Contexts, Goals, Beliefs, and Learning Mathematics*

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*The research reported in this paper was supported by a grant from the Spencer Foundation The opinions expressed are solely those of the author and do not necessarily reflect the position of the Spencer Foundation.

The last few years have witnessed an almost exponential growth in research into students' mathematical beliefs. This research emphasis has been sparked, in part, by "horror stories" of the apparently nonsensical things that students at all grade levels do when they attempt to solve mathematical problems. These include examples of students with relatively sophisticated concepts who use low level, primitive methods in certain situations. Further evidence for the importance of beliefs comes from findings that demonstrate that students at higher grade levels often use superficial methods or accept "impossible" answers rejected by children at lower grade levels.

At the primary level, for example, the relational approach to arithmetic word problem solving typically evidenced by first graders is "replaced by the superficial analysis of verbal problems found in many older children as they attempt to decide whether to add, subtract, multiply or divide" [Carpenter, Hiebert, & Moser, 1983, p. 55]. The older children were second and third graders. Further, failure to relate tasks such as 6+7, 6+8, 6+9, . , 6+15

> does not imply that the principle is not known. For example, some children might have refrained from using their knowledge of principles to short-cut computations because they felt it was "cheating." Indeed, a number of children seem to have interpreted looking at the used pile and using a short-cut as "naughty" ... This attitude seemed to persist despite the efforts to counter it. [Baroody, Ginsburg, & Waxman, 1983, pp. 167-168]

Similarly, at the high school and college level, students who are quite capable of producing formal, deductive proofs to solve geometry problems typically fail to generate such arguments when they are asked to solve geometry problems that involve making constructions [Schoenfeld, 1985]. These and numerous other examples indicate that students' apparently bizarre behaviors frequently cannot be accounted for solely in terms of conceptual limitations. To use Schoenfeld's phrase, there is a need to move beyond the "purely cognitive."

The primary purpose of this paper is to advance the hypothesis that students reorganize their beliefs about mathematics to resolve problems that are primarily social rather than mathematical in origin. Thus, I am suggesting that research into students' beliefs is complemented by research into the social aspect of mathematics instruction, at least at the level of classroom social interactions. I will first argue that beliefs are an essential aspect of meaning making in general and of mathematical meaning making in particular. Attention will then turn to the influence of mathematics instruction as a socialization process on students' beliefs.

The contextuality of cognition

I will draw on work from a variety of disciplines to support the contention that cognition is necessarily contextually bounded The discussion will also illustrate the intimate relationship between contexts, goals, and beliefs.

Contemporary philosophy of science provides an initial example of the crucial role that context plays in problem solving There is a growing consensus that the scientist's actions are guided by a largely implicit contextual framework, be it called a world view or paradigm [Kuhn, 1970], a research program [Lakatos, 1970], or a research tradition [Laudan, 1977]. Kuhn hinted at the essential role played by paradigms when he observed that "in the absence of a paradigm or some candidate for a paradigm, all the facts that could possible pertain to the development of a given science are likely to seem equally relevant" [p. 15]. In other words, a paradigm or world view restricts the phenomenological field accessible to scientific investigation. The general context within which the scientist operates therefore constrains what can count as a problem and as a solution [Barnes, 1982]. Analogously, the general contexts within which children do mathematics delimits both what can be problematic and how problems can be resolved [Cobb, 1985a].

The second example, cited by Schoenfeld [1985], demonstrates that the psychological context within which one gives a situation meaning can radically affect subsequent behavior Kahneman and Tversky [1982] presented subjects with the following personality sketch.

> Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Respondents were then asked which of the following two statements about Linda was more probable: (A) Linda is a

For the Learning of Mathematics 6, 2 (June 1986)
FLM Publishing Association, Montreal, Quebec, Canada

bank teller; (B) Linda is a bank teller who is active in the feminist movement. It was found that 86% of undergraduate students and 50% of psychology graduate students judged the second statement to be more probable.

