety of stated definitions and then exploring the new constructions. In short, the abstract algebra course should require a new type of mathematical activity for students. Moreover, in his unpublished doctoral dissertation, Findell [1] argued that the abstract algebra course is the

> place where students might extract common features from the many mathematical systems that they have used in previous mathematics courses. (p. 12)

Ideally, students would be dedicated to their studies and would attempt to develop their mathematical understanding whether or not they were required to do so. Similarly, in an ideal situation, abstract algebra teachers would be making use of materials that require students to develop their understanding of the mathematical concepts. Instead, it is an imperfect world that we must navigate, with students who often fail to study, curricula that are not ideal, and teachers who, in an effort to make the course accessible to all students, provide a series of hints or scaffolding that may account for much of the mathematics in the original problems. I will argue that, in this real world, our courses do not require that students come to an understanding of the mathematical concepts. They can pass courses via mimicry and symbol manipulation.

Thurston (1994) gave a summarization that seems to allude to the same ideas – less original thinking is required of students in mathematics courses than is ideal:

> we go through the motions of saying for the record what we think the students "ought" to learn [...] Books compensate by giving samples of how to solve every type of homework problem. Professors compensate by giving homework and tests that are much easier than the material "covered" in the course, and then grading the homework and tests on a scale that requires little understanding. (pp. 165-6)

I will argue below that Thurston's critique is valid, and that it equally applies to advanced undergraduate courses, even though our best wishes might be otherwise.

My support

Let me examine three instances of decisions made by teaching faculty that decreased the cognitive demands of students, and a line of research on proof that raises troubling questions.

Classroom observations

We, as instructors, often create situations in which students can survive, perhaps prosper, via mimicry or mechanical symbol manipulation just as they do, it is said, in their previous courses. I do not mean this to be an indictment of faculty. I would argue that these are very real attempts to meet students' needs that slip past the bounds of helpful scaffolding.

I include two different classroom vignettes to illustrate how instructors decrease the cognitive complexity of the tasks they assign. In the first example, I observed students in an introductory abstract algebra class at a prestigious public university who were discussing the concept of group isomorphism. As will become apparent, students were asked to mimic the demonstrated proof and were able to prosper without developing conceptual understanding.

> Dr. K began the class by reminding the students of previous work with group operation tables and the notion of sameness. He stated that there was a formal way to describe this intuitive notion called isomorphism.
>
> On the board, he began writing a standard definition of isomorphic groups: Two groups are isomorphic if there exists a function f…
>
> He paused and said,
>
>> for starters, we need this function to make sure there are the same number of elements in each set
>
> and wrote,
>
>> f, that is one-to-one and onto
>
> he continued writing,
>
>> and if a, $b \in$ G, then f(a*b) = f(a)• f(b).
>
> He said,
>
>> this property allows the elements to combine in the operation tables correctly. If you take two elements in the first group and operate on them, you need to land in the correct space in the second group.
>
> Next, Dr. K demonstrated how to prove that the cyclic multiplicative group of base 7 is isomorphic to the integers under addition. Dr. K then had the students form groups and assigned each group an exercise in which they were to determine if the proposed function between two groups was an isomorphism. The students were to post a proof or show which property failed. One student was assigned the same problem he had just used as an example, with a 10 in place of the 7. He went to the board and, in Dr. K's work, where the '7' was the student wrote a '10' by drawing a '1' and rounding out the '7' into a '0'.

Another manner in which faculty often decrease the cognitive complexity of problems is by providing a series of scaffolds that tells the students what must be done and shows them the means to do it, leaving little more to be done than writing out in symbols what the instructor has said. At another institution, while observing a class focused on quotient rings, I observed the following interaction:

> Students walked away from Monday's class with the assignment to demonstrate that **Q**[x]/(x^2 - 2) is not isomorphic to **Q**[x]/(x^2 – 3).
>
> The next class meeting began with Dr. R noting that some students had come to visit her between classes about the problem and that she wanted say a few more things about it. She began by reminding the students of the fact that each of the two structures was a quotient field. Dr. R followed this by asking the students how they might describe the elements in the first field (asking them to recall a fact that she had stated two class periods prior), to which one student responded,
>
>> The set a plus b root 2 where a and b are in Q.
>
> Dr. R continued,
>
>> Yes, and that means that this other set is a plus b root 3, right? You're more used to thinking of expressions with real numbers than with classes, so why don't we think of the problem like this… Let's write [writes: $\sqrt{2} = a + b\sqrt{3}$], and try for a contradiction. Why don't we square that and see if you can't get it. You can square it out and solve for root 3, then what type of number will it be?
>
> A student responds, "Rational." and Dr. R continues,
>
>> And, rational, bad… Or you force a or b to be zero, but you know that they're supposed to be non-zero…

In this case, Dr. R gave her students so much explicit help that all they needed to do was translate her words into symbols without grappling with the concept that the quotient fields each contain a root of the appropriate polynomial.

