REASONING ABOUT INTENSIVE QUANTITIES IN WHOLE-NUMBER MULTIPLICATION? A POSSIBLE BASIS FOR RATIO UNDERSTANDING

MARTIN A. SIMON, NICORA PLACA

Teaching proportional reasoning is one of the greatest pedagogical challenges in elementary and middle school mathematics (Chung & Lew, 2007; Lamon, 2007). Proportional reasoning has been considered the culmination of elementary school mathematics and a "watershed" concept, necessary for access to high school mathematics (Lesh, Post & Behr, 1988). The concepts of ratio and proportion are often never learned adequately (Capon & Kuhn, 1979). One of the challenges of ratio [1] is that it involves *intensive quantites* (elaborated below), a type of quantity that is more conceptually demanding than those that are evaluated by counting or measuring. From a mathematics educator's perspective, intensive quantities are not only found in ratio situations, but also in whole-number multiplication and division situations. However, current curricular approaches generally fail to promote student reasoning about intensive quantities during instruction on whole-number multiplication and division.

This article is based on a particular theoretical exploration [2] of the possibility of promoting intensive quantities during the teaching of whole-number multiplication. We use the verb *promoting* to indicate intentional acts in planning and enacting lessons designed to enhance the probability that students will develop a particular concept. Our exploration is grounded in a measurement-based instructional design that builds on the Elkonyn-Davydov (E-D) curricular approach. The larger project, Measurement Approach to Rational Number (MARN), is a research and development project that aims to elucidate a measurement-based learning progression from whole-number multiplication to advanced concepts of ratio. It builds on *Measure Up,* a US version of the E-D curriculum, and involves collaboration with *Measure Up*'s PI, Barbara Dougherty.

Measurement-based approaches, such as the E-D (and *Measure Up*) approach, are a compelling basis for this work because of the central role given to quantities and units throughout the curriculum. We work from the conjecture that reasoning about intensive quantities can be built on the foundation offered by a measurement approach to the development of quantities.

In this article, we explain intensive quantities, review the affordances of the E-D measurement-based approach, discuss how the E-D approach teaches whole-number multiplication and division without developing reasoning about intensive quantities, and describe our effort to design a measurement approach to whole-number multiplication and division that lays a foundation for reasoning about intensive quantities. We conclude by discussing implications derived from this design effort and the larger effort to develop understanding of ratio.

Background on intensive quantities

In order for students to gain access to ratio and proportion, they must have a solid understanding of multiplicative reasoning. Schwartz (1988) argued that multiplicative reasoning involves new quantities that are integral to multiplication: *intensive quantities.* In contrast to extensive quantities, such as length, mass, area or volume, which can be measured directly, or the number of items in a set, which can be counted, intensive quantities, such as speed or concentration, cannot be measured directly or counted. Rather, intensive quantities express the relationship between two quantities, which can themselves be either intensive or extensive. Schwartz (1988) described intensive quantities as the "generalization of the notion of an attribute density" (p. 43); in other words, the intensity of a trait (*e.g.,* speed, color, temperature, populated-ness).

Thompson (1994) emphasized the importance of distinguishing students' reasoning about intensive quantities from students' reasoning about the association of two extensive quantities. For example, many students may not conceptualize a price as an intensive quantity (rate). The price for them is an amount of money (extensive quantity) associated with a particular amount of a product to be purchased. Thus, 3 kg of bananas at $1.50 per kg is seen as $1.50 associated with the first kg, $1.50 associated with the second kg, and $1.50 associated with the third kg. When reasoning only about extensive quantities, the total price is the sum of these three amounts of money. Schwartz (1988) pointed out that multiplication is a referent transforming operation. In a conventional multiplication word problem, each of the three quantities involved is characterized by a different referent or unit. For example, 3 kg of bananas x 1.5 dollars/kg = 4.5 dollars. Students reasoning only with extensive quantities solve the problem using a referent preserving operation (addition of three extensive quantities). We work from the assumption that reasoning of this latter type is typical of elementary students in many education systems (see Hart, 1981).

