# REVISITING VAN HIELE

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When I first developed an interest in young children's understandings of shapes, I was led to the van Hiele model of geometrical thinking. That was when my brother had his first child, so at about the time I was observing my niece learn, I found myself reading *Structure and Insight* (van Hiele, 1986). My observations of my niece are partly why I rebelled when I got to the point where van Hiele states that "thinking without words is not thinking" (p. 9). How can there be learning without thinking? And how could my niece be learning when she was not thinking, oblivious as she was of any words? Now I have two daughters of my own. During infancy, most of the time I had no idea what they were thinking, but I was certain that they were thinking because I watched them learn new things, day in, day out.

My interest in young children and shapes originally arose from the numerous opportunities I had of observing young children in early childhood education settings, exhibiting knowledge about shapes in numerous ways (through construction, drawing, gestures), which they seemed unable to express in words. For me it was obvious that young children knew a lot about the structure of shapes even though they could not express their understandings verbally. Their inability to express themselves verbally did not lead me to question their understandings, but to grow an interest in investigating the nature of thinking without words.

Now, more than a decade after I first encountered the van Hiele model, I still feel the need to return to the matter. Despite a growing body of research investigating new epistemologies and ways of thinking and constructing meaning (*e.g.*, through construction, gestures, computer programming), the picture presented in the literature on young children and shapes seems to have stayed intact. I still feel the need to argue my opposition to the consensus, which has dominated the literature for almost three decades, that young children view shapes as a whole and pay no attention to shape structure and that geometrical thinking can be described through a hierarchical model formed by levels.

My first reaction when I got to p. 9 of *Structure and Insight,* was to close the book, put it aside and refuse to read any more. I soon realized, however, that I could not claim to be studying children's understandings of shapes without reading the whole thing. It turned out to be a wise move. Reading van Hiele's work led to the surprising realization that his theory was full of ideas that could open the same door that was blocked by the rejection of non-verbal thinking as a "real" form of thinking. This was due to the paradoxes that characterize his theory. I discovered a van Hiele who has a lot to offer to a more constructive approach to geometrical thinking; a van Hiele who could offer many new perspectives on the investigation of thinking without words.

In this article, I invite you to join me in revisiting van Hiele. In the following sections, I will present two van Hiele paradoxes and the ways in which these paradoxes seem to have influenced van Hiele-based research. I conclude this visit by trying to address the question of whether the van Hiele model is hierarchical in nature and by discussing the implications of describing thinking through hierarchies.

## Van Hiele Paradox A: thinking with and/or without words

In an attempt to differentiate his model from Piaget's developmental hierarchy, van Hiele (1986) states that language is the only reliable route to children's thinking and states that "thinking without words is not thinking" (p. 9), a statement that seems to have been placed by many, implicitly or explicitly, in the center of his model. Nevertheless, further on in his book, van Hiele takes a positive stance towards non-verbal thinking. A substantial part of his book is devoted to emphasizing the value of intuition, a fact that has been eliminated by most van Hiele-based research. So here is the first van Hiele paradox: on the one hand, in his opening claim, van Hiele acknowledges verbal thinking as the only valuable way of thinking, while on the other hand, he supports intuition, which he defines as a way of thinking without words. Van Hiele analyses in detail what he calls "the intuitive foundations of mathematics" and strongly supports the idea that "all rational knowledge begins with intuitive knowledge (p. 125). He explains how "the intuitive approach begins with the presentation of a structure" (p. 117) and supports the point of view that "man is able to react directly—without the intermediary of a language—to visual structures" (p. 127).

