SCALE, PERSPECTIVE, AND NATURAL MATHEMATICAL QUESTIONS

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Gerofsky (2010) highlights a constitutional tension at the core of the usual ways of posing real-world problems: Students are expected to see through a real-world veneer to recover specific mathematical exercises. They learn this game from their earliest encounters with story problems, when they are asked to make sense of nonsensical situations involving people that do not exist, doing things a person would not do, to answer questions that a mathematically-minded person would not bother to pose (Gerofsky, 1996). As students progress from primary to secondary schools, the real-world veneer can become more engaging or even more authentic, but it is still the case that the experience of being in the world is subordinated to a mathematical account of that experience—in fact, producing such an account is the purpose of real-world or modeling or applications-based problems. We aim to explore how to deliberately breach the contextualized problem genre by posing problems that aren't problems but that, instead, convey an authentic sense of being someplace in the world. We argue that this sense of place creates a context for asking natural mathematical questions, which we exemplify and explore below.

We begin by describing a place, a sand bar in the Atlantic Ocean, 2.5 miles from the southern coast of the Island of Eleuthera—a long, thin island in the Bahamas. It is a small island of *oolitic* sand [1] that is surrounded by shallow water (5–20 feet, depending on the tide). Below, we describe this place in a generic way, with the goal of conveying to the reader a sense of what it is like to stand on the sand bar and take in one's surroundings.

The place

> Imagine we are on an island. It is near midday with good visibility. A warm breeze sweeps over us and the tide is rushing out. The air is salty. Beneath our feet, we feel the smooth, oblong grains of a special kind of sand that forms in the ocean. Over eons, the sand has built up into a bank, the very top of which peaks out of the water and provides a place for us to stand and take in the scene. We are atop a tiny, uninhabited island, surrounded on all sides by blue and turquoise water. Other than us, the only other object on the sandbar is an obelisk that is intended to warn boaters of the shallows (see Figure 1). Spend a few moments surveying our surroundings. Imagine the sights, sounds, smells, and feels of the sandbar. What do you notice? What are you wondering about? What do you want to find out about where we are?

Figure 1. A picture of the south Eleuthera sandbar.

This description of the south Eleuthera sandbar does not fit into the genre of contextualized mathematics problems. It contains neither numbers nor key words with which to process those numbers. The questions it poses are not, themselves, mathematical problems, but rather are questions that suggest possible problems, including problems that we would argue are natural questions that emerge from one's experience in the world.

What natural questions suggest themselves from the description of the sandbar? In the spring of 2018, we traveled to the sandbar with four experienced secondary mathematics teachers and asked them what being at the sandbar caused them to wonder about. The teachers were curious about the sand itself: Where did it come from? How did it form into a bar? They were curious about how they could determine where they were and how far away they were from the nearest land. Eventually, their curiosity about where they were led them to pose the question, 'How far away is the horizon?'. We refer to this question as the 'horizon problem'.

The horizon problem is a compelling example of a natural mathematical question because it is a question worth asking that is grounded in curiosity and that, most significantly, can only be answered by mathematical analysis. This sets the horizon problem apart from the other questions the teachers raised at the sandbar, which would have been difficult to investigate without additional tools (*e.g.,* a sextant) or knowledge from other disciplines (*e.g.,* geology). Furthermore, empirically measuring the distance to the horizon is impracticable: The true horizon is visible at sea, where waves, tides, and currents would challenge any effort to gauge the distance between where an observer is and the farthest point away that the observer could see. A mathematical analysis of the horizon problem, on the other hand, could

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produce a general model that would determine the distance to the horizon for any observer.

We explore the horizon problem by describing the teachers' mathematical activities at the sandbar. In particular, we focus on how the teachers used different perspectives to negotiate different scales of space. We analyze how the teachers translated the natural question, 'How far can we see?', into a mathematical problem they could solve.

Scales of space and mathematics

Herbst and colleagues (2017) suggest we connect the teaching and learning of geometry to students' experiences of being in the world. They propose a modeling approach, wherein conceptions of geometric figures are grounded in explorations of space. The overarching goal of their project is to use the historical roots of geometry—the mathematical science that was developed in concert with our ancestors' efforts to measure and describe the Earth (van Brummelen, 2012)—to inform the design of new kinds of geometry learning experiences.