Thompson and Saldanha (2003) stressed: "Envisioning the result of having multiplied is to anticipate a multiplicity. One may engage in repeated addition to evaluate the result of multiplying, but envisioning adding some amount repeatedly cannot support conceptualizations of multiplication" (p. 103). They emphasized that conceiving of 5 fours as a number 5 times as big as 4, is different from seeing 5 × 4 as a direction to add 4 five times and involves proportional reasoning. However, they did not explore the origins of reasoning about intensive quantities specifically.

Vergnaud's (1988) discussion of the *multiplicative conceptual field* characterized multiplication and division situations into three categories: *isomorphism of measures, product of measures,* and *multiple comparisons.* Vergnaud considered a conventional multiplication problem involving two measure spaces to be an isomorphism of measures, which he defined as a simple comparison between measure spaces, where 1 is always the basis for one of the ratios. By *two measure spaces,* we mean two quantities that cannot be measured with the same set of units (*e.g.,* length units and time units in a velocity problem). In isomorphism of measures situations, the intensive quantity corresponds to the functional relationship between the two measures. An example of this type of situation is buying four toy cars that cost $5 each. Although students can reason additively about the problem as the price of one car ($5) added four times, Vergnaud saw their additive reasoning as limited. He pointed out that such situations could be seen as rudimentary proportional reasoning situations (*e.g.,* $5/1 car = $20/4 cars).

Our interest in finding ways to develop deep understanding of multiplication, fractions, and proportional reasoning, led us to explore the potential of a measurement-based approach as embodied in the E-D and *Measure Up* curricula. This paper focuses on three questions:

1. What is the relationship between existing measurement-based approaches and the development of reasoning about intensive quantities in the teaching of multiplication and division?

2. Might there be more effective ways to develop reasoning about intensive quantities in the teaching of multiplication and division?

3. What implications does the investigation of the first two questions have for the teaching of proportional reasoning?

Measurement-based approaches to early arithmetic concepts

Our research team conducted an intensive analysis of the E-D first through third grade curriculum (aged 6-9 years; see Davydov, Gorbov, Mukulina, Savelyeva & Tabachnikova, 1999) and data from the *Measure Up* project [3]. The E-D curriculum, developed (and developing) in Russia, is grounded in an activity theoretical perspective (Leontiev, 1981) derived from the work of Vygotsky. A central tenet of this perspective is attention to developing instruction from the general to the specific. The most unique aspect of the E-D curriculum, building on measurement rather than counting is an application of this principle. Measurement is seen as sufficiently general that specific quantities of all kinds can be developed without changing the original concept of a quantity. Numbers, rather than being the primitive element, are the result of measuring a quantity with a unit of measure. In traditional curricula, students often experienced difficulties each time they encountered a new type of real number, such as fractions, and irrational numbers (Davydov, 1975). In each such transition, students were forced to alter their conception of number to accommodate these new types of real numbers. Building on measurement was an attempt to develop from the outset a conceptual basis for all real numbers. Davydov and colleagues worked from the notion that whole numbers and other real numbers have a "common root" (p. 121): measurement.

The E-D curriculum begins with the exploration of quantities. At first, students engage in direct comparison of concrete objects in terms of their mass, length, area and volume. Measurement is introduced as a way to compare quantities indirectly. A number *n* is introduced as the number of times the unit of measure E measures the quantity Q. E and Q may represent any discrete or continuous quantities in any relationship, which makes this concept of number general; *n* can be any real number. Students represent this measurement, starting in first grade, with the arrow notation in Figure 1. Particular attention is given to measuring a quantity with different units and to how different numbers result from measuring the same quantity with different units, as well as how different quantities can be represented by the same number. Students' initial experiences are first with positive integers. However, the concept of a number does not have to change as rational and irrational numbers are explored.

$$E \xrightarrow{\quad n \quad} Q$$

Figure 1. Measurement representation in the E-D curriculum.

Affordances of a measurement approach as a foundation for teaching ratio

Before we discuss the issue of the development of intensive quantities, we will describe a few of the particular affordances of the E-D measurement approach that made it a compelling foundation for teaching ratio.

