LEARNING MATHEMATICS IN TIME AND SPACE

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All research in mathematics education is, at some level, about learning. Much research is, of course, explicitly focused on students' mathematics learning or on ways to bring about or facilitate or improve the conditions for mathematics learning. Many theories of learning are invoked in this research. Some that spring immediately to mind include Vygtoskian theory, enactivism, or cognitive psychology. These theories generally propose mechanisms that explain how learning happens. For proponents of a Vygotskian theory of learning, for example, higher mental functions, such as mathematical reasoning, are first developed on a social plane, through participation with more knowledgeable others. They then develop internally to become features of individual cognition.

Thinking about my own learning, however, I realise that established theories do not map perfectly onto my experience. Of course, no theory of learning pretends to provide a definitive account of learning. Nevertheless, most theories are developed from a mixture of empirical data analysis and conceptual development. Furthermore, the development of any theory of learning depends on accounts of learning: that is, descriptions of learning are fundamental to the development of the theory. For example, Vygotsky's work includes numerous accounts of his observations of children solving problems and learning new concepts. Accounts of learning may be given by the learner, by teachers observing learners, or by researchers observing learners or teachers or both. Learning, as a facet of human experience, is largely inaccessible; there is no way to directly read off anyone else's learning (or one's own). Accounts of learning are, therefore, the closest we can get. Such accounts could be in the form of narratives, descriptions, or more scientific versions involving statistics and graphs. As such, accounts of learning are widespread in reports of mathematics education research. The nature of accounts of learning within such reports rarely forms the object of inquiry. So what are they like? How are they organised? How are they constructed? What features do they have? These questions are important, because the construction and presentation of accounts of mathematics learning are as close as we can get to a sense of what learning mathematics is like and, as such, influence the theories of our field.

In this article, I focus on a subset of these questions. In terms of accounts of learning, I examine self-generated first-person descriptions. I am interested, specifically, in how the organisation of such descriptions may be understood. To do so, I draw, in particular, on Bakhtin's (1981) notion of the chronotope. Bakhtin's work offers a way to go beyond the general category of 'description' to look at different dimensions and structures of such accounts. The notion of the chronotope particularly highlights the significance of space-time in the organisation of different genres of writing. In this article, then, I explore the role of space-time in accounts of mathematics learning. This writing extends an earlier study, in which I showed how descriptions of mathematics learning presented within published research reports involve carefully constructed accounts of student behaviour that serve to permit particular interpretations or claims about these students' learning of mathematics (Barwell, 2009). I do not claim to offer a definitive or fully-worked-out theory or epistemology; my aim is to highlight the potential of Bakhtin's ideas and suggest some general directions for further work.

An example might be helpful at this point: an account of my own learning of mathematics. It is written as a simple account, rather than anything deliberately theoretical, although it necessarily bears traces of some of my assumptions about learning. I will make some initial observations and then illustrate the theoretical development that follows in subsequent sections. In the latter part of the article, two further accounts are introduced and analysed in order to draw out similarities and differences. These accounts can be described as well-rehearsed— they are stories I have related many times rather than texts constructed entirely for the purposes of this article. Their well-rehearsed nature justifies my use of them as 'data' sufficient for the exploration of the theoretical ideas at play.

> I generally found mathematics easy at school and one of the puzzles of my story of learning mathematics is that I stopped finding it easy during the latter part of my university studies. At school, I enjoyed solving problems and completing exercises, whether it be primary school arithmetic practice, algebra or linear equations or later, solving calculus problems. I would even create my own problems. I liked the fluency I developed, and would work out patterns and short cuts.
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> The highlight of my mathematics learning would have to be my sixth-form (aged 16-18) pure mathematics course. We were a small group of around a dozen strong students. What I remember was the extensive discussion. We discussed problems or aspects of what the teacher presented or information shown in our textbooks. Outside of class I also worked with classmates on homework exercises as well as on problems of our own interest.

At university things were different. I attended lectures on foundations, analysis, and group theory in large tiered lecture theatres with two hundred or more students present. There was little discussion or interaction of any kind. Most lecturers worked through blackboards of theorem-proof. Some hardly turned round to face us. A few had a more conversational style (Christopher Zeeman, for example) and would at least throw out the odd question to try to engage us. Weekly lectures were followed by problem sheets and we were assigned to a postgraduate student who would meet with two or three of us at a time.

In the first year, I did okay. We had looked at some of the principles of analysis in my sixth-form pure mathematics group, so I could make sense of things based on that. In the second year, I applied myself more thoroughly, and found some satisfaction in topics like metric spaces. On the other hand, I struggled with Galois theory, never having really got to grips with other algebra courses. I found the way of approaching proofs perplexing, and remember other students who would complete the problem sheets with ease, while I would not know where to begin. Through a combination of basic understanding and memorisation of some key proofs, I did well enough, but I was losing the strong connected understanding I had developed at school and at sixth form. Extending what I had learned at sixth form to more advanced ideas no longer worked.