Kahneman and Tversky argued that these judgements ignore baseline statistics and are therefore not perfectly rational (i.e. the set of all bank tellers who are active in the feminist movement is a subset of the set of all bank tellers). They suggested the subjects who chose (B) used what they called the representativeness heuristic by which judgements are based on the degree to which the characteristics of X (e.g. Linda) match archetypal characteristics of Class A. A consideration of the subjects' intentions and purposes extends the analysis and allows us to understand why they might have used this heuristic. As Kahneman and Tversky noted, the subjects who chose (B) were not attempting to play the statistics game at all—they were operating in a different context. To play the statistics game successfully, one must learn to view individuals as members of classes. Save for the properties that specify their class membership, their individuality is of no interest. However, the subjects who chose statement (B) seemed to re-present Linda's individuality—their goal was to build a detailed re-presentation of Linda. And the process of modeling, be the product a scientific theory or a re-presentation of another person, involves searching for that which fits while adapting to the incongruous. Knowing that Linda is a bank teller is, at best, neutral with respect to their re-presentations of Linda. Although it is possible that she could be a bank teller given what they know about her, they would not have predicted it. In contrast, knowing that she is active in the feminist movement fits their re-presentations—this is what they would expect or, in other words, it is probable.

This analysis indicates that to merely say that the subjects' behavior is not completely rational misses an important point. Within the constraints of the modeling context, their behavior was perfectly rational—they know from past experience that it works. It is the process by which we each construct models of others. The students' failure was not necessarily due to a lack of understanding of elementary probability. Instead, it reflected the goals they established and the contexts within which they interpreted the term "probable."

The next example drawn from research on everyday reasoning was also cited by Schoenfeld [1985]. Again, I believe that more can be said. Perkins, Allen, and Hefner [1983] gave adult subjects two of the following four questions and asked them to reach a position.

> Would restoring the military draft significantly increase American's ability to influence world events?
>
> Does violence on television significantly increase the likelihood of violence in real life?
>
> Would a proposed law in Massachusetts requiring a five-cent deposit on bottles significantly reduce litter?
>
> Is a controversial modern sculpture, the stack of bricks in the Tate Gallery, London, really a work of art?

Subjects were aksed to state and justify their position and follow up questions were asked. It was found that approximately three fourths of the flawed arguments did not reflect faulty reasoning. Instead, they resulted from a failure to conduct a sufficiently elaborate analysis by, say, considering various counterexamples. Further, most subjects could produce counterexamples to their own arguments when asked to do so. Perkins *et al* concluded that what they call naive reasoners

> ... act as though the test of truth is that a proposition makes intuitive sense, sounds right, rings true. They see no need to criticize or revise accounts that do not make sense—the intuitive feel of it suffices. [p. 186]

The above characterization of the everyday reasoning context can be compared with what might be called the *context of academic reasoning*. When we, as academic reasoners, write to produce publications, to share our ideas informally, or to clarify our own thinking we anticipate possible counterarguments because we know from experience that they will almost certainly be offered. We also buttress our arguments against potential criticisms when we have discussions with colleagues. A crucial feature of these situations is that critical scrutiny is the rule rather than the exception. Further, we usually have time to reflect and conduct an internal dialogue before translating thought into action.

In contrast, the overall goal of everyday reasoning is not to construct compelling arguments in the face of potential scrutiny. Instead, it is to act so that the individual can achieve his or her particular goals in a specific situation.

> In many cases, the more systematic and precise approach would result in less effective practical action since it would take more effort to develop and would be less flexible in the face of unanticipated opportunities or constraints. Effective practical problem-solving may proceed by using tacit knowledge available in the relevant setting rather than by relying on explicit propositions. [Rogoff, 1984, pp. 7-8]

Consequently,

> what is regarded as logical problem-solving in academic settings may not fit with problem-solving in everyday situations, not because people are "illogical" but because practical problem-solving requires efficiency rather than a full and systematic consideration of alternatives. [p. 7]

The error made by the subjects of the Perkins *et al* study was not that they failed to construct carefully reasoned arguments per se. Instead it was that they failed to think of producing such arguments—they did not anticipate that their arguments would be scrutinized and, consequently, they failed to play the *academic reasoning game*.