The inability of van Hiele-based research to detect his conflicting claims concerning verbal and nonverbal thinking, and thus a more dynamic and pioneering side of the van Hiele model, led to the reinforcement of a research culture in which verbal data were used to assign children to levels. Shaughnessy and Burger (1985) used children's verbal claims to conclude that the van Hiele thinking levels are adequate to describe children's geometric thinking. For example, according to Shaughnessy and Burger, children that belong to Level 0 (visualization), see a geometric figure as a whole and pay no attention to its components. It is children's verbal descriptions that characterize them as belonging to this stage. These descriptions are "purely visual": "If asked why he or she called a figure a rectangle, a student might reply, 'Because it looks like a rectangle. It is like a window or a door'" (p. 420). Clements *et al.* (1999) followed a similar approach (they analysed children's verbalisations) and thus reinforced the idea that children, at the first level of geometric thinking, view shapes as wholes and see no relationship between different shapes or between the properties of shapes. These descriptions of how (young) children view shapes dominate in many later publications by Clements and his colleagues (Clements & Sarama, 2000, 2009; Sarama & Clements, 2009), for they insist on describing the visual level as the level where children identify shapes based on their appearance:

> […] at first children can't distinguish between one shape and another. Later, they can, but only visually–they recognize shapes as wholes. They might call a shape a "rectangle" because it looks like a door. They do not think about the defining attributes or properties of shapes […] Not until later, often middle school or later, do students see relationships between classes of figures. For example, most children incorrectly believe that a figure is not a rectangle because it is a square. (Clements & Sarama, 2009, p. 124)

This type of research has had deep consequences for the portrayal of children's understandings which dominate research even today. Making lists of children's misconceptions has led to a restricted and limited understanding of what children know and are capable of. Burger (1985) presents one misconception as a natural consequence of another when he claims that since many children think of shapes as a whole without explicit reference to their components, they consider shapes such as squares and rectangles to be completely different. This is indeed a neat and tidy point of view; but does it do justice to children's minds?

Van Hiele scholars have not only based their research on the belief that "thinking without words is not thinking", they have also ignored van Hiele's extensive reference to intuition and his conviction that "all rational thinking begins with intuitive knowledge". Even though van Hiele's well-formed theory of intuition could be an important contribution to the existing literature on the role of intuition in mathematical and scientific thinking (diSessa, 2000; Fischbein, 1987), it has been totally ignored. In an extensive review of literature on "geometry and space", Clements and Battista (1992) make no reference to van Hiele's ideas about intuition, even though they do have a section on intuition.

## Van Hiele paradox B: subjectivity and/or objectivity

The second paradox I have detected in *Structure and Insight* is also connected with van Hiele's claim that "thinking without words is not thinking". This claim constitutes in itself a paradox. Van Hiele somehow admits that there is such a thing as thinking without words but he finds it difficult to label it as thinking. This sentence makes sense only if what is before the "not" is different in some way from what comes after it. One way of completing the sentence is by using, as van Hiele (1986) did at some point, the word "real". Even though one might argue that it is not clear what van Hiele meant when using the world "real", the choice of this word led scholars to fit van Hiele within the culture of "misconception theory", which overemphasizes the importance of making a list of all that children think or know and which, compared to "real" knowledge, is wrong.

However, whereas van Hiele (1986) refers to "real" thinking, he maintains a positive stance towards the existence of a more subjective side to mathematics. Once more, I refer to how van Hiele attempts to differentiate himself from Piaget:

> Piaget was not aware that those [theoretical] concepts are only human constructions, which, in the course of time, may change. So development with some theory as a result always must be understood as a learning process influenced by people of that period. (p. 6)
>
> The conclusion "Every square is a rhombus" is not a result of maturation, it is the result of a learning process. An intelligent person need not conclude that every square is a rhombus; this is only submission to a traditional choice. In some Greek philosophies, a square could not be a rhombus, for it had some properties a rhombus could not have. (p. 50)

If it were not for the claim in favor of the existence of "real" thinking, van Hiele's epistemological stance would have been very clear. In his meticulous philosophical discussion on objectivity and subjectivity, van Hiele (1986) accepts the dual nature of mathematics:

> Some judgments are discussable and testable within a large, private group. Within the group, these judgments have great objectivity. Still there is the possibility that other people outside the group are not willing to accept the presumptions of the group and therefore do not agree with the judgment. Norms that have been valid for centuries now appear not to suffice in newer applications. Statements that were unassailable a century ago are now considered out of date and incorrect. (p. 218)

Most importantly though, van Hiele (1986) pursues this discussion and declares that for the practice of teaching mathematics, "the 'better' mathematics has its foundation in not yet mathematical knowledge" (p. 219). Van Hiele, therefore, accepts that from a pedagogical point of view, it would be more fruitful if mathematics were treated as a subjective field, a lesson we get from the subject's phylogeny.