Quantitative reasoning

Thompson (1993) postulated two types of reasoning, quantitative and numerical. Quantitative reasoning involves reasoning about quantities (*e.g.,* the heights of two people) independent of particular numerical values for those quantities. Such reasoning involves quantitative operations in which two or more quantities are related to produce a new quantity (*e.g.,* comparing two people's heights results in a new quantity, the difference in their height). Numerical reasoning involves evaluating the particular size (numerical value) of a quantity. Development of quantitative reasoning is particularly important, because it is the basis of mathematizing real situations (Thompson, 1993) and understanding the relationships among quantities that allow for problem solving and growth in conceptual understanding. Further, it is the basis for generalization, the foundation of algebraic thinking (Ellis, 2007; Smith & Thompson, 2007).

The E-D approach develops quantitative reasoning early in the curriculum. Starting in first grade, students are engaged in reasoning about the relationship between quantities. For example, they investigate how to make unequal quantities equal (physically adding the difference, *e.g.,* adding water to one container) without determining the amount of those quantities or the quantity representing the difference. Further, beginning in first grade, letters are assigned to represent quantities, so that quantitative relationships and operations can be symbolized (prior to symbolizing numerical relationships).

Attention to units

It is well documented that a significant part of the challenge of learning multiplication and division of whole numbers (Steffe, 1988) and the learning of fractions, particularly multiplication and division of fractions (Behr, Harel, Post & Lesh, 1992), is the difficulty students have with units. The E-D curriculum, with its foundation of physical measurement, is focused on units throughout. Unlike traditional curricula, where units are seemingly irrelevant during the development of whole-number addition and subtraction, the E-D curriculum engages students deeply in the key role of the unit from early in the first grade.

Multiplication

Researchers have specified different intuitive models subsumed under multiplication [4], including equivalent sets, multiplicative comparisons, Cartesian products and rectangular arrays (Fischbein, Deri, Nello & Marino, 1985; Greer, 1992; Kouba, 1989). Nonetheless, students seem to have a dominant view of multiplication as the summing of the elements of equivalent sets. Fischbein *et al.* (1985) observed that students tend to develop primitive intuitive notions such as "multiplication makes bigger". If we have a length of 2 meters, does it get longer when we multiply by 100 to get 200 cm? If we have 5 items and we sell them for 6 dollars each (assuming a fair trade), has the buyer's net worth increased? In each case, only the *number* gets larger [5]. Thus the primitive notion is based on a view of multiplication as a numerical operation and is not grounded in quantitative reasoning.

The E-D curriculum, continuing its emphasis on quantitative reasoning, introduces multiplication as a change in unit. Using the example above of changing 2 meters to centimeters, multiplication is the quantitative operation that determines the number of centimeters in a quantity that was measured in meters. More generally, multiplication relates the number of units in an *intermediate unit* (*e.g.,* 100 cm in a meter) with the number of intermediate units in a quantity (*e.g.,* 2 meters in the quantity) to determine the number of units in the quantity. The general relationship can be expressed with letters. If the quantity Q is measured by intermediate unit I, and I is measured by unit E, multiplication determines the number of units E that would measure Q.

The arrow notation that was introduced earlier is used to develop the notation for multiplication and division (a composition of three arrows). The diagram shows the intermediate unit I, measured by the unit, E, the quantity, Q, measured by E, and the quantity, Q, measured by I. The numbers over the arrow always represent the number of those units in that quantity (see Figure 2).

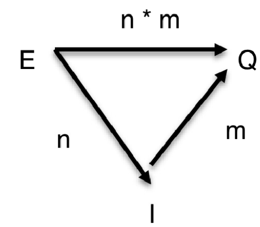

Figure 2. *General multiplication representation in the E-D curriculum.*

To represent our example of the number of centimeters in 2 meters, the diagram shown in Figure 3 is used.

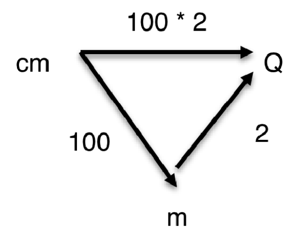

Figure 3. *Multiplication representation for the number of centimeters in 2 meters.*

Likewise, if we have 3 dozen eggs, we can represent the number of eggs we have using the diagram in Figure 4.