The examples given thus far illustrate that behaviours that might initially be dismissed as irrational begin to make sense when the contexts within which the subjects operated and the goals they attempted to achieve are considered. The focus on the subject's rationality is compatible with

Smedslund's [1977] contention that

> the only defensible position is always to presuppose logicality in the other person and always to treat his understanding of given situations as a matter of empirical study. From this point of view, people are always seen as logical (rational) given their own premises, and hence behavior can, in principle, always be understood. This also applies to small children. [p. 3]

The work of Wilker and Milbrath [1972] also illustrates that seemingly irrational behaviors have an underlying rationality within the goal-directed contexts that frame them. In contrast to Kahneman and Tversky and Perkins et al., Wilker and Milbrath attempted to account for behavior that is produced spontaneously by subjects in the course of their everyday lives. The phenomenon of interest to these political scientists was that "large numbers of people show by questioning on policy issues that they cannot link the policy outcome that they desire with the stands of public officials that they support" [p. 41]. Such people will explain why they disagree with the policies of a particular political candidate and yet vote for that candidate. From the perspective of the "informed citizen" who engages in political activity with the overall goal of pursuing such tangible things as tax reform, improving U.S. military power, or increasing environmental controls, this behavior seems illogical.

Wilker and Milbrath suggested that many citizens' political activity is not carried out within the context defined by the instrumental pursuit of desired policy goals. These individuals are playing a different game—their political action is expressive rather than instrumental. In particular, election campaigns

> give people a chance to express discontents and enthusiasms, to enjoy a sense of involvement (in the political process). This is participation in a ritual act, however; only in a minor degree is it participation in policy formation. [Edelman, 1964, p. 3]

"By voting an individual "proves" the veracity of the civic myths that give him a sense of well-being" [Wilker & Milbrath, p. 53]. Their action corroborates the belief "that the order is a rational one, that we all control our destiny, and that the world is indeed a friendly place" [p. 53]. In other words, their overall goal is to be good citizens. And by voting, these individuals "mobilize the body of myth that lies behind society's definition of what it means to be a good citizen" [pp. 56-67].

The work of Wilker and Milbrath and the two preceding examples demonstrate that an analysis of an individual's goals, intentions, and purposes is a crucial feature of contextual analysis. To infer the context within which an individual is operating is to infer the overall goals that specify the framework within which action and thought is carried out. This conclusion is reminiscent of Lewin's [1951] field theory, a field being a close relation of Kuhn's world view.

> Field theoreticians contend that a field can only be described by referring to the goals, purposes and needs that are involved. These purposive factors are the prime motivating forces and provide the impetus for structuring the field. [Wilker & Milbrath, p. 51]

The relationship between goals or purposes and beliefs has been clarified by the pragmatist philosophers Peirce and James. They argued that a "willingness to act, and, in the case of James especially, the assumption of some risk and responsibiity for action in relation to a belief, represent essential indices of actual believing" [Smith, 1978, p. 24]. The general goals established and the activity carried out in an attempt to achieve those goals can therefore be viewed as expressions of beliefs. In other words, beliefs can be thought of as assumptions about the nature of reality that underlie goal-oriented activity. With regard to Wilker and Milbrath's analysis, for example, to say that an individual's overall goal when voting is to be a good citizen clarifies the individual's beliefs about the political process. The individual believes, at least implicitly, that the democratic process is a rational one and that he or she can substantiate this rationality by voting. The alternative possibility of pursuing policy goals does not seem to arise—it is a separate context.