It has been stressed in reviews of van Hiele's work (Clements & Battista, 1992; Hoffer, 1983), that there are two important aspects of the van Hiele model; the levels of thinking and the phases of learning. In differentiating himself from Piaget, van Hiele states that Piaget's stages are age-related, whereas his level system is linked with a learning process. For van Hiele (1986), "this difference is of great importance" (p. 56). The levels of thinking and the phases of learning are not two autonomous aspects of the model but are indistinguishably related. Pegg and Davey (1998) see many commonalities between van Hiele and Vygotsky:

> […] van Hiele offered no developmental timetable for growth through the levels. In particular, he questioned the notions of growth being linked with biological maturation. Instead, in ways that have much in common with Vygotsky (1978), he saw development in terms of students' confrontation with the cultural environment, their own exploration, and their reaction to a guided learning process. (p. 112)

Similarly, Clements and Battista (1992) recognize that:

> the phases of instruction are inextricably connected with the levels of thinking, and potentially more important for education; therefore, it is surprising and unfortunate that little research other than the van Hieles' has examined the phases directly. [...] Additional studies are sorely needed, especially given unresolved questions and concerns regarding the phases. (p. 434)

Of course, we should keep in mind that we are now talking about another era, when different learning tools and "languages" are available and different learning cultures are constructed. Van Hiele's acknowledgement that different learning cultures and philosophies might lead to different, equally acceptable mathematics reinforces the need to revisit his work in the light of new epistemologies which, for example, the era of the computer has brought.

As in the case of Paradox A, van Hiele-based research has failed to detect van Hiele's position on subjectivity and objectivity. Van Hiele has often been interpreted as supporting a misconception theory perspective. The study by Burger and Shaugnessy (1986) is an example of a project which had van Hiele at the core of its theoretical background and aimed to understand children's geometric thinking by locating their misconceptions, in an effort to classify them into a level system. Van Hiele-based studies have not taken into consideration van Hiele's support for subjectivity, or the role that van Hiele attributed to the learning phases. This omission has not only eliminated an important aspect of van Hiele's theory, but has misleadingly led researchers to see the assessment of the thinking levels as an independent component, thus diminishing the van Hiele model to a developmental hierarchy, similar to that of Piaget.

This approach has led to paradoxes within van Hiele-based research. The fact that many studies were driven to the conclusion that the "assignments to levels did not seem to be strictly related to age or to grade category" (Burger & Shaughnessy, 1986 p. 43) did not seem to lead researchers to question their methodology. There was instead a continuous attempt to evaluate the van Hiele level system (Burger & Shaughnessy, 1986; Gutierrez *et al.*, 1991; Mayberry, 1983) as independent from the setting in which it occurred. As Noss and Hoyles (1996) state, however, "the acknowledgment that setting is intimately bound up with performance on mathematical tasks precedes the widespread acceptance of the situated view of cognition" (p. 30). Through the methodology followed by these researchers, children's thinking was characterized by a sense of sameness and uniformity. Researchers such as Burger and Shaugnessy (1986) and Gutierrez *et al.* (1991) failed to incorporate "the notion of individual differences into the van Hiele model" (Pegg & Davey, 1998, p. 114), even though this was their original intention. Pegg and Davey's (1998) methodological suggestion that "what is needed is a closer examination of student understanding with an eye to the seeking and documenting of diversity" is based on their conviction that "there is clearly great variability in the ways students learn, the structures on which students build their understandings, the roles teachers play, and the techniques teachers use" (p. 115).

As a result of neglecting paradox B, van Hiele has been mistakenly, but most importantly, fruitlessly and misleadingly, placed on the wrong side of the discussion concerning the nature of mathematical knowledge. I emphasize the need to develop new epistemological and methodological perspectives in relation to children's understanding of shapes, by emphasizing important dynamic parts of the van Hiele model that have, until now, been eliminated and ignored. In an effort to revisit van Hiele from a different perspective, it is important to address the question of whether the van Hiele model is of a hierarchical nature.

