Concepts, Processes and Mathematics Instruction

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> *The point is to see that the more adequate our grasp of what we understand as "knowledge", the more we can consciously, responsibly, and morally play the educator* [*Soltis, 1981, p 104*]

In mathematics education, the term "concept" repeatedly surfaces, often in contrast to the term "skill" Yet the teaching of mathematics as skills still predominates in our schools, partially because advocates of conceptual learning often assume the value of concepts without explicitly defending it by defining precisely what they are Until an adequate response to this question is given, the question of how to teach concepts will remain unanswered and the techniques of skill teaching will continue to dominate mathematics teaching In this article, I will briefly present a set of characteristics of concepts which contribute to their significance in education, and, in particular, to mathematics education In doing so, I will point out a need to focus on the processes inherent in acquiring concepts in order to understand what a concept is I will suggest and demonstrate that relevant approaches to concepts are embedded in the work on conceptual change in the philosophy of science from Toulmin, Wittgenstein, and Vygotsky Finally, I will suggest how this focus allows one to unite work on problem solving, and mathematical abilities and processes, and linguistic analysis, to yield a rich area for examining and exploring conceptual instruction of mathematics

The need to define a concept is increased as one begins to see how difficult a task it is to do The use of the term "concept" predominates, whereas its meaning is avoided Toulmin [1972] warned:

> The term "concept" is in danger of becoming an irredeemably vague catch-all, for the same reasons as its forerunners "impression", "idea", "notion", "essence" and "substance". Even supposing we set aside popular corruptions of the term — such as "a new concept in packaging" — it already carries as much intellectual load as it can safely bear and possibly more [p 9]

In this paper I will not attempt to provide a complete definition of a concept; however, I will begin this endeavor by offering a set of characteristics of concepts that provide a rationale for their significance in education, and that begin to point toward a possible theory of conceptual instruction.

1 Concepts serve both a public and a private role In surveying the definitions of concepts offered in the literature, one finds a distinction between those that locate concepts in the mind and those that locate concepts in the world as apart from human beings Accordingly I am suggesting that any definition of a concept that does not refer to an individual who is "conceiving" be labeled a public view of a concept Often, according to such a view, concepts represent the truth as it exists out there in the world; they are cast from the inherent logical relations in this external existence For much of the population, this is their view of mathematical concepts Mathematical concepts are the necessary components of the inherent order in the world, and they are discovered by people, not invented

In contrast, concepts are also frequently defined in such a way to stress that they are human constructions that allow the individual to make sense of the world Since a person is assailed by an infinite variety of sensory and mental stimuli, a person must order, select, interpret that infinite variety, and does so by the use of concepts Thus concepts are not naturally occurring "joints" in the world but are human creations for making sense of the world According to such a view, the truth of a concept is not assessed by external or absolute standards, but by its adequacy in serving a function for an individual or set of individuals People holding such a private view of mathematics see mathematical concepts as human responses to problems found in mathematical situations

It seems likely that a definition of a concept ought to consider both of these roles of concepts In looking for a definition one can consider concepts as having both a private (individual) role and a public (collective) role In their private role concepts allow people to organize and select information and make sense of their experience; in their public role concepts must be subjected to collective scrutiny in order to judge their precision, their effectiveness, and their correspondence with the world as experienced by others

The power of such a view of concepts for education is that it allows one to cast a role for education as bridging the private concepts of individuals with the public or collective disciplinary concepts. To take an example in mathematics education, consider the concept of a function A function has various precise public meanings, one of which can be stated as "a relation such that for each element in the domain there is exactly one element in the range" However, if students are asked informally to discuss what a function is, a variety of private concepts are revealed These include such ideas of a function as an action, an operation, a rule, a relationship, a graph, or a picture Mathematics education can be seen, then, as an attempt to take these private concepts of a function and to transform them so they evolve into the public definition given above Value is placed, therefore, on both the private and public role of concepts Thus the role of mathematics teaching would be not only to consider the private conceptions of the students, but to show developmentally how the public definitions evolved out of a need for precise communication, to eliminate certain special cases, to include others, or to cast the definition within a certain axiomatic framework Thus a public definition of a concept is presented as the current state of evolution of a series of increasingly adequate private conceptions, or as

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one possibility among a variety of alternative conceptions.