As a starting point, they consider how different scales of space (Acredolo, 1981) frame students' conceptions of mathematical figures. 'Microspace' refers to the immediate personal space around an agent, such as the space in front of one's face when reading a book. Mathematical figures are typically encountered as representations in microspace—*e.g.,* on the pages of textbooks, on personal display screens. Figure 2 shows a reproduction of a microspace representation from a geometry textbook.

'Mesospace' refers to "the space of objects that can be captured by gaze"—*i.e.,* objects that range in size from "about half the size of the agent to 50 times its size" (p. 54). A table would be an example of a mesospace object, as would some monuments—*e.g.,* the obelisk monument the teachers encountered at the sandbar. Whereas microspace can be perceived and controlled from a fixed point, to interact with objects in mesospace requires movement (Montello, 1993)—*e.g.,* walking from one part of a room to another. The mesospace can thus facilitate interactions among different agents and allow people to use whole-body movements (crouching, standing on one's toes, walking, *etc.*) for exploration and sense-making (Kelton & Ma, 2018; Ma, 2016).

'Macrospace' refers to "spaces too expansive to be captured in their totality by gaze" (p. 53).Whereas mesospace can be visually surveyed (though not manipulated) from one vantage point, macrospace requires one to switch vantage points simply to see everything in the space (*e.g.,* walking through the rooms of a house). Furthermore, exploring macrospace through movement generally requires modes of transportation beyond what bodies alone can provide. Real world mathematical activities tend to involve representations of macrospace (*e.g.,* a map of a city) that facilitate perception of some qualities of macrospace in their entirety (its network of roads) at the expense of other qualities of the space (the height of the city is flattened). The history of geometry is a history of how humans learned to model macro- and mesospace objects with microspace inscriptions (Herbst *et al.* 2017).

Perspective also frames interactions with mathematical representations. The small, two-dimensional diagrams typi-

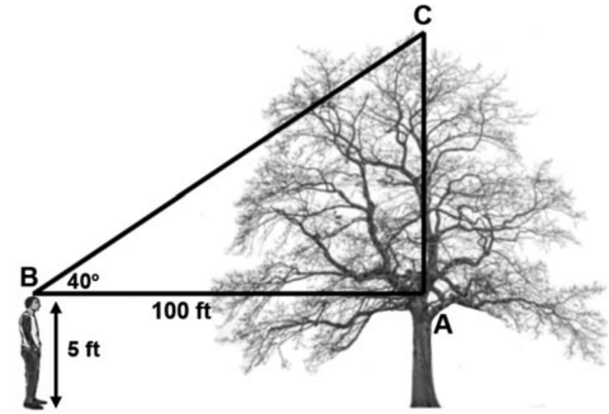


Figure 2. A reproduction of an exercise from a high school geometry textbook (Cummins, Kanold, Kenney, Malloy, Mojica, 2001, p. 566).

cally encountered in textbooks are viewed from a fixed, third-person perspective: The viewer is external to the diagram, the diagram is viewed from a specific angle, and the mathematical figures represented in the diagram are constrained by the boundaries of the display pane (*e.g.,* a diagrammatic realization of a line is a stroke of finite length). The control over perspective inherent to two-dimensional diagrams is both an affordance and a limitation. A mark of one's mathematical literacy is the ability to choose a strategic perspective from which to construct a two-dimensional representation of a three-dimensional figure; strategic in this case referring to the ability to mathematically operate on a constructed representation. But the work of creating such representations is generally done for students, as in Figure 2. There, the viewer is positioned orthogonally to the tree and the person; this framing enables a three-dimensional situation to be modeled with one-dimensional lengths.