## Is the van Hiele model hierarchical?

After struggling to understand van Hiele for a long time, I was relieved when I discovered a paper published in 1999, were he clearly admits that he was wrong in arguing that thinking without words is not thinking:

> Thinking without words is not thinking. In *Structure and Insight* (van Hiele, 1986), I expressed this point of view, and psychologists in the United States were not happy with it. They were right. If nonverbal thinking does not belong to real thinking, then even if we are awake, we do not think most of the time. Nonverbal thinking is of special importance; all rational thinking has its roots in nonverbal thinking, and many decisions are made with only that kind of thought. (p. 311)

Van Hiele's change of mind reinforced my belief that *Structure and Insight* is characterised by paradoxes. In the light of van Hiele's rejection of his original claim, I will attempt to address a question that has troubled researchers in the past: the question of whether the van Hiele model is hierarchical, as well as the implications of assigning a hierarchical dimension to thinking.

Clements and Battista (1992), almost two decades ago, raised a similar issue, in a different and quite interesting manner: do the levels form a hierarchy? After a search in existing literature for an answer to this question, they concluded that "the levels appear to be hierarchical, although there remains a need to submit this hypothesis to rigorous tests" (p. 429). In more recent publications, Clements and Sarama (2000) continue to describe geometrical thinking in a hierarchical manner:

> Children at different levels think about shapes in different ways, and they construe such words as square with different meanings. To the pre-recognition thinker, square may mean only a prototypical, horizontal square. To the visual thinker, squares might mean a variety of shapes that "look like a perfect box" no matter which way they are rotated. To a descriptive thinker, a square should be a closed figure with four equal sides and four right angles. But even to this child, the square has no relationship to the class of rectangles, as it does for thinkers at higher levels. (p. 482)

More recently still, even though they emphasise proposed learning trajectories, Sarama and Clements (2009) still talk about children passing through "levels of thinking" and the existence of "developmental progressions".

The question posed by Clements and Battista (1992) was formulated in an interesting manner. Is it not interesting to wonder whether a level system may or may not form some kind of a hierarchy? That is why in the question driving this section, I did not use the word "level". The question is whether the van Hiele model (and not the van Hiele level system) is of a hierarchical nature. Thus, in order to question the hierarchical nature of the van Hiele model, one must first question the van Hiele model as a level system. Of course, van Hiele himself talked about a series of levels. But as I argued earlier, his theory was characterised by a number of paradoxes which allowed van Hiele-based research to interpret his work in a certain way and to eliminate other important aspects of his theory. My argument is that if we focus on the eliminated aspects of the van Hiele model we can assign a different interpretation to his theory that moves away from the idea of levels and allows us to proceed with a more constructive view of how children think about shapes.

Clements and Sarama (2000) describe each level as a lower form of thinking compared to higher levels of thinking, characterise each level by what children can(not) do and explain that as they go up the level system, children know more. This is clearly an approach that describes the van Hiele model as a hierarchy. The continuous use of the word "level" does not leave any room for doubt that, for Clements and Sarama, the van Hiele model is hierarchical. The only way to develop an alternative perspective on the van Hiele model that is not hierarchical is to move away from the idea of levels. If we remove any reference to levels from claims made by van Hiele scholars, what will we be left with? According to Clements and Sarama (2000):

> Children at different levels think about shapes in different ways, and they construe such words as square with different meanings. (p. 482)

If we remove any reference to the levels we are left with the following claim:

> Children think about shapes in different ways, and they construe such words as square with different meanings.

Moving away from the assumption of a level system makes a big difference. It allows us to think not about higher and lower forms of thinking, but different forms of thinking. The idea that children (and adults) may think about shapes in different ways and assign different meanings to a specific geometric concept is the way one could think about geometric understanding within the sphere of epistemological pluralism; the idea of "accepting the validity of multiple ways of knowing and thinking" (Turkle & Papert, 1992). This new perspective on the van Hiele level system is more compatible with some of van Hiele's hidden convictions and can better explain the common findings of most van Hiele-based research. Van Hiele-based research has concluded that the different thinking levels are not related to age or maturation, that children may move backwards or forwards between levels and might simultaneously belong to different levels. So if we substitute the word "levels" with the phrase "diverse modes of understanding", we can support the point of view that children may think in different ways independently of their age and may think in different ways simultaneously.