To summarize the discussion thus far, firmly held beliefs constitute, for the believer, current knowledge about the world. They are a crucial part of the assimilatory structures used to create meaning and to establish overall goals that specify general contexts. The act of formulating a goal immediately delimits possible actions; the goal, as an expression of beliefs, embodies implicit anticipations and expectations about how a situation will unfold. For example, students who have constructed instrumental beliefs about mathematics [Skemp, 1976] anticipate that future classroom mathematical experiences will "fit" these beliefs. They intend to rely on an authority as a source of knowledge, they expect to solve tasks by employing procedures that have been explicitly taught, they expect to identify superficial cues when they read problem statements, and so forth. Alternative ways of operating do not occur to them. Consequently, an examination of the situations in which a student's expectations are corroborated and contradicted by experience provides valuable information about his or her beliefs. If, for example, we are interested in a student's beliefs about the role of the mathematics teacher, we might focus on teacher-student interactions. Observations of the student asking the teacher to verify his or her work as soon as an answer is produced indicate that the student regards the teacher as an authority—the student expects the teacher to say whether or not the answer is correct. This inference would be further substantiated if the teacher responds by attempting to initiate a Socratic dialogue and the student shows irritation or frustration.

To conclude this discussion of context, I will consider two further examples that deal with mathematical behavior. The first is drawn from the work of Lave, Murtaugh, and de la Rocha [1984]. These researchers compared adults' ability to solve arithmetical problems that arose while they were shopping for groceries in a supermarket with their performance on a paper-and-pencil arithmetic test. The subjects' "scores averaged 59% on the arithmetic test, compared with a startling 98% — virtually error free— arithmetic in the supermarket" [p. 82]. Lave *et al* conducted a correlational analysis between these scores and certain demographic variables and concluded that

> arithmetic problem-solving in test and grocery shopping situations appears quite different, or at least bears different relations with shoppers' demographic characteristics. Analysis of the specific procedures utilized in doing arithmetic in the supermarket lends substance to this conclusion. [p. 83]

It would seem that the test and grocery shopping situations were separate contexts for most of the subjects. One can speculate that they approached the arithmetic test with the intention of solving tasks by attempting to recall and use procedures they had been taught in school. In the shopping situation, arithmetical problems arose when the subjects could not immediately make a practical decision such as which of two items is the better value. And in these problematic situations, they did not try to recall a general method. Instead, they used self-generated methods that were tailored to the concrete decisions that had to be made. In other words, the shoppers' practical arithmetic procedures were constructed within the constraints of specific contexts narrowly defined by their on-going practical activity. Their overall goal was simply to select the items they wanted by making appropriate practical decisions. As Scribner [1984] observed,

> skilled practical thinking is goal-directed and varies adaptively with the changing properties of problems and changing conditions in the task environment. In this respect, practical thinking contrasts with the kind of academic thinking exemplified in the use of a single algorithm to solve all problems of a given type. [p. 39]

The final example comes from Schoenfeld's [1985] investigations of the beliefs held by high school geometry students. His data indicate that high school students, almost without exception, take an empirical approach to geometry tasks that involve making a construction. Their sole criterion for accepting or rejecting a solution to problems of this type is the accuracy of the construction — whether or not it looks right. Schoenfeld presented a wealth of evidence to demonstrate that many of these students could make deductive arguments that would have allowed them to deduce an appropriate construction without difficulty. It appears that the construction and argumentation or proof situations were separate contexts for the students. From their perspective, proof had little if anything to do with either discovery — the generation of conjectures — or verification. However, they could invoke proof either when the teacher explicitly demanded it or when they believed that such demands had been made implicitly. This suggests that their overall goal in the proof context was simply to satisfy the expectation of an authority — they attempted to do what they thought they were supposed to do. In contrast, their overall goal in the construction context did not seem to involve a strong reference to others. They wanted to produce constructions that satisfied their criterion of acceptability — they must "look right."

The examples discussed in this section of the paper lend weight to Sigel's [1981] admonition that "*decontextualizing* the child's cognitive development is just as much in error as denying the role of internal processing by the individual" [p. 216]. As Rogoff [1984] put it, "context is an integral aspect of cognitive events, not a nuisance variable" [p. 3].

Meaning-making in context

To say that cognition is context-bounded is to say that beliefs are intimately involved in the meaning-making process. In other words, the elaboration and coordination of contexts is essential to the achievement of the most general of goals, the construction of a world that makes sense. Weizenbaum's [1968] analysis of the process of conducting a conversation and Minsky's [1975] reflection on problem solving processes illustrate this aspect of meaning-making.