2. Concepts allow one to go beyond the information given. Another characteristic of concepts that makes them so powerful in education is that they allow one to go beyond the particular instances of the concept with which one has personal experience. By doing so one can identify new instances of a concept, can efficiently remember the variety of instances experienced, and predict new occurrences. In this sense concepts are somehow generalizations, abstractions, idealizations, prototypes, or models. Pointing out this characteristic of a concept emphasizes that conceptual learning entails the development of a student's meta-structure. That is, a student must learn to recognize that one can abstract or generalize beyond the instances with which one has had direct experience, and then must be able to do so. Thus the recognition that $2y + x = 3$, $y = -x + 4$, $x = 2/3\, y + 1$, $x = y$ and $y = 2$ are all lines requires the determination of which components are variables in an equation for a line, and which are fixed. I offer the suggestion that the flexibility encouraged by this meta-structure of concepts is a distinguishing characteristic of conceptual as against skill knowledge.

3. Another important characteristic of concepts is their relationship to other concepts. One could hardly begin to imagine a student understanding the concept of "area" without some understanding of the "unit" concept. And yet, working in schools with ninth-grade algebra and general mathematics students, I have seen a severance of these two concepts. Students who are asked how many pattern units are in a rectangle that is 3 units by 4 units (where a pattern unit is a triangle one half the area of a square unit) can tell me the area is 12, but resort to counting, often by twos, to find the number of pattern units. They cannot identify the number of square units with the area, having memorized the algorithm: "length × width equals area".

Moreover, all of us who have taught mathematics recall frequent occasions where students appear to have learned a concept, only to be unable, in another context, to see its relationship to a new concept. Bruner [1956] called the ability to switch contexts "transfer", a term which I find connotes too holistic a movement, as though it required only recall, rather than a significant amount of analysis and reconstruction on the part of the student. Nonetheless, this tendency to integrate across contexts seems to be a defining feature of a concept.

This characteristic of concepts being embedded in a configuration, and the recognition that the configuration is fluid, not static, poses another special concern for the teaching of concepts, and that is, how to encourage this integration of knowledge so that each lesson or unit is not mastered in an isolated fashion, but adds to the configuration of the whole, transforming it not simply into a bigger whole, but one with a variety of new possibilities.

4. A fourth characteristic of a concept is its relationship to a specialized set of terms. In much of the research on concepts, the ability to use a concept in a linguistically correct manner is considered as a condition or criterion of conceptual grasp. The issues raised concern the precise relationship between linguistic skills and conceptual grasp, but for the purposes of this paper only a rather abbreviated version of this discussion follows. (For a more detailed and fascinating discussion of this, see Toulmin [1969].)

One discussion presented by Toulmin in the paper that relates concepts and language is a description of a distinction made by Wittgenstein between "expressions" and "language games". An expression is a name, term, or sentence that conveys the analytic meaning of a word, likened to a dictionary definition. This is argued to be inadequate alone for understanding the meaning of a word because it tends to give one equivalent or synonymous expressions but fails to explain their meaningfulness or function within a larger context. Hence, returning again to the concept of function, a precise definition fails to communicate the importance of the concept function within the conceptual framework of set theory, analysis, or algebra. Consideration of these larger relationships are referred to by Wittgenstein as "language games".

The role of language in teaching mathematical concepts has become increasingly fundamental in my thinking. This role functions at two levels: one is the use of ordinary terminology in a precise manner, such as concepts of lines, sets, functions, etc. The second role of language in mathematics is the use of symbols. Much research needs to be done on how students come to see mathematical language and symbols, and how these structures have an impact on the acquisition of concepts.

The same distinction by Wittgenstein can be applied as a criticism of our attempts to introduce terminology in mathematics. We often give students a precise analytic definition of a word or an "expression" but fail to discuss how this term is a part of a larger context, or how it is interconnected with other expressions, actions and concepts through language games.