We raise the issue of perspective because it has been suggested as a lens for understanding embodied mathematical activity. In a study of how students used gestures to represent graphs of functions, Gerofsky (2011) identified character viewpoints and observer viewpoints to capture differences in the perspective that students would adopt when using their bodies to act out a graph. Some of the students in the study made small, precise gestures with their hands or fingers, almost directly in front of their faces, in an effort to create an ethereal sketch of the graph that they could see in the air. They were, in essence, using their fingers to create microspace representations that adopted a third-person perspective (observer viewpoint). Other students used their entire bodies to move through space as if they were the graph. They created mesospace representations that they actively explored from a first-person perspective (character viewpoint). Gerofsky (2011) concluded that the different viewpoints corresponded to different inclinations toward mathematical activity. The observer viewpoint tended to be adopted by students who valued precision, correctness, accuracy, and who relied on algorithmic reasoning to gain control of the mathematical representations they were creating. The character viewpoint tended to be adopted by students that were imaginative, flexible, and for whom the whole-body movements provided "multiple potential entry points for sense-making and the creation of more robust mathematical conceptual objects" (p. 253).

In our exploration of the horizon problem with secondary mathematics teachers, we observed participants adopting both the character (first person) and observer (third person) perspectives. As we will see below, the teachers at the sandbar moved their bodies in different ways in mesospace to understand the first-person experience of seeing the horizon. The first-person perspective generated the natural question that would become the horizon problem. As they used their bodies to explore the problem, they transitioned from first-person, mesospace representations—in which their bodies were integrated into the representations—to third-person, microspace representations. The move from a character perspective to an observer perspective was effected by making gestures and drawing diagrams.

Teachers' explorations of the horizon problem

We took a boat with four experienced teachers (Gemma, Fred, Tommy, and Dwight [2]) from South Eleuthera to the sandbar. The teachers all taught pre-calculus in the same school and were taking part in a 4-day immersive professional development experience that was hosted at the Cape Eleuthera Institute, a research outpost in South Eleuthera [3]. The waterproof housing for our video and audio recorder failed during the portage from the boat to the sandbar. As a backup, we documented the activity that transpired with field notes. The field notes included documentation of movements the teachers used to investigate the horizon problem. During an afternoon session that followed the excursion to the sandbar, we worked with the teachers to recreate and photograph the diagrams they drew on the sandbar to model the horizon problem. The teachers also wrote about their work on the horizon problem at the sandbar in journals and answered questions about their experience in video diaries. Teachers' journals and diary entries were transcribed by the researchers.

Teachers' initial ideas about a solution method

The teachers split up into pairs and began working on the horizon problem. Gemma and Fred went with Amanda to the end of the sandbar with the obelisk, while Tommy and Dwight stayed with Justin by the other end. Both pairs of teachers explored the problem with their bodies to generate initial ideas. Tommy and Dwight used their gaze to attempt to bound the distance to the horizon between other, known distances. Gemma and Fred attempted to relate the apparent height of the obelisk (measured via hand-lengths) to the apparent height of a freighter they could see in the distance in an attempt to set up a proportion that would yield the distance to the horizon. After about 10 minutes, both pairs of participants abandoned these initial efforts because they were unsatisfied with how much their methods relied on estimation—they wanted something more exact.

A spark of inspiration

After some time, Dwight said aloud "So what we want is like the length of the line of sight." As he said this, Dwight stuck his arm out toward the horizon and lowered his chin to gaze along his arm. Here is an instance where Dwight was attempting to translate his first-person experience of gazing toward the horizon into a mathematical representation that he could measure. He conceptualized his gaze as a line segment that joined his eyes to the farthest point away that he could see. Dwight had converted the initial, natural question, 'How far away is the horizon?' into a specific mathematical problem, 'What is the length of the line of sight from his eyes to the horizon?' Dwight captured the fact that there is a limit to how far we can see by making the gesture shown in Figure 3. The gesture is shown from a third-person perspective to provide a clear illustration of what the gesture looked like as seen by Tommy and Justin.

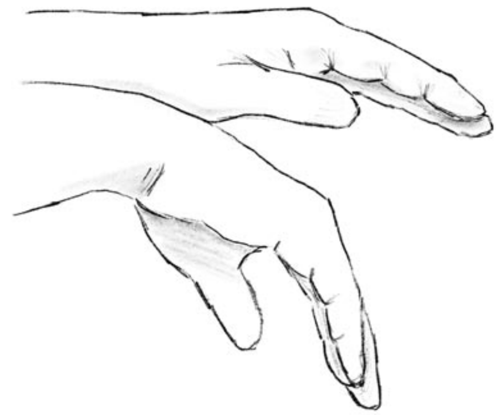


Figure 3. Re-creation of the gesture used by Dwight to show how the line of sight to the horizon related to the curvature of the Earth.