> In real conversation, global context assigns meaning to what is being said in only the most general way. The conversation proceeds by establishing subcontexts, sub-subcontexts within these and so on. [Weizenbaum, p. 18]

> At each moment one must work within a reasonably simple framework. I contend that any problem that a person can solve at all is worked out at each moment in a small context and that the key operations in problem solving are concerned with finding or constructing these working environments. [Minsky, p. 119]

It would be misleading to imply that the meaning-making process proceeds in an orderly top-down manner with the establishment of less and less general contexts culminating with the specific context of on-going activity. On purely intuitive grounds, one can argue that activity is always on-going — the individual is always in a specific context. Further, the ability to change perspective (i.e. context) when problem solving illustrates that contextually bound experiences can precipitate contextual reorganizations. This phenomenon is also evidenced by the manner in which most children modify their beliefs about mathematics as they proceed through the elementary school grades.

The view that meaning-making is a top-down process also leads to a contradiction. A problem arises when one asks how general contexts are established. The only alternative to the conclusion that the construction of viable contexts is a trial and error process is that the establishment of contexts is triggered by the perception of cues in an external environment. But the act of perceiving such cues is itself contextually bounded, for otherwise all possible observations would seem equally relevant [cf. Kuhn, 1970]. In short, it is necessary to assume the prior establishment of a specific context in order to account for the establishment of a more general one.

A viable alternative is to regard more general contexts associated with overall goals and the specific contexts of on-going activity as being mutually interdependent. As Lave *et al* [1984] put it, "neither setting [or general context] nor activity exists in realized form, except in relation with each other; this principle is general, applying to all levels of activity-setting [i.e. activity-context] relations" [p 74]. "In short, activity is dialectically constituted in relation with the setting" [p. 73]. This interdependence is exemplified by the dialectical relation between theory and observation. Theory constrains observation but unanticipated observations (i.e. anomalies) can precipitate theoretical refinements which in turn modify what will be observed A similar analysis holds for the relation between a scientist's general world view and specific theories Thus, in line with Kuhn's [1970] thesis, the mutual interdependencies between observations, theory, and world view suggest that the scientist's conversion from one world view to another can be precipitated by anomalies that are either empirical or conceptual in origin [Laudan, 1977]. Current debates in the philosophy and sociology of science indicate that the task of accounting for the specifics of this process of conversion is itself extremely problematic Nonetheless, the general emphasis placed on anomalies indicates that the scientist changes allegiance from one world view to another in an attempt to resolve problematic experiences. Similarly, children's modification of their general beliefs about mathematics, the analogues of the scientist's world view, can be viewed as attempts to deal with problematic situations This contention extends Vergnaud's [1984] argument that concepts are solutions to problematic situations from what might be called the cognitive to the metacognitive domain. Thus, a child's reorganization of his or her beliefs represents an attempt to restore a temporary stability to experiential reality—to make sense of seemingly incoherent situations. This sought-for stability is itself indicated by the ability to achieve one's goals. Since beliefs constrain what can be problematic within a context, the child who reorganizes his or her beliefs and changes more general goals deals with problematic situations by modifying what can count as a problem rather than by striving to find ways to solve the original problem. This implies that the task of understanding why children reorganize their beliefs involves inferring what might have been problematic for them. In the next section, I will suggest that many of the problematic situations that precipitate children's reorganization of their beliefs about mathematics are social rather than mathematical in origin.

Learning in interactive situations

Thus far, the discussion has considered the construction of meaning by the individual without reference to interactions with others. Sociologists of science have alerted us to the limitations of this approach [e.g. Brannigan, 1981; Barnes, 1977, 1982; Bloor, 1976; Knorr, 1980; Knorr-Cetina, 1981]. Barnes [1977] argued, for example, that the generation of scientific knowledge "must be accounted for by reference to the social and cultural context in which it arises" [p. 2]. Similar comments apply to the development of the individual's knowledge. As Sigel [1981] put it, "to understand the source and course of cognitive growth, the detailed analysis of social experience is necessary—it is the interaction that is crucial" [p. 216].