This new perspective allows us to place van Hiele among those who have proposed the existence of multiple ways of thinking, an approach that has opened up new perspectives on understanding "understanding". It allows us to provide a dynamic alternative interpretation of the van Hiele theory, which shares the same belief as Noss and Hoyles (1996) in the existence of "diverse kinds of mathematics", the same belief as Turkle and Papert (1992) in the existence of "epistemological pluralism" and the same belief as diSessa (2000) in the existence of ways of knowing "beyond the stereotypes of knowledge we have culturally institutionalized in school and even in our common sense" (p. 71).

In a study referred to earlier, Clements *et al.* (1999) analyzed children's verbalizations to describe "children's concept of shapes" but were led to the conclusion that there is a need for research using manipulatives and construction tasks. The extensive work conducted by Clements and his colleagues since then makes use of manipulatives, construction and technology. Nevertheless the idea that children (at an earlier stage) view shapes as wholes and pay no attention to shape structure stays intact. This will not change as long as we investigate children's understanding in conjunction with a level system. The question is what will happen if we investigate children's understandings of shapes within a level-free research culture; what will happen if we step outside of the level-culture box?

In my doctoral research (Papademetri, 2007), the aim was to describe the ways children express their understandings about shapes within and in correlation to specific settings. I asked 52 5-year-olds to construct a square by selecting wooden sticks from a pile (construction task) after successfully completing a classification task where they had to identify squares among a variety of cut-out shapes and explain why these shapes belong in the same group (description task). Thus children's understandings were investigated within two different settings (the description task and the construction task) where different tools were involved and different techniques were employed.

In designing the description task, I borrowed methods from other van Hiele-based studies (Shaughnessy & Burger, 1985; Clements *et al.*, 1999), which tried to evaluate children's understandings of shapes through their utterances. To be more precise, after completing the classification task, the children were asked, as in other studies: "How did you know these were squares?" "Why do you say these are squares?" After analyzing the data from the children's involvement in the description task, I concluded that:

> if this study was to follow the same methodological framework as Shaughnessy and Burger (1985) it could reconfirm the existence of van Hiele level 0 […] The difference between this study's methodology and the methodology followed by Shaughnessy and Burger (1985) lies in the fact that here the aim is to describe the ways children express their understandings about shapes within and in correlation to a specific setting and not to evaluate children in order to place them in levels independently to the setting in which they express their understandings. That is why the existence of level 0 cannot be reconfirmed at least not in a way similar to most existing studies. But the study's findings can support the position that within a setting

restricted to classification, recognition and description tasks children often use appearance based descriptions (p. 114).

The data analysis from the children's involvement in the construction task reconfirmed my opposition to the idea "that children's limited and often appearance-based descriptions of shapes indicate that children view shapes as a whole and lack understanding of shape structure" (Papademetri, 2007, p. ii). Based on the choices they made at the beginning of their attempt to construct a square, the children exhibited rich intuitive structural understanding about squares. The majority of children showed an understanding of the two properties which, in combination, distinguish the square from other quadrilaterals. To be more precise 57% of the children (30/52) exhibited a combined understanding of the fact that a square has four equal sides and four right angles (beginning their attempt by intentionally selecting equal sticks and arranging them so as to construct right angles). The percentage of children who exhibited understanding of only one of these two characteristics was much lower: 19% (10/52) exhibited understanding of the fact that a square has right angles (beginning their attempt by randomly selecting sticks of different lengths and arranging them so as to construct right angles) and 12% (6/52) exhibited understanding of the fact that a square has four equal sides (beginning their attempt by intentionally selecting equal sticks). Finally, 12% (6/52) of the children simply exhibited understanding of the fact that a square has four sides (beginning their attempt by randomly selecting four sticks). The previous descriptions were actions which the children did at the beginning of their attempt to construct a square and that stayed intact until the end of their attempt, which in most cases also included experimentation. At the end of their attempt 62% (32/52) of the children ended up with a square, 19% (10/52) with an oblong and 19% (10/52) with a quadrilateral with only some or no right angles and equal sides.