This account is a version of my learning of mathematics up until the end of my undergraduate studies. It is organised as a chronological narrative, starting with primary school mathematics, and ending with my final year at university. It describes and contrasts times in which mathematics came easily and times of struggle. It is somewhat unremarkable and resembles, I imagine, how many people would describe their experience of mathematics schooling (although for many the point at which mathematics became more difficult comes much earlier in the account).

Space-time

In his essay *Forms of time and of the chronotope in the novel,* Bakhtin (1981) uses a wide-ranging discussion of literary forms from the classical period to the early twentieth century to examine the different ways in which time and space are invoked in novelistic discourse. As with so much of Bakhtin's work, his analysis of literature embeds deeper and more general theories of language and discourse.

Bakhtin's broad epistemological stance can be characterised as dialogic. This is to say that Bakhtin focuses on the relations between things, rather than on the things in themselves. For example, in his writing about genre, he emphasises the fluid nature of genres as they interact with other genres over time. Similarly, in his writing about language, he underlines how at every level—phonological, lexical, verbal, *etc.*—it is the relations between forms that are crucial. There is no absolute definition of formal mathematical discourse, for example: mathematical discourse can only be discerned in relation to other discourses (Barwell, 2016). This dialogic perspective extends to the relations between authors, texts and readers. For Bakhtin, any utterance (verbal or written), is written for someone and by someone. We hear the voice of the author, mixed with other voices, in the utterance. We can also see the influence of the addressee in the formulation of the utterance (a principle important to other theories of discourse, such as conversation analysis). Author, text and reader are all also situated in particular sociohistorical moments. Thus, while it is possible to analyse texts (such as novels), such analysis reveals or proposes something about the relation author-text-reader, as both individuals and as sociohistorical instances, rather than coming up with any kind of 'direct' or 'absolute' or 'final' interpretation. Bakhtin's general approach is often described as being about 'possibility': the idea is that no utterance, no text, can ever be completely interpreted; there is always more to say.

Based on these ideas, my account of my learning must be seen as addressing someone and as being written by someone. For example, it is written (or rewritten) in the context of this monograph, based on what I presented at the symposium on learning in honour of Laurinda Brown. I studied with Laurinda for my master's and doctoral degrees. The style of writing (of my account) reflects one of the first tasks she assigned in the first master's course I took with her, a course which was broadly about theories of learning mathematics. The writing is thus in relation with a history of interactions between Laurinda and I from the time of my studies with her to the present. While my account does not reflect any well-established genre of writing, this article does—it is in the form of an essay, of a type commonly published in FLM. Such articles often include short accounts of experience and my own account reflects this kind of writing. My account, then, must be understood through its relations to addressee, author and the situation of its publication, among other relations.

In *Forms of time and of the chronotope in the novel,* Bakhtin turns his attention to a way of making distinctions between novelistic texts from many different epochs. He sets out an extensive argument for the idea that the treatment of time and space varies significantly across novelistic genres (historical and contemporary) and argues further that they are not distinct dimensions, but represent a single unified space-time, for which his translators use the term 'chronotope' (which actually means 'time-space'). Not for the first time, Bakhtin seems to have been influenced by physics—in this case, Einstein's theories of relativity and the notion of space-time. As with Bakhtin's (1981) adoption of centripetal and centrifugal forces within his theory of language, these terms are used largely metaphorically. Nevertheless, Bakhtin appears to have been inspired by the idea that time and space are fused together in a single system within literary texts, leading to a formidable reading of literary forms across multiple traditions. An initial reading of the short account of my learning of mathematics shows that space and time are relevant features of its construction. I highlight different periods of my education, as well as different contexts and locations. The notion of the chronotope could, therefore, be valuable in looking more closely at the construction of accounts of learning.

Through Bakhtin's examination of many different novelistic forms, the idea of the chronotope becomes clear (or as clear as anything can be in Bakhtin's work—he avoids clear-cut definitions, formulae or categories). For example, in his treatment of the Greek romance, he proposes the notion of 'adventure-time' as a characterisation of the form. Greek romances begin with a young man and a young woman meeting and falling in love. They then undergo various trials and tribulations before ending in happy marriage. Bakhtin explains, through illustration, how in such stories time and space are treated in particular ways. Time is of little consequence: the young couple do not undergo any change despite the various challenges. The marriage is consequent on their falling in love and the time in between is of little temporal significance. Bakhtin highlights the significance of chance in this form, with events happening to the characters in seemingly unpredictable ways. The plot is driven by chance encounters, and fortunate or unfortunate occurrences. Similarly, space is treated in a particular way: while strange countries and their particularities are evoked and the action shifts from one location to another, these geographic changes are largely symbolic. The specific nature of real locations have little bearing on the story or its outcome; they are in a sense interchangeable.