The importance of social interactions as sources of constraints that delimit possible avenues of development is readily apparent when one notes that schools in general and mathematics classrooms in particular are socializing institutions [Bishop, 1985a; Stake & Easley, 1978]. Saxe, Guberman, and Gearhart [1985] offered an analysis of the process by which interactions with others influence the child's development As their analysis focuses on goals, it is highly relevant to the problem of explaining how and why children reorganize their beliefs. Both Saxe *et al* and Scribner [1984] noted that the socialization process involves engaging the child in socially and culturally organized activities. For preschoolers these typically include counting collections of items and setting a table for a meal and, for first graders, joining and partitioning collections and completing addition and subtraction worksheet. By engaging the child in these activities, the more mature member of the culture influences the goals that the child generates This is not to say, however, that others set the child's goals. Instead, "children transform the goal structure of cultural activities into forms that they understand and these forms may differ over the course of early development" [Saxe *et al,* p. 5] The authors' analysis of one-on-one interactions between mothers and their children as they engaged in numerical activities indicates that there is typically a negotiation of the activity. The mothers' overall goal seemed to be the achievement of a coherent interaction or dialogue To achieve and maintain this, they monitored their child's behavior and frequently intervened by giving implicit and explicit cues that included temporarily restricting the activity to a sub-goal of the original activity. They might, for example, initially ask the child to count just one row of an array of items. In general, the goal structure that the child constructed was dynamic and emerged in the course of the interaction as he or she attempted to give meaning to the mothers' actions. More specifically the mother and child each continually adapted to the behavior of the other. The mothers' interventions were adaptive, purposeful responses to observed but unexpected or undesired behaviors. The child, for his or her part, adapted to these interventions by modifying goals, giving the mother opportunities to make further observations. These interactions exemplify the general process that occurs in communicative situations when individuals attempt to construct an understanding of each other and thus establish what Maturana [1980] calls a consensual domain. With regard to school mathematics, the question that arises is how do social interactions in the classroom influence the overall goals that students establish and thus their beliefs? An adequate answer to this question will involve analyzing what, in the classroom setting, might be problematic for students.

D'Ambrosio [1985] called the cultural activities that typically constitute school mathematics *academic mathematics*. "The mechanism of schooling replaces these [self-generated] practices by other equivalent practices which have acquired the status of [academic] mathematics, which have been expropriated in their original form and returned in codified version" [p. 47]. Two aspects of academic, codified mathematics can be distinguished [Hiebert, 1984]. The first concerns the conventional mathematical symbol systems per se and the second refers to the interrelated mathematical objects that are expressed both by the symbols themselves and by procedures for manipulating symbols. The academic mathematics of high school geometry, for example, involves both abstract, geometrical objects (e.g. points, planes) and deductive argumentation as expressed in formal proofs. Schoenfeld's [1985] study indicated that the mechanism of schooling has limited success in replacing students' empirical geometry with academic geometry. At the first grade level, academic arithmetic includes but is not limited to numerals, number words, addition and subtraction sentences as well as the conceptualization of addition and subtraction in terms of the uniting and partitioning of sets, the commutative property of addition, and the inverse relationship between addition and subtraction. A major goal of first grade arithmetic instruction as it is typically practiced is to replace children's self-generated, counting arithmetic with academic, set-theoretical arithmetic.