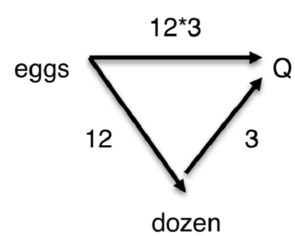

Figure 4. *Multiplication representation for the number of eggs in 3 dozen.*

Intensive quantities and E-D approach to multiplication

The E-D curriculum focuses throughout on quantities and units. So how well does the E-D approach develop reasoning about intensive quantities in whole-number multiplication and division, in preparation for proportional reasoning? The E-D curriculum (and the *Measure Up* adaptation) builds on extensive measurement, using a physical unit that can be iterated to measure a physical quantity. Intensive quantities are, by definition, not the result of direct measurement. Experience with extensive measure does not necessarily lead to the conceptualization of intensive quantities. Let us examine this more closely.

In the E-D model [6] for multiplication, the arrows making up the diagram always indicate the extensive measurement of a particular quantity with a particular unit. Thus, each of the three arrows has the same meaning, albeit with different unit-quantity relationships. If we return to the task in which 2 m is changed into 200 cm, theoretically the units of the number 100 should be cm/m. 100 is the size of the intensive quantity in the problem. However, the arrow diagram (see Figure 3) and the reasoning it represents contain no intensive quantity. Each arrow indicates an extensive measurement: measuring 1 meter with a cm, measuring the quantity with a meter, and measuring the quantity with a cm. It would not make sense to talk about measuring 1 meter with 100 cm/m. Thus, multiplication as modeled in the E-D curriculum involves only extensive quantities (Gorbov, personal communication; Dougherty, personal communication).

The E-D model of multiplication has another related limitation, its inability to represent word problems involving two measure spaces. Consider the earlier problem about the cost of purchasing 2 kg of bananas. It involves the measure space of money (dollars) and the measure space of weight of bananas. We would identify 1.50 dollars per kg of bananas as the intensive quantity, price. Not only does the arrow diagram exclude intensive quantities, it excludes problems that involve two measure spaces. It would make no sense to consider measuring a kg of bananas using a unit of a dollar. To use the arrow diagram, the problem must be reduced to extensive measures *and* reduced to one measure space. The would-be intensive unit, dollars per kg of bananas, is replaced as the intermediate unit by $1.50 (the amount of money associated with a kg of bananas). See Figure 5 for the arrow representation.

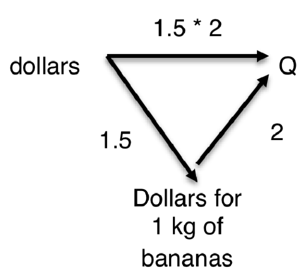

Figure 5. Representation of the cost of 2 kg of bananas.

Measuring the quantity Q with an extensive quantity rather than reasoning about an intensive quantity is reminiscent of Thompson's (1995) analysis of JJ's early work on a speed-distance-time problem. JJ conceptualized speed as an extensive quantity (our language) and measured the total distance in what Thompson called "speed lengths" to find the time elapsed. It was not until later in the teaching experiment that she began to construct speed as an intensive quantity, which was essential for her understanding of ratio. Thompson's analysis indicates the need for a development of intensive quantities. Note, however, that Thompson fostered the development of intensive quantities in the context of rates and not during the development of whole-number multiplication and division.

To summarize, the E-D approach to the initial teaching of whole-number multiplication does not promote a way of reasoning about or representing intensive quantities or give students access to situations involving more than one measure space since the model of multiplication is one of nested (extensive) measures (a unit measuring an intermediate unit and an intermediate unit measuring the quantity). One can debate the importance of reasoning about intensive quantities in understanding whole-number multiplication and division. However, there is no doubt that such reasoning is central to proportional reasoning and the concept of rate (Lesh, Post & Behr, 1988; Schwartz, 1988; Thompson, 1994). When and how should intensive quantities be developed, including intensive quantities that span two measure spaces?

The absence of reasoning about intensive quantities in many traditional curricula

As we indicated earlier, reasoning only about extensive quantities when learning and doing whole-number multiplication is common in many elementary mathematics systems. In some, students use equal groups of manipulatives to represent a multiplication situation, others see pictures of groups of objects in their text. The total number of objects is first done by counting all, then by skip counting, and finally by using memorized multiplication facts. As a result, multiplication is a numeric operation that is grounded in a model of repeated addition. It is not developed as a referent transforming operation involving intensive quantities.