In this gesture, the top hand represents the line of sight. The bottom hand represents the curvature of the Earth. Where the base of the top thumb touches the top of the bottom wrist is the point where the line of sight is tangent to (and consequently blocked by) the curve of the Earth. Dwight produced this gesture spontaneously, as he reframed the question about how far he could see into the problem of measuring the length of a segment. The gesturewas the seed that allowed the teachers to begin to gain mathematical control over the experience of gazing out to the horizon.

Justin worked with Tommy and Dwight to investigate Dwight's line-of-sight gesture and to determine what the length of the line-of-sight depended on. Justin invited the pair to notice the bands of blue and turquoise water visible as one gazed toward the horizon. Then he invited them to lie down on their stomachs and look again. Once down on their chests, Tommy said, "They disappear." After rising again to their feet and brushing themselves off, Justin asked, "So what does that tell you?" Dwight responded, "That how far you can see depends on your height." At this point, Justin asked "Okay, so what else might matter when you are trying to figure out how far you can see?" By then, Gemma and Fred had returned from surveying the obelisk. Amanda walked Gemma and Fred through a similar line of questioning leading them to the same moment of lying on their stomachs and gazing toward the horizon, only to notice the

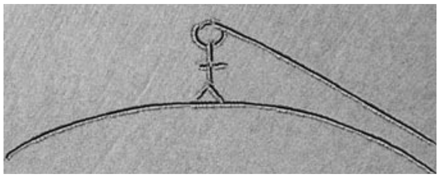


Figure 4. A re-creation of Gemma and Fred's initial sand sketch of the problem situation.

bands disappear. Fred noted that the answer must depend on one's height.

Modeling macrospace in microspace

Gemma and Fred lingered a bit longer on the ground and began drawing models in the sand. The two began by sketching a curved line with a stick figure standing on it and the line of sight to the horizon (see Figure 4). This sketch is a microspace representation of the experience of seeing the horizon. Like Dwight's gesture, the diagram represents the line of sight as tangent to the curve of the Earth. The diagram fixed Dwight's gesture into a representation that could be viewed, analyzed, and modified. The shift from gesture to diagram also entailed a shift in perspective, from the character viewpoint of the first-person gesture to the observer viewpoint of a third-person diagram.

Fred began wondering aloud about how similar triangles could be useful. Illustrating in the sand, Fred described a process to create similar triangles where an individual would take stock of an object in the far off distance (like another island) and walk backwards until he could no longer see that object, noting how far he had walked (Figure 5). Fred recorded this idea in his journal as his next progression of thought. He wrote, "using pyramid height approach" and drew a figure like the one the two had drawn in the sand (see Figure 6).

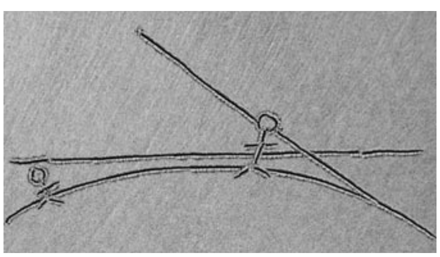


Figure 5. A re-creation of Fred's second sand sketch of the problem situation.

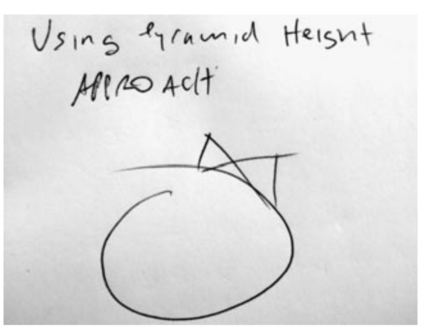


Figure 6. Entry in Fred's journal.