At the broadest level of a language game, we seldom even discuss with students the role that language — terminology and symbols — serves in mathematics. These relationships among the language, the symbols, and the underlying concepts, are extraordinarily important. Consider, for example, the limit concept in calculus. James Kaput [1979] describes a case where the language, symbols, and concepts interact in the calculus curriculum.

> Perhaps the most obvious is the motion metaphor that gives
>
> $$\lim_{x \to a} f(x) = L$$
>
> its primary meaning. "As x moves towards a, $f(x)$ moves towards L." This is frequently written:
>
> $$\text{as } x \to a,\ f(x) \to L$$
>
> Nowhere is the schizoid relationship of mathematics to metaphor more blatant than here. On the one hand the role and responsibility of the metaphor are openly acknowledged in the symbols, especially in the use of the arrow, which symbolically carries the meaning of the motion metaphor. It is more important than Cupid's arrow. On the other hand, for very good reasons, gen-

erations of brilliant mathematicians struggled for hundreds of years in an attempt to get logic to control and clarify the meaning of this metaphor They finally succeeded in squeezing out the metaphor, leaving us with the familiar (timeless) ϵ-δ conditional statement as the ultimate definition They also left us with all the arrows used to denote it In view of the importance of motion (both physical and more abstract) in the creation of the calculus, I am less surprised that it took so long to disentangle the logical, formal definition from motion than I am that it was done at all When we try to squeeze the motion metaphor from an undergraduate's understanding of limit and replace it with ϵs and δs, then, of course, the ϵs start moving (towards zero, of course). If we can stop the ϵs, then the δs start moving. If finally, through coercion and threats, we are able to stop the ϵs and δs, everything stops, especially thinking. The ϵ-δ definition was designed to bar monsters that cannot fit through such an undergraduate's conceptual door. [p 294-5]

We need to consider the same interactions among terminology, symbols, and concepts, in other areas of mathematics in order to promote conceptual instruction

Up to this point I have summarized four characteristics of concepts: having public/private roles, going beyond particular instances, being embedded in configurations, and relating intimately with language These characteristics of concepts are not particularily original, but in considering them as a collection one can begin to discern some fundamental claims about concepts which may lead toward a theory of conceptual instruction

Primarily one discovers that each of these characteristics highlights the fact that concepts have processes and content merged within them. What I am suggesting here is that if one scrutinizes each of these characteristics of concepts, embedded in it is not only meaning but also process. The private/public roles of concepts do not simply entail dual functions, but suggest a process of movement between these two roles. Expanding on this, one can see highlighted in this view of concepts the need to focus on the processes of movement by which students successfully negotiate the transition from their private concepts to more public ones. Work of this kind has begun through a series of research projects intended to discover students' individual conceptions. Although the majority of the work in this area has been undertaken in science [see Nussbaum, and Novick, 1981; Smith and Sendelbach, 1979; Posner, Strike, Hewson, and Gertzog, 1979], researchers in mathematics education are undertaking similar studies [Erlwanger, 1975; Confrey, 1980, 1981; Clement, 1980, 1981; Marton, 1978; Lybeck, 1979]

Recognition and description of these individual conceptions is a starting point, but only that. From there we need to continue on to consider a fundamental question posed by Toulmin [1972] for the philosophy of science:

> A (wo)man demonstrates his(her) rationality, not by a commitment to fixed ideas, stereotyped procedures, or immutable concepts, but by the manner in which, and the occasions on which, (s)he changes those ideas, procedures and concepts. [p. x]

Thus we need to find out how concepts are acquired, and under what conditions they are modified. In doing so, one might appeal to the work of Piaget on accommodation and assimilation or other work on cognitive dissonance. Equally essential to investigating these issues is an emphasis on determining "why" students believe what they believe, or why they are willing to change their beliefs. The reasons for believing or modifying beliefs are essential to examine as we offer a theory of conceptual change.