Whole-number division is likewise done without reasoning about intensive quantities. For example, quotitive division problems are properly understood as problems in which the amount per group is known and the number of groups is to be determined. These problems are often referred to as "measurement division." The term *measurement division* is based on a notion of measuring the dividend quantity with the divisor quantity (Lo, McCrory & Young, 2009). Consider the problem, "How many 20 cm pieces of ribbon can be made from a 140 cm length of ribbon?" The idea behind measurement division is that one can measure 140 cm with a 20 cm piece of ribbon. Whereas this measuring can be done physically and can provide a basis for abstracting one aspect of division (*i.e.,* quotitive), it involves reasoning with only extensive quantities. Such an approach to division does not involve the intensive quantity, 20 cm/piece. If we change the problem to an isomorphic problem, such as, "If you ride 20 km/hr, how long will it take to ride 140 km?", we have the reasoning enacted by Thompson's (1995) subject, JJ, discussed above (measuring with "speed-lengths"). The notion of "measurement division" derives from a model of multiplication and division in which all quantities are thought of as extensive. On one hand, this conception is easily accessible and often a spontaneous approach of students. However, on the other hand, it does not develop understanding of intensive quantities.

An attempt to design instruction for reasoning about intensive quantities in the context of learning whole-number multiplication

The fact that the E-D curriculum does not develop intensive quantities during the learning of whole-number multiplication is particularly striking given its continuous focus on units and the commitment to developing a concept of number that does not have to change when studying more advanced mathematics. We began our design work with the idea that there must be a way to build intensive quantities through working with a measurement approach to whole-number multiplication. We now document our attempt to do so, not to emphasize the particular attempts we made, but to use it as a paradigm case to support a particular conclusion.

What is an intensive quantity?

Having decided to design a ratio unit built on the basis of measurement concepts that focuses in part on the development of reasoning about intensive quantities, we realized that we had to better articulate the notion of an intensive quantity. Schwartz (1988) provided the main description of intensive quantities: they (1) cannot be measured directly, (2) express the relationship between two quantities, and (3) can be thought of as measuring "attribute density". This description, however, was not sufficient as a basis for our instructional design. We sought a specification that related intensive quantities to extensive quantities, including intensive quantities that span two measure spaces, and focused on essential conceptual elements. One common specification is the amount of one quantity per one unit of another co-varying quantity (*e.g.,* price could be thought of as the number of dollars per unit of product and speed as the number of kilometers per hour). *Co-varying* refers to the fact that both extensive quantities can vary and still maintain the same relationship. For example, the number of kilos of bananas can vary and the total price can vary (the covarying quantities) and the price (the intensive quantity) can stay the same. However, while appropriate and part of the story, this specification did not do all the work that we needed it to do. As mentioned above, students tend to interpret per-one quantities as the extensive quantity associated with 1 unit of another extensive quantity, thus, continuing to think exclusively about extensive quantities. Further, we do not assume that the per-one notion of intensive quantities brings with it other important ideas, such as the invariant multiplicative relationship between co-varying quantities. We are therefore currently working with the following idea (in addition to the per one articulation above):

> An intensive quantity is the relative size of the magnitudes (number of units) of measures from two different measure spaces, or of the measures of two different quantities from the same measure space, given particular units of measurement for each measure.

This specification of intensive quantity has several advantages.

1. It articulates the relationship between extensive measuring and intensive quantities.
2. It describes the relative magnitude of measures in an equivalence class, specifying a relationship within pairs of quantities without reducing the intensive quantity to per one.
3. It emphasizes the role of the extensive units in determining the magnitude of the intensive quantity. For an extensive quantity, specifying the unit determines the magnitude. In an intensive unit (as specified above), specifying the two extensive units determines the magnitude of the intensive quantity. This implies that, for the purpose of determining the magnitude of the intensive quantity, two units (perhaps from different measure spaces), are put on the same scale, where a unit of measure A is equivalent (in its contribution to determining the magnitude) to a unit of measure B. In extensive quantities, changing the unit changes the magnitude inverse proportionally. Likewise changing an intensive unit results in an inverse proportional change in the magnitude.