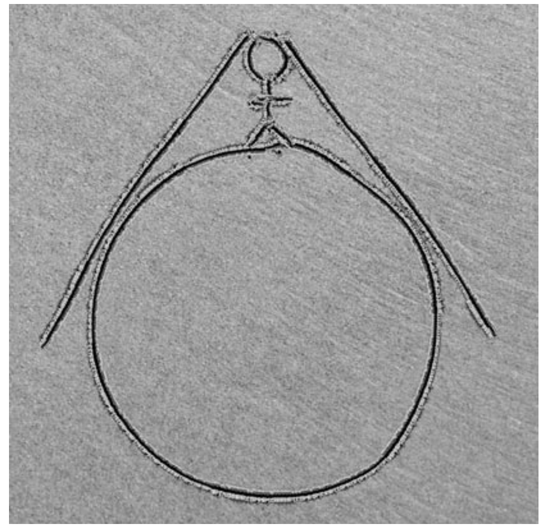


Figure 7. A re-creation of Gemma's second sand sketch of the problem situation.

Fred represented his idea in a diagram, but it was ultimately an empirical approach to the problem that relied on traversing a local section of the Earth. He had moved from the character perspective to an observer perspective by including people in the diagram, but the activity the diagram represented was a character-focused experiment. Gemma produced the first diagram that included the whole Earth. Gemma and Fred were thinking about the lines of sight both in front of and behind a person and trying to imagine ways they could use those to establish two triangles (see Figure 7).

Pointing to Gemma's second sand sketch, Justin said, "How about the surface you are standing on? How does that matter? Like here it looks a little like you've drawn the Little Prince."[4] Both Gemma and Fred laughed a bit, with Gemma admitting, "Yeah the scale is a bit off". With a smile,

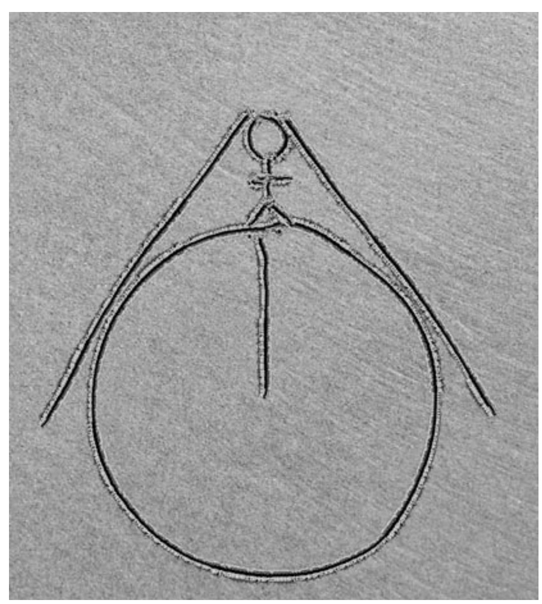


Figure 8. A re-creation of Gemma's second sand sketch with Fred's modification of the Earth's radius.

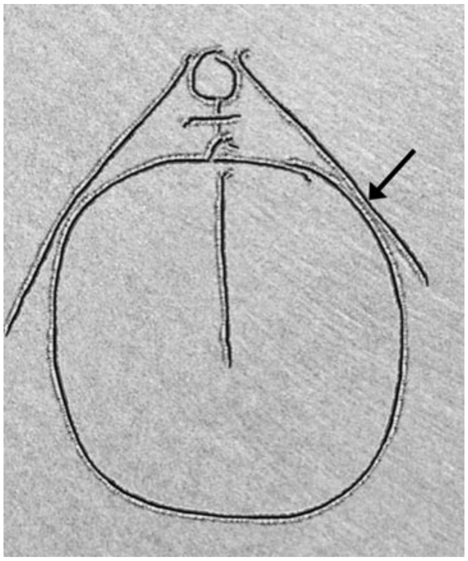


Figure 9. A re-creation of Gemma's second sand sketch including an arrow to show the spot where Gemma pointed to when asked to clarify which distance.

Justin asked: "How far could the Little Prince see on his world?" Gemma said, "Not very far at all". Immediately Fred responded with, "Oh, the size of the globe. It matters! How could I have overlooked that variable. Of course that matters. I wasn't even thinking about that!" At this point, Fred added a stroke to Gemma's drawing to represent the radius of the Earth to the point on the globe where the stick figure was standing (see Figure 8).