Hiebert [1984] observed that "many children experience difficulty in learning school mathematics because its abstract and formal nature is much different from the intuitive and informal mathematics the children acquire" [p. 498]. Two potential sources of problems can be identified that stand apart from the abstract nature of academic mathematics per se (i.e. difficulties arising from a mismatch between the child's concepts and those implicit in symbolic forms.) The first concerns the general contexts of self-generated and academic mathematics. D'Ambrosio [1985] contrasted academic mathematics with what he called ethnomathematics. This is the "mathematics which is practiced among identifiable cultural groups .... Its identity depends largely on focuses of interest, on motivation, and on certain codes and jargons which do not belong to the realm of academic mathematics" [p. 45]. In particular, a crucial feature of ethnomathematics is that the ideas or concepts are put to use for practical purposes. The analogy between ethnomathematics and children's informal, self-generated arithmetic is thought-provoking, although it stretches credibility to talk of, say, first graders as an identifiable cultural group. Like ethnomathematics, child-generated mathematics is constructed in the general context of pragmatic problem solving. The criterion of acceptance for self-generated methods is that they work, that they allow the child to attain his or her practical goals. Ideally, academic mathematics is also constructed within a problem solving context. However, the pragmatic criterion that a method must work is not sufficient. In addition, concepts and procedures must be expressed in terms of conventional symbol systems that sociocultural history has provided as tools for cognitive activity. A primary motivation for doing so is to facilitate the communication of mathematical thought. There is thus an important difference between the role and intent of self-generated and academic mathematics that is analogous to that between everyday and academic reasoning as discussed by Lave *et al* [1984] and Rogoff [1984].

Self-generated mathematics is essentially individualistic. It is constructed either by a single child or a small group of children as they attempt to achieve particular goals. It is, in a sense, anarchistic mathematics. In contrast, academic mathematics embodies solutions to problems that arose in the history of the culture. Consequently, the young child has to learn to play the academic mathematics game when he or she is introduced to standard formalisms, typically in first grade. Unless the child intuitively realizes that standard formalisms are an agreed-upon means of expressing and communicating mathematical thought, they can only be construed as arbitrary dictates of an authority. Academic mathematics is then totalitarian mathematics. The child's overall goal might then become to satisfy the demands of the authority rather than to learn academic mathematics per se. This goal can be achieved, at least in the short term, by either covertly constructing and using self-generated methods or by attempting to memorize superficial aspects of formal, codified procedures. If the latter approach is adopted, mathematics becomes an activity in which one applies superficial, instrumental rules.

The second, related source of difficulty in learning academic mathematics concerns the nature of the interactions between teacher and student. The socialization process usually involves engaging the child in joint activity with more mature members of the culture who attempt to regulate the activity in accordance with sociocultural patterns. As Bishop [1985b] noted, there is a necessary power imbalance in the learning-teaching relationship. The central question is then how the teacher translates his or her power into action. Bishop [1985a] discussed two general means of doing so, negotiation and imposition, that represent endpoints of a continuum. The analysis in Saxe *et al* of the interactions between a mother and her child exemplifies negotiation. The mother's overall goal was to maintain a coherent interaction or dialogue and, to this end, she continually adjusted her interventions in light of the child's responses. The mother's activity was characterized by an attempt to communicate, to construct shared meanings. In general, teaching by negotiation is "more concerned with the initiation, control, organization, and exploitation of pupils activity. There is .. a dynamic, organic-growth, feeling in the classroom" [Bishop, 1985b, p. 26]. It was also noted that the goals established by the mother and child were dynamic and emerged in the course of the interaction. Similarly,

> the teacher has certain goals and intentions for the pupils and these will be different from the pupils' goals and intentions in the classroom. Negotiation is a goal-directed interaction, in which the participants seek to [modify and] attain their respective goals [p. 27].

Negotiation therefore involves concerted attempts by the participants to develop their understandings of each other.

This type of interaction can be contrasted with that in which the teacher's overall goal is to regulate students' activity by attempting to impose his or her knowledge of academic mathematics on them. The teacher's primary focus is on "a compartmentalized list of specific knowledge or skills to be taught from nothing, and to be finished in a set time" [pp. 26-27]. Such "mathematics classrooms are places where you do mathematics not where you communicate or discuss mathematical meaning" [p. 27]. Here, teachers

> exert their power through their control of mathematical knowledge, which can be extremely abstract and opaque to pupils. In that sense they would be like teachers of a foreign language in whose presence the pupils would feel relatively powerless and dependent on the teacher. [Bishop, 1985a, pp. 11-12]