Our design attempt

Our task design process consisted of various attempts to use a measurement approach to foster the development of intensive quantities in the context of developing whole number multiplication and division. [7] Our most intriguing attempt involved the use of parallel number lines to teach multiplication [8]. The parallel number lines could be used to represent two measure spaces, which was not possible in the arrow diagram (intermediate-measure) model. For example, if 1 kg of chicken cost $7, the two number lines in Figure 6 would be created.

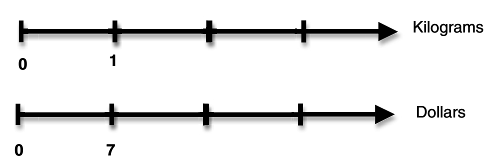

Figure 6. Parallel number lines representation of cost of kilograms of chicken.

Students could then use the two number lines to reason about the cost of 6 kilograms of chicken, as well as the cost of 2.5 kilograms of chicken. We initially believed this representation addressed the problems that occur when multiplication is taught as nested measures (discussed earlier). It had a number of advantages. There was no need to change the model to represent multiplication of fractions. The model did not reduce the tasks to operations with extensive quantities only. Rather, the intensive nature of the quantity was implicit in our model: the relationship between the corresponding quantities on the two number lines. The model represented the continuous invariant relationship between the two quantities represented. Further, this representation was also advantageous because of its use in many curricula to teach ratio. It could be used continually from whole-number multiplication and division, through fractions and ratio. We were excited about the possibilities.

However, as we began to work further with the double number line model, carefully considering potential student reasoning with the model, we began to see that the model would likely not have the intended effect on promoting students' reasoning about intensive quantities. There were two related problems with using this model. First, instead of reasoning about the relationship between the two quantities (the number of dollars is 7 times the number of kilograms in the example above), the student could use the amount of the first quantity for 1 unit of the second and iterate it (for example, one kg is 7 dollars so we combine the 7 dollars corresponding to each kg). This is not appreciably different from the ways that students often think about these problems in terms of two associated extensive quantities, what we referred to earlier as a referent-preserving operation. Furthermore, the realization that students would focus on the value of the second variable for *one* unit of the first led us to concerns about the model's effectiveness for later developing ratio concepts. If the students had considerable experience thinking about the double number line in terms of the amount for one unit, it would likely reduce the flexible ways they might reason about ratio problems and therefore the model's effectiveness in promoting proportional reasoning. [9]

Conclusions

As we indicated earlier, the purpose of describing our exploration of the double number line for teaching multiplication and division was not to highlight the particular attempt, but rather to use it to discuss some more general conclusions. The idea of promoting reasoning about intensive quantities the first time they appear in the curriculum (*i.e.,* appear from the mathematics educator's perspective) seems worth pursuing. However, our example has led us to the conclusion that this is likely not feasible.

Our example was significant because it built on measurement, thus allowing an emphasis on units. It represented the continuousness of the two extensive quantities and the relationship among them at every value. It avoided representational features that required extensive-only thinking. However, what it was not able to do is to create a need for reasoning about intensive quantities in the context of whole-number multiplication and division problems. This does not seem to be specific to the model we generated. As discussed earlier, Vergnaud (1988) identified whole-number multiplication problems involving two measure spaces as a subclass of ratio, *isomorphism of measures,* and pointed out that 1 is always the basis for one of the ratios. That simplification of ratio seems to allow for the adequacy of extensive-only reasoning for such problems. Our work with this design situation and additional work in designing for the development of ratio concepts have led us to the following conclusions [10]:

1. It is unlikely that a model can be created that necessitates reasoning about intensive quantities in whole-number multiplication and division problems.

2. Reasoning about intensive quantities must be developed during the development of ratio. This requires developing situations for which students are not likely to use per-one thinking.

3. Reasoning about intensive quantities is closely tied to the development of a functional concept of ratio. [11]

We will elaborate on this third point. Whereas the development of proportional reasoning often begins with reasoning about the parallel accumulation of two quantities often represented as parallel movement along two coordinated number lines (see van den Heuvel-Panhuizen, 2000), an important aspect of understanding ratio involves developing a concept of ratio-as-measure (Simon & Blume, 1994). Conceptualizing ratio-as-measure involves knowing that the ratio *m/n* implies an invariant multiplicative relationship between *m* and *n and* that *m/n* is a *measure* of the strength of that invariant relationship, that is, the measure of a particular attribute. This pedagogically challenging concept (Kaput & West, 1994) is referred to as a *functional* relationship as opposed to the more easily developed *recursive* relationship, involving parallel (linked) accumulation of two extensive quantities. It is only in the context of ratio-as-measure, that an intensive quantity makes sense. An intensive quantity is the unit by which the strength of an attribute of this type (*e.g.,* speed, density) is measured.