With this diagram, Gemma and Fred have moved closer to an ideal, external observer perspective. There is only one human figure in the diagram whose position is fixed at one end of three different segments: two lines of sight and the radius of the Earth. They have gained more control over the situation of standing on Earth by drawing a microspace representation that includes a radius of the Earth—a segment that they have no direct, real-life access to. Dwight's gesture and Figures 4, 5a, 6, and 7 show a progression of mathematical control that is facilitated by transitioning from a first-person, character perspective to a third-person, observer perspective.

Fred and Gemma stared for quite some time at the sketch. Amanda asked, "What are you trying to figure out?" Gemma restated the question, this time referencing the diagram, saying "We want to know how long this is", pointing to the tangent line on the right side of the diagram. Amanda responded by asking, "And what's the 'this' you are referencing?" Gemma responded, "The distance from the person to this *[pause]* spot" hovering her pointer finger over the point of tangency and moving it back and forth as if tracing a small segment (see the arrow in Figure 9).

Amanda asked, "Is there a way we could find out exactly where that point is?" In reply, Gemma added another radius to the diagram (Figure 10).

Fred added the radius to the other point of tangency and again the two stared at the diagram for sometime. Fred broke the silence this time and said, "Well we'd need to know the radius of the earth to solve this, but for now (to Amanda) is it okay if we just call that r?" To which Amanda nodded her head. Then Fred spoke again saying pensively "What do we know about this diagram?" Silently, Lisa added the markings representing a ninety degree angle (see Figure 11) and asked Fred, "Is that true? I can't recall. It's been so long since I have taught geometry."

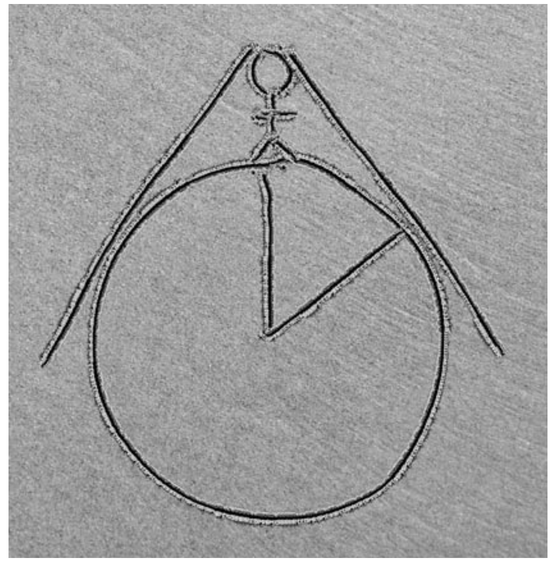


Figure 10. A re-creation of Gemma's second sand sketch with an additional radius drawn by Gemma.

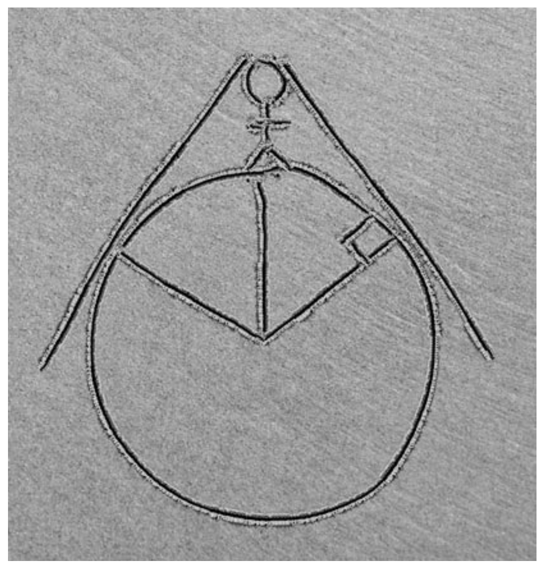


Figure 11. A re-creation of Gemma's second sand sketch with the additional radii and 90° marking added.

At this point, Fred affirmed Gemma's conjecture, with the two of them recalling the following theorem: *If a radius of a circle is drawn to a point of tangency of a tangent line, then the radius is perpendicular to the tangent line.* Fred exclaimed, "Okay, so we have everything we need then" (labeling the diagram with the variables *r, x,* and *h* as shown in Figure 12).