In both of these cases, the instructors made a pedagogical decision to reduce the cognitive complexity of the students' work. To use Weber and Alcock's (2005) language, both of these instructors have eliminated the need for students to consider the issue of warrant. Weber (2001) and Weber and Alcock's (2005) work can be read as suggesting that the most difficult tasks in proof creation are deciding on the format of the proof, calling into mind the necessary facts and deciding if these facts are warranted. The incidents described above show instructors who have made the nature of the assigned task into one requiring symbolic fluency and little else.

My goal is not to criticize the decision as unfounded or somehow incorrect, but rather to illustrate that well-intentioned scaffolding can, and does, sometimes work against our goals for the course. While I have suggested that the actions of instructors do sometimes contribute to students' ability to complete courses without content mastery, recent research on proof suggests that the curricula may also be contributing to this phenomenon.

Research on proof

The second reason I suggest that the work we require of students does not require conceptual understanding is drawn from the recent literature on proof. Weber's [2] recent work is just such an example; he wrote a 'proof' program into which he was able to load a small set of facts (definitions and theorems) from abstract algebra in an attempt to replicate and understand students' proving skills. In a similar vein, Schoenfeld (1985) described a computer-based exploration in which the facts were types of integrals.

Both individuals decided to load an incomplete set of facts into the program in order to simulate better the type and amount of knowledge accessible to students when attempting to create an algebraic proof. Both programs had, as their input, a statement to be proven and provided as output a completed proof. Weber [2] noted that his algorithm was unsuited to producing proofs by contradiction or proofs in which a novel construction is called for, but in the case of 16 statements approachable via proofs that can be completed by beginning with the hypothesis and working in a linear manner to arrive at the conclusion, the program was successfully able to produce 13 proofs. That would be accounted successful in many classes.

Weber [2] also demonstrated that he can teach students to employ his algorithm in an unthinking manner and expect similar rates of return. That is, Weber has created an algorithm that will allow students to simulate understanding and successfully complete a large percentage of algebraic proofs. The students now have a behavior pattern on which they can rely to execute a large number of algebraic proofs. What is most interesting is the type of problems that Weber's subjects are assigned; they complete proofs that would be standard in most introductory abstract algebra courses.

Another article by Weber (2005) showed a concrete instance of a student, Erica, successfully completing proofs, such as demonstrating that the sequence $(n - 1)/n$ converges to 1, yet not having conceptual understanding. Weber wrote,

> I asked her if she could explain why this proof established the sequence [same as above] converged to 1, but she was unable to answer this question. Later comments from this interview revealed that Erica had very little understanding of limits of sequences. (p. 7)

In short, the student was able to complete the required work by mimicking her instructor's examples.

In effect, the work of Schoenfeld (1985) and Weber ([2], 2005) suggest that students could be quite successful in advanced mathematics without making any of the constructions that Dubinsky *et al.*(1994) claim are required.

Conclusion

In light of the above evidence, it seems quite reasonable to suggest that in the study of advanced and abstract mathematics, the requirements to be successful still do not include developing conceptual understanding. That is, the emperor has no clothes. Many school mathematics teachers have found recent curriculum conversations useful in thinking about how to support students while maintaining the level of cognitive complexity of the tasks. Whilst we at the undergraduate level are experiencing an upsurge in the number of conversations focused on innovative curricula and pedagogy, it seems that unless those conversations are coupled with serious thought about how best to employ them in the classroom we will not be addressing this issue.

For example, as suggested by Weber and Alcock (2005) we may want to consider how wrestling with warrants changes the cognitive demands of a problem (and results in different types of mathematical development). Might proof-validation tasks be a type of activity that is more accessible to students while, at the same time, requiring much of the same cognitive activity as proof creation?

At the very least, more study is needed to explore the idea that teaching faculty may be providing too much, or the wrong kind of scaffolding for their students and that the exercises in mathematics texts do not require students to understand the content in order to complete them.

Notes

[1] Findell, B. (2000) *Learning and understanding in abstract algebra*, unpublished doctoral dissertation, The University of New Hampshire. For further information, questions or comments, e-mail bfindell@uga.edu.

[2] Weber's article is to be included in a forthcoming volume of *Research in Collegiate Mathematics Education*, and is currently available in pre-print by emailing K. Weber at khweber@rci.rutgers.edu.

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