Notes

[1] We use "ratio" as a generic term to include both concepts of *rate* and *ratio*.

[2] "Theoretical exploration" denotes the fact that the exploration involved planning for instruction (including thought experiments) and not instruction.

[3] In this Hawaii implementation, the concepts taught in three years in Russia, were taught over the span of grades one through five.

[4] In an attempt to avoid redundancy, division is not discussed explicitly in this paper since the issues raised regarding multiplication are essentially the same for division.

[5] The latter case can also be seen as unit conversion. 5 items are equivalent to $30. The conversion rate is 6:1. The latter situation is no more appropriately characterized by "getting bigger" than the former.

[6] We use "model" in the Realistic Mathematics Education sense of *model of* and *model for* (Gravemeijer, 1999).

[7] This was a design process only and was never used with students.

[8] The parallel number lines representation is similar to the double number line used in some curricula. The latter puts one quantity in digits above the line and the second quantity in digits below the same line.

[9] Thompson and Thompson (1996) demonstrated the power of using a compatible approach to developing a concept of rate (thinking about parallel accumulation of two quantities). Therefore, it offers no indications for developing reasoning about intensive quantities *during* the learning of whole-number multiplication.

[10] Our conclusions are based on the assumption that it makes sense to teach whole-number multiplication and division before developing explicit (*i.e.,* beyond intuitive) proportional reasoning.

[11] This idea is consistent with the E-D approach currently in use in Russia (Gorbov, 2011).

References

Behr, M. J., Harel, G., Post, T. &Lesh, R. (1992) Rational number, ratio, and proportion, In Grouws, D. A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning,* pp. 296-333. New York, NY: Macmillan.

Capon, N. & Kuhn, D. (1979) Logical reasoning in the supermarket: adult females' use of proportional reasoning strategy in an everyday context. *Developmental Psychology* **15**(4), 450-452.

Chung, I. & Lew, H.-C. (2007) Comparing Korean and U.S. third grade elementary student conceptual understanding of basic multiplication facts.

In Woo, J.-H., Lew, H.-C., Park, K.-S. & Seo, D.-Y. (Eds.) *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education,* vol. 2, pp. 161-168. Seoul, Korea: PME.

Davydov, V.V. (1975) Logical and psychological problems of elementary mathematics as an academic subject. In Steffe, L. P. (Ed.) *Soviet Studies in the Psychology of Learning and Teaching Mathematics, Vol. VII: Children's Capacity for Learning Mathematics,* pp. 55-107. Chicago, IL: University of Chicago.

Davydov, V. V., Gorbov, S., Mukulina, T., Savelyeva, M. & Tabachnikova, N. (1999) *Mathematics.* Moscow, Russia: Moscow Press. [Translated, Grades 1–3, 2000]

Dougherty, B. J. (2007) Measure up: a quantitative view of early algebra. In Kaput, J. J., Carraher, D. W. & Blanton, M. L. (Eds.) *Algebra in the Early Grades,* pp. 389–412. Mahwah, NJ: Erlbaum.

Ellis, A. B. (2007) Connections between generalizing and justifying: students' reasoning with linear relationships. *Journal for Research in Mathematics Education* **38**(3), 194-229.

Fischbein, E., Deri, M., Nello, M. S. & Marino, M. S. (1985) The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education* **16**(1), 3-17.

Gravemeijer, K. (1999) How emergent models may foster the constitution of formal mathematics. *Mathematical Thinking and Learning* **1**(2), 155–177.

Greer, B. (1992) Multiplication and division as models of situations. In Grouws, D.A. (Ed.) *Handbook of Research on Mathematics Teaching and Learning,* pp. 1-295. New York, NY: Macmillan.

Hart, K (1981) Hierarchies in mathematics education. *Educational Studies in Mathematics* **12**(2), 205-218.