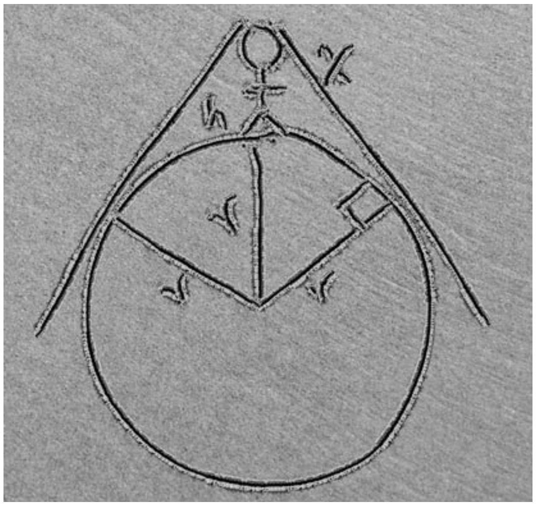


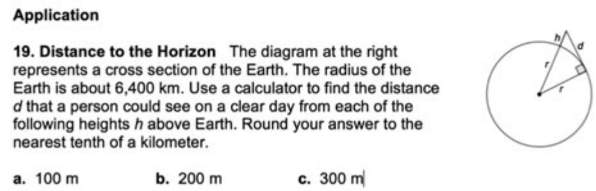
Figure 12. A re-creation of Gemma's second sand sketch with the additional radii and variables added by Fred.

Discussion

The final figure produced by the teachers at the sandbar was a complete mathematical model of the horizon problem. The diagram included letters that named constants (r), inputs (h), and outputs (x), as well as markings that showed angle relationships. The generation of this diagram involved a gradual shift in perspective from first-person, character viewpoints on a macrospatial scale to third-person, observer viewpoints in microspatial diagrams. It took more than 30 minutes from when the teachers initially posed the horizon problem to get to the point where they were modeling the situation with diagrams.

Two bodily movements provided breakthroughs for the teachers that allowed them to model the macrospace of the Earth in a microspace diagram. The first movement was Dwight's line-of-sight gesture that is reproduced in Figure 2. Dwight's gesture captured how the curvature of the Earth, eventually, blocks one from seeing any farther. The second movement was the teachers moving their bodies down onto the sand, lying on their stomachs, and looking out. From this vantage, the teachers could see that as they changed their height, the distance they could see out on the water contracted. When they stood up, they could see that the distance to the horizon increased. By using their bodies, the teachers were able to identify that the distance to the horizon depended on the height of one's eyes above sea-level. The insights from Dwight's gesture and lying on the sandbar allowed the teachers to see the mathematics in the situation.

We see the richness of the teachers' explorations as evidence that linking the experience of gazing at the horizon (first-person, macrospace) to a diagrammatic representation of the horizon (third-person, microspace) is a significant act of mathematical modeling. That is, negotiating the differences in scale (meso- and macrospace vs. microspace) and perspective (first person versus third person) is the mathematical work of the problem. Once they were able to represent the situation in a diagram, and recall a theorem about tangent segments, the solution to the problem was a matter of calculations they knew how to do.



Application

19. Distance to the Horizon The diagram at the right represents a cross section of the Earth. The radius of the Earth is about 6,400 km. Use a calculator to find the distance *d* that a person could see on a clear day from each of the following heights *h* above Earth. Round your answer to the nearest tenth of a kilometer.

a. 100 m **b.** 200 m **c.** 300 m

Figure 13. A reproduction of an application exercise from a high school geometry textbook (Cox, 1992, p. 391).

That generating a representation, as opposed to completing a calculation, could be the end of a mathematical activity contrasts sharply with standard modes of presenting real world problems. This is the case not only because of the difficulties of getting to a place like the sandbar, but also because of the time and content constraints on school mathematics. We recognize that a mathematical activity such as the horizon problem—where students could spend 90 minutes or more to draw a single diagram—would be a challenging commitment for teachers, given the limited resources of school. But we also believe that the rich, embodied exploration of a real-world question that we reported here represents a higher-quality mathematical experience than those that are entailed by other modes of posing real-world problems. For example, Figure 13 shows how the horizon problem could be represented in a textbook.