As we have seen, negotiation is characterized by the manner in which teacher and students mutually adjust their goals and activity as they interact. Consequently, the teacher's continually changing expectations are unlikely to be construed as demands by students. Imposition, in contrast, places a greater onus on students. The teacher's expectations are far more rigid—it is expected that the students will solve certain sets of tasks in prescribed ways. In this situation, it is the students' responsibility to adjust their goals and activity to fit the teacher's expectations. Expectations can then be construed as demands and the students' primary goal might well become to find a means—any means—of satisfying them. If this occurs, the students' overall goal becomes to solve problems that derive from social interactions. Their activity is directed towards the goal of either bringing about or avoiding certain responses from the teacher. Well-meant interventions made with the intention of facilitating students' construction of mathematical knowledge then become, in and of themselves, a source of major problems for students. In the absence of dialogue, there is a gross mismatch between the goals that the teacher thinks he or she is getting for students and the goals that students actually seek to achieve. In other words, the teacher believes that the students are operating in a mathematical context when their overall goals are primarily social rather than mathematical in nature.

The dangers of this situation become apparent when it is observed that, to satisfy the teacher's demands, students have to behave as though they know how to play the academic mathematics game. To succeed, the students have to produce evidence that they have performed appropriate symbolic manipulations. As Schoenfeld's [1985] analysis of high school geometry students' problem solving activity indicates, it is quite possible that this can become an end in itself. He observed that the students attempted to write formal proofs only when they perceived explicit or implicit demands to do so. However, their goal was not to construct and express a deductive argument in terms of conventional formalisms. Instead, it was to produce an acceptable form per se. The conceptual aspects of geometry that underlie the form were not a major focus of the students' activity. Cobb [1985b] reported a similar finding when he investigated first graders' beliefs about arithmetic. As the school year progressed, the children increasingly produced appropriate addition and subtraction forms (i.e. numeral, operator sign, numeral, equals sign, numeral) that did not make sense when interpreted in terms of sensory-motor activity (e.g. $7 + 4 = 3$, $4 - 7 = 3$).

In general, students who come to believe that school mathematics is an activity in which one attempts to produce appropriate forms and thus satisfy the perceived demands of the teacher thereby increase their dependence on the teacher. This is because they have lost three valuable sources of feedback that could guide their construction of mathematical knowledge. These are the inconsistencies that arise when individuals attempt to share mathematical meanings, when concepts and procedures are expressed in terms of actual or re-presented sensory-motor actions, and when the results of related procedures are compared. In their absence, students have no option but to appeal to an authority in order to know whether their solutions are appropriate. Unfortunately, this phenomenon is only too well known to mathematics educators [e.g. Confrey, 1984; Hoyles, 1982; Peck, 1984; Wheatley, 1984; Wheatley & Wheatley, 1982]. With regard to the primary grades, many children

> follow what they remember to be an appropriate rule and, on that basis, believe the answer must be correct regardless of what the symbols say. The result is that unreasonable responses such as $8 + 21 = 13$, become quite common in the primary grades [De Corte & Verschaffel, 1981]. This is especially disconcerting, since beginning first graders do not make these kinds of errors. [Hiebert, 1984, p. 503]

It should be stressed that imposition is but one end of a continuum that captures the various ways in which teachers exert their power. However, Bishop [1985a] suggested and observations made by Goodlad [1983] and Stake and Easley [1978] indicate that most teachers' activity is far closer to the imposition than the negotiation pole. Further, the manner in which most students reorganize their beliefs as they progress through the elementary school grades is compatible with the suggested influence of teaching by imposition on students' goals and thus on the contexts within which they do academic mathematics. The analysis therefore constitutes an initial framework within which to relate children's beliefs about mathematics to their classroom experiences of doing mathematics. In particular, the analysis suggests that it is necessary to consider the social as well as the purely cognitive aspects of classroom mathematics in order to account for the meanings students give to the formalisms of academic mathematics. This, it should be clear, does not mean that meanings and beliefs are communicated directly from the teacher to students. Instead, I have argued that the beliefs students construct, the overall goals they establish, and the contexts in which they do mathematics are their attempts to find a viable way of operating in the classroom. The are expressions of students' underlying rationalities, of the ways they try to make sense of classroom life. In short, students' beliefs about mathematics are their attempted solutions to problems that arise as they interact with the teacher and their peers.

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