Kaput, J. & West, M. M. (1994) Missing value proportional reasoning problems: factors affecting informal reasoning patterns. In Harel, G. & Confrey, J. (Eds.) *The Development of Multiplicative Reasoning in the Learning of Mathematics,* pp. 237–287. Albany, NY: SUNY Press.

Kouba, V. L. (1989) Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education* **20**(2), 147-158.

Lamon, S. J. (2007) Rational numbers and proportional reasoning: toward a theoretical framework for research. In Lester, F. K. Jr. (Ed.) *Second Handbook of Research on Mathematics Teaching and Learning,* pp. 629–667. Reston, VA: National Council of Teachers of Mathematics.

Leontiev, A.N., (1981) *The Development of Mind.* Moscow, Russia: Progress Publishers.

Lesh, R., Post, T. & Behr, M. (1988) Proportional reasoning. In Heibert, J. & Behr, M. (Eds.) *Number Concepts and Operations in the Middle Grades*, pp. 93-118. Reston, VA: National Council of Teachers of Mathematics.

Lo, J., McCrory, R. & Young, S. (2009) Developing prospective teachers' knowledge of elementary mathematics: a case of fraction division. In Swars, S.L., Stinson, D.W.R. Lemans-Smith, S. (Eds.) *31st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA),* vol. 5, pp. 1125-1132. Atlanta, GA: Georgia State University.

Schwartz, J. (1988) Intensive quantity and referent transforming arithmetic operations. In Heibert, J. & Behr, M. (Eds.) *Number Concepts and Operations in the Middle Grades,* pp. 41-52. Reston, VA: National Council of Teachers of Mathematics.

Simon, M. A. & Blume, G. (1994) Mathematical modeling as a component of understanding ratio-as-measure: a study of prospective elementary teachers. *Journal of Mathematical Behavior* **13**(2), 183-197.

Smith, J. & Thompson, P. W. (2007) Quantitative reasoning and the development of algebraic reasoning. In Kaput, J. J., Carraher, D. W. & Blanton, M. L. (Eds.) *Algebra in the Early Grades,* pp. 95-132. Mahwah, NJ: Erlbaum.

Steffe, L. P. (1988) Children's construction of number sequences and multiplying schemes. In Hiebert, J. & Behr, M. (Eds.) *Number Concepts and Operations in the Middle Grades,* pp. 119–140. Reston, VA: National Council of Teachers of Mathematics.

Thompson, A. G. & Thompson, P. W. (1996) Talking about rates conceptually, part II: mathematical knowledge for teaching. *Journal for Research in Mathematics Education* **27**(1), 2-24.

Thompson, P. (1993) Quantitative reasoning, complexity and, additive structures. *Educational Studies in Mathematics* **25**(3), 165-208.

Thompson, P. (1994) The development of the concept of speed and its relationship to concepts of rate. In Harel, G. & Confrey, J. (Eds.) *The Development of Multiplicative Reasoning in the Learning of Mathematics,* pp. 181-234. Albany, NY: SUNY Press.

Thompson, P. W. (1995) Notation, convention, and quantity in elementary mathematics. In Sowder, J. T. & Schapelle, B. P. (Eds.) *Providing a Foundation for Teaching Mathematics in the Middle School Grades,* pp. 199-221. Albany, NY: SUNY Press.

Thompson, P. W. & Saldanha, L. (2003) Fractions and multiplicative reasoning. In Kilpatrick, J., Martin, W. G. & Schifter, D. (Eds.) A *Research Companion to Principles and Standards for School Mathematics,* pp. 95-113. Reston, VA: National Council of Teachers of Mathematics.

Van den Heuvel-Panhuizen, M. (2000) Mathematics education in the Netherlands: A guided tour. Freudenthal Institute CD-ROM for ICME9. Utrecht, The Netherlands: Utrecht University.

Vergnaud, G. (1988) Multiplicative structures. In Hiebert, J. & Behr, M. (Eds.) *Number Concepts and Operations in the Middle Grades,* pp. 141-161. Reston, VA: National Council of Teachers of Mathematics.

Measurement is fundamental to the teaching and learning of mathematics because it provides a natural 'way in' both to the development of number concepts and also to the application of mathematics over a very wide field. (p. 78)

Cockcroft, W. H. (1982) *Mathematics Counts (The Cockcroft Report).* London, UK: HMSO.