The problem appears in a numbered list of other problems under the heading 'Application'. The labeled diagram includes all the necessary variables. The opening text, featured in bold type, telegraphs the purpose of the exercise. The remaining text conveys the numbers students will need to complete calculations—the radius of the Earth, different values for an observer's height above the surface of the Earth—and instructions for how students should format their answers ("Round your answer to the nearest tenth of a kilometer"). The statement of the problem is a pretext for students to set up and solve a specific equation:

$$d = \sqrt{(r + h)^2 - r^2}$$

The accompanying diagram ensures that students will not be expected to ponder the shape of the Earth and generate the two-dimensional representation of that shape that would facilitate doing mathematics. Nor is the student expected to develop a geometric interpretation of the concept of a line of sight as a tangent to a sphere. Even the question of whether there is a visual horizon or how the horizon could be modeled geometrically is somehow overlooked, despite "Distance to the horizon" being the title of the problem. The work of analyzing one's experiences of space and building a geometric model of the activity of gazing off into the distance is completed for the student in the statement of the exercise.

The contrast between the sandbar and textbook representations of the horizon problem illustrates a distinction Dewey draws between *genuine* and *simulated* problems.

Dewey (1916) notes that "the following questions may be useful" for discriminating between the two:

> Is there anything *but* a problem? Does the question naturally suggest itself within some situation of personal experience? Or is it an aloof thing, a problem only for the purposes of conveying instruction in some school topic? Is it the sort of trying that would arouse observation and engage experimentation outside of school?
>
> Is it the pupil's own problem, or is it the teacher's or textbook's problem, made a problem for the pupil only because he cannot get the required mark or be promoted or win the teacher's approval, unless he deals with it? (italics ours, Dewey, 1916, p. 182).

Dewey asks whether the problem is worth posing independently of the instruction it is intended to convey and whether the problem would matter outside of school. He also asks whether the question "*naturally* suggests itself within some situation of personal experience," and it is this sense of a naturally occurring question that arises from real (and potentially shared) experiences of human curiosity we mean to invoke when we describe natural mathematical questions. By contrasting the rich, varied experiences of teachers as they posed and solved a natural question with the narrow, algorithmic representation of that question in a contextualized problem, our goal is to highlight that radical disruptions of the contextualized problem-genre may be necessary to create more authentic mathematical problem posing situations for students.

Conclusion

Our goal in reporting teachers' work on the horizon problem at the sandbar was to highlight how attending to scale and perspective can provide underpinnings for mathematical modeling activities. The third person, microspace representation of the horizon problem shown in Figure 12 is a stark departure from the first-person, macrospace experience these teachers' had when looking out to the horizon. As we illustrated, even seasoned mathematics teachers found it challenging to switch between scales and perspectives when engaged in the work of modeling a situation mathematically. Yet, we see in their efforts the kind of productive struggle that draws mathematical activity closer to practices found in the discipline of mathematics and for these reasons we see affordances for providing such opportunities to students as well as teachers.

To be clear, we see the sandbar as just one interesting place, among many, that can be fruitful for cultivating the practice of asking and investigating natural questions. We recognize that some such places may be challenging for teachers and students to get to. And yet even without visiting the sandbar itself, it is possible to capture some semblance of what it is like to be there in the description of the place we provided above. Furthermore, immersive virtual environments can provide access to hard-to-reach places that could be the basis for generating questions that could be turned in to problems. Rather than providing students with already mathematized situations in the form of diagrammatic models or optimization problems, teachers could offer them opportunities to engage in similar kinds of productive struggle by inviting students to consider the characteristics of different places and asking them to engage in inquiry about natural questions that come from taking various perspectives on such spaces.

Notes

[1] *Oolitic* sand is smooth, egg-shaped sand that forms in the ocean itself, as the result of chemical or biological mechanisms that are still being studied. See https://www.hakaimagazine.com/news/the-mysterious-origin-of-oolitic-sand/

[2] Pseudonyms

[3] This report is based on a research and professional development initiative supported in part by a gift to the University of Michigan. The views expressed herein are those of the authors and are not necessarily those of the university or the donor.

[4] Referring to *The Little Prince* by Antoine de Saint-Exupéry.

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