In my doctoral study, I describe in detail the 52 different routes that the children followed in their attempt to construct a square, thus providing a very detailed description of young children's intuitive understandings and the way these children build new understandings on their original intuitions. In describing the children's construction of meaning there is strong evidence that in most cases the children were not thinking in words since they did not have the vocabulary that could accompany their actions. It seems that the picture of how young children think about shapes changes radically if we move away from the idea of levels and hierarchies and allow children to express themselves through alternative means (*e.g.,* construction).

## Discussion

In the preface of his book *Intuition in Science and Mathematics,* Fischbein (1987) states that because of the vague and tacit nature of intuitive knowledge, "rich sources of information based on experimental findings have been ignored by most of the theorists" (p. x). This is exactly how I felt when I first started reading van Hiele-based research. As I stated at the beginning of this article, I originally became interested in young children's understandings of shapes because of the numerous opportunities I had to observe young children exhibiting structural knowledge about shapes "without being able to tell". Preschool children construct shapes with accuracy but "have no knowledge" (if we measure knowledge by verbal language) of concepts like angle, right angle, parallel lines *etc*. I have many times seen children make the right choices when constructing a square; like "intuitively" selecting 4 equal sticks out of a set of numerous sticks of different lengths (the term "intuitively" is intentionally used here as an alternative to the phrase "without real thinking" that others might have used in this case). What would have happened if I had tried to fit these children into "boxes"? Would I place these children in boxes based on their correct choices or their failure to express their understanding in words? The simplistic, linear description of how children think about shapes within van Hiele-based research did not fit with my everyday experiences, which indicated that thinking is more complex and diverse.

This idea of data being discarded keeps reappearing within research concerning children's understanding of shapes. Within research cultures which aim to categorize children into level systems, important, rich understandings which children express about the properties of shapes are ignored and children are judged mostly in relation to what seems wrong about their replies. It is interesting how the same data can be interpreted in different ways within different frameworks. Whereas Lehrer *et al.* (1998) interpret children's answers like "it's pointy" as evidence that children focus on attributes that result from a shape's properties, Burger (1985) uses the same utterance ("it's pointy") as evidence that "many children rely on imprecise qualities to identify shapes" (p. 53). Interpretations like the one by Lehrer *et al.* (1998) are quite rare within research concerning young children and shapes.

This discussion reminds me of the way Davis and Sumara (2000) talk about their years as teachers in the 1980s. They describe their experience of completing year plans which were based on "orderly, sequential, grid-like structures" (p. 822). They acknowledge that this kind of planning is "easy to make", "commonsensical, familiar, reassuring". These are exactly the assets I recognize in research studies were children are assessed in order to be placed in "boxes", which are arranged in "orderly, sequential, grid-like structures". In contrast to the way Davis and Sumara (2000) reflect on their years as teachers, they recognize that:

> projects that been characterized more in terms of Euclidean forms (*i.e.,* lines, grids, spirals and so on) might be seen as incommensurate with the diversity and complex texture of activity present in any learning setting. Such imperatives as the pre-specification of learning outcomes and the articulation of comprehensive lesson plans, we suggest can eclipse the richness embodied in any moment of engagement with a subject matter. (p. 830)

In this article, I have shared the story of how I came to understand van Hiele and van Hiele-based research. I have argued my opposition to the consensus that dominates the literature that geometrical thinking can be described through a hierarchical model formed by levels. I have argued that any attempt to fit children into linear level systems consequently leads to the discarding of the richness and the complex texture (to borrow Davis and Sumara's words) of children's understandings. In the light of the paradoxes which I have identified in van Hiele's theory, I have argued in favor of an alternative interpretation that moves away from the idea of levels.

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Women, perhaps the majority of women, prefer to discuss moral problems in terms of concrete situations. […] Faced with a hypothetical moral dilemma, women often ask for more information. It is not the case, certainly, that women cannot arrange principles hierarchically and derive conclusions logically. It is more likely that they see this process as peripheral to or even irrelevant to moral conduct. They want more information, I think, in order to form a picture. Ideally, the need to talk to the participants, to see their eyes and facial expressions, to size up the whole situation. Moral decisions are, after all, made in situations; they are qualitatively different from the solution of geometry problems.

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