One important clarification must be made here. That is, although I have presented the discussions of private conceptions and conceptual change as independent endeavors, they should not be thought of as distinct. As Toulmin has pointed out, it is often at times of change that one is most likely to reveal some of one's most tenaciously held beliefs. Thus in studying processes of change one can often find insightful results about private conceptions. (For further discussion of this, see Confrey [1981]

If the private/public roles of concepts lead to research into student conceptions and conceptual change processes, what can be seen about the processes and content relationship posed by the characteristic of concepts of "going beyond the information given"? Once again, the importance of process is stressed Here the processes include those of generalization, abstraction, and prediction. That is, if concepts allow one to go beyond one's experience with particular instances, then one must recognize this potential and develop the abilities to handle this

Research conducted on abilities has the potential to enlighten us on some of these processes Krutetskii [1976] did an extensive study of the abilities that seemed to distinguish exceptional and less able students His list comprised six abilities: to generalise, to employ reversibility, to be flexible and elegant, to curtail reasoning, to discern mathematical structure, and to hold items in memory. If one considers abilities not as innate potential but as variably developed psychological and mathematical processes, these abilities can be seen as the basic processes in the acquisition of concepts in mathematics.

In a study of ninth-grade algebra and general mathematics students, using clinical interviews, we* are finding considerable differences among students in the development of these abilities as we look across various concepts [Confrey *et al*, 1981] As a result we are beginning to conceptualize these abilities as processes of learning mathematics which can enhance or impede the acquisition of concepts Again it's important to recall that these abilities or processes are not learned in isolation from concepts, but that the processes and the concepts are bound together.

The third property offered above as a characteristic of concepts was that they are suspended in a network of other concepts. Formerly, as mentioned earlier, the process of relating and integrating concepts was identified with Bruner's concept of transfer However, issues of transfer have recently evolved into issues of problem solving; part of the reason for this is that bridges between and among con-

*This study is being completed at the Institute for Research on Teaching. Michigan State University. I have directed the clinical interview portion, and been aided in the inquiry by Sister Chrisanne Weisbeck. Anne Nason, Dr Bruce Mitchell, and Dr Perry Lanier

cepts are often a function of the problems that unite them.

Toulmin [1972] has offered a similar suggestion, writing that one can describe a genealogy of problems that a discipline strives to solve, and that may be more revealing of the continuity of a discipline than any fixed set of content or methods:

> In this way, the problems around which successive disciplines focus their work form a kind of dialectical sequence; despite all the changes in their actual concepts and techniques, their problems are linked together in a continuous family tree Analyzed in these terms, the continuity of a scientific discipline then rests as much on the considerations governing the changes between successive "generations" of affiliated problems — with all their associated theories and ideas — as on any considerations leading to the survival of unchanging problems, theories, and concepts. [p 148]

Thus one can begin to see that the processes of problem solving may serve to unite concepts and lead to an integration of knowledge.

Lesh [1980], in writing on problem solving, also rejects the separation of problem solving and concept formation:

> Students do not first learn an idea, then learn to solve problems using the idea, and finally learn to solve applied problems Rather there is a dynamic interaction among basic mathematical concepts and important applied problem solving processes.
>
> It seems questionable to claim that a person can apply a concept before the concept has been learned, but it seems equally questionable to claim that a concept is *known* if it cannot be used in simple applications. This apparent paradox is resolved because it is naive to think of ideas as being either understood completely or else not understood at all. It is more accurate to think of ideas as gradually becoming more and more meaningful as they become more complex, as they are connected to more and more other ideas and events, and as they become embedded in progressively more powerful representational systems. [p 4-5]

Concepts were also characterized as being tied to a specialized set of terms In looking for the processes that are required by such a claim about concepts, I will turn again to Toulmin's [1969] discussion of the relationship between conceptual grasp and linguistics. First Toulmin argued that, in order to study the nature of a concept, one needs to study the process of its acquisition.

> All scientific experience indicates that one cannot analyze the criteria for recognizing when a process is *completed,* in a final and definitive form, until the actual *course* of the process has been studied Rather, the two investigations must proceed *pari passu*. We start out with a first, rough criterion of "completion"; but, as our understanding of the process improves we progressively refine that criterion — developing, as a result, more satisfactory conceptions *both* of the actual course of the process *and* of its completion [p. 80]

Such a statement supports the contention that in order to define a concept, one must consider the processes inherent in it, especially those of acquisition. This discussion is, however, qualified by a discussion of Vygotsky's contention that "language plays different roles in relation to behavior at the *learning* stage, and in the subsequent *employment* of concepts" As Vygotsky puts it, "at the learning stage, language is a "means" or "instrument", while in the subsequent employment of concepts, it is a "symbol"." [p. 81] In mathematics, an example of this might be in the learning of fractions — the words denominator and numerator are used to provide a label for the fractional parts, and serve as a means for learning to add fractions; once students can successfully add fractions, the loss of recall of these terms does not necessarily mean a loss of the practical ability to add. The implications of this loss for conceptual understanding are less certain. The processes connected with the linguistic characteristics nonetheless merit careful examination.

In summary, in the second portion of the paper I have examined each of the four characteristics of concepts and suggested how content and processes are *both* inherent in this view of concepts. In each case I have identified particular processes which are being researched currently and/or need further investigation in order to unite them into a theory of conceptual instruction These include the areas of individual conceptions, conceptual change, mathematical abilities (seen as learning processes), problem solving strategies and language acquisition and development.

What this focus on the merging of meaning and process in concepts leads me to believe is that one must reject phrases such as "having a concept", replacing it with phrases such as "developing" or "acquiring" a concept, which implies both content and process Toulmin [1972] wrote on the difficulties inherent in the former phrase:

> But the heart of the whole problem lies in the fact that there are no such simple, unambiguous and unchanging criteria for "having concepts". The definitive test for judging whether a dog or an infant, a child or a schoolboy, a normal adult or an asphasic, knows/recognizes/has learned what "red" is (or "four" or "self") are not obviously unitary or univocal In this context defining "having a concept" verbally as "having a certain form of knowledge" merely begs the question by encouraging us to think of "conceptual grasp" as being simple and static — even algorithmic. [p. 41]

The alternative, then, is to examine more closely the processes of conceptual change, mapping both the route of meanings a concept is transformed through, the alternative routes that are likely, and the processes of belief and justification that are undertaken. As a part of this, the processes of problem solving, language acquisition, and generalization, need to be included. The combination of all of these features is what I intend when I call for the investigation of conceptual change in mathematics education.

I believe that uniting these disparate inquiries into one area can provide mathematics education with a theory of conceptual instruction. Such a theory might start with the following plan:

1. Identify relevant concepts to be taught;
2. Determine students' alternative, private conceptions of the concepts, perhaps through their responses to a problem;
3. Identify terminology and symbols attached to those public concepts, and to those private conceptions;
4. Propose possible routes from private to public concepts through a series of developmental stages — these should be both conceptual and linguistical stages;
5. Apply a theory of conceptual change. One possible method is to construct or search out problems that unite the concepts to be learned and to pose these as challenges. To be effective problems for the students it's likely that these should conflict with the students' privately held conceptions. (For an excellent example of this, see Nussbaum and Novick [1981].)
6. Devote attention to the processes necessary to form the concepts, such as generalization, prediction, abstraction, curtailment, etc.
7. Assess the students on problems that involve flexible and original instances of the concepts, and that require problem solving strategies as well as recall of previous instances

Before becoming useful, such a theory needs to be demonstrated on a particular set of concepts. Furthermore, the development of a particular array of concepts needs to be tied into a discussion of the overall purposes of conceptual instruction. Finally, particular attention needs to be paid to the uniqueness of mathematics as a discipline in which the relationship of concepts and empirical evidence is unique, and in which the role of images, symbols, and terminology varies from other disciplines. Skemp [personal communication, 1981] has suggested that in mathematics we form "concepts of concepts." Others have suggested that mathematics be seen as attempting to distill out the images associated with them, leaving only the symbols to be manipulated. Without an in depth discussion of these issues, theories of conceptual instruction specific to mathematics will inevitably omit these important "language games" and their conceptual equivalent, and the theories will be able to be reduced to a set of techniques as in skill teaching. However, I am hopeful that this uniting of various strands of research around concepts and processes can begin to provide a basis for further development of conceptual instruction

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