While Bakhtin's analysis focuses on an ancient literary form, it seems to be alive and well in the form of adventure movies. In the 2018 Tomb Raider film, for example, there is no real chronological development in the life of Lara Croft or the other characters. She undergoes a series of events that happen to her (falling bridges, being swept down rivers towards a waterfall), in a range of locations that are not of crucial importance. The course of the story is affected by chance encounters (she accidentally finds one key character while escaping from a gang of thieves) that she must navigate to the inevitably victorious outcome. It seems clear that in the account of my learning of mathematics, space and time do not function in this way. I describe a change over time and this change is significant and meaningful with respect to my learning. Similarly, the places or locations are significant; university is organised differently from school, for instance.

While *adventure-time* is a broad chronotopic category used by Bakhtin to distinguish an entire genre of novelistic storytelling, he also invokes more specific chronotopic features that occur in different genres. Among these, for example, he mentions the chronotope of the *road*, a place and time in which a diversity of people meet, encounter each other and travel through. The road is, of course, a widely used feature of much fiction, precisely for the way it permits chance and human encounters to arise, within the particular temporal context of a journey. Bakhtin also mentions the specific chronotopes of the *alien world* and of the *salon*. These chronotopes often feature regularly occurring motifs, such as that of the meeting, or of the epiphany. Meetings often arise on roads, or in salons; epiphanies often happen in alien worlds. A discussion of the chronotope of the *agora* or public space, leads into a lengthy discussion of the changing concept of the individual, from an ancient concept of the entirely public individual (both socially and psychologically), to modern conceptions of private and inner lives. This last example also underlined how Bakhtin's conception of space is about more than physical location; it is also about the social context that accompanies it.

This brief summary of some of Bakhtin's treatment of the concept of chronotopes hints at its potential power in examining different story forms and relating them to wide-ranging sociohistorical themes, such as the nature of individual identity. The account of my learning of mathematics, for example, is an example of a particular type of account in which facility with mathematics is at some point lost. Although I cannot refer to examples, based on reading many such accounts written by pre-service teachers, I conjecture that the point at which mathematics becomes difficult is often associated with a change of space—either a change of school, or of grade (often associated with a change of teacher). This would suggest that there is a common chronotope associated with learning mathematics. To explore these ideas further, in the next section, I introduce two further self-authored accounts of learning, followed by some analysis of them using Bakhtin's ideas.

Chronotopic interpretations

The opening account in this article describes something about my learning of mathematics. The first account in this section describes something of my learning to teach mathematics.

> I did my PGCE [post-degree teacher education program] in Wales and then taught in Yorkshire. On the PGCE course, I enjoyed working on mathematics and I left with a clear sense of wanting children to both understand and enjoy mathematics. I did not want to teach rules and pages of repetitive exercises. In Yorkshire, I went through the learning curve of being a new teacher but I'm not sure that I had much time to work on my mathematics teaching in a systematic or coherent way.
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> In 1995, I was accepted by Voluntary Service Overseas to take up a position in northern Pakistan. I was to teach mathematics, science, and English in an English-medium village school in the Karakoram mountains, as well as offer professional development workshops for teachers in the area. The region was culturally distinctive, with several minority languages spoken, and a history of separation and isolation from the rest of the country. Most people spoke one or more local village languages and Urdu, the national language.
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> In my teaching, and particularly in my work with teachers, I came to an important realisation. While we may be using the same (English) words, these words could mean very different things. When we discussed something like the use of concrete materials to teach mathematics, for example, our understanding of the discussion was not the same. I became aware of this difference by observing the teachers in the classroom. They might be using materials in ways we had discussed, but my interpretation of these discussions had been quite different. I realised that the words we used to talk about teaching mathematics were informed by our experiences of learning mathematics, culture (in general, of education, and of mathematics), and languages.

> Since I and the teachers had very different experiences, our interpretations were also different. This awareness led to a shift in how I thought about my teaching. My goal was no longer to lead students or teachers to a particular outcome. Rather, it was to work together on activities that would allow our thinking and understanding to grow. With teachers, for example, I would facilitate work on mathematics together, and then lead a discussion of their experience of the activity. My hope was that the participants would develop new ideas and new insights into their ways of teaching mathematics, that they would then try out in their own teaching. The key shift was that I no longer sought to determine what these new ideas or insights would be.

The final account describes aspects of my learning to be a researcher.

> After Pakistan, I attended the University of Bristol and completed my master's and doctoral degrees in education. It was a time of great fulfilment. In Pakistan, I had formulated some clear interests, particularly in the role of language in learning mathematics. I had many questions. At the University of Bristol, I discovered a place where everyone had questions and read a lot and did research. I was inducted into an academic culture in which I felt at home.
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> I learned about how theory can be used to interpret experience and vice versa, whether my own or that of others. In one of the first courses I took, I was asked (by Laurinda) to write about some aspect of my experience of learning mathematics, and then read around theories of learning used in mathematics education to interpret this experience. As a result, I encountered many different ideas about learning, including Vygotsky's *Thought and Language,* to add to ideas I had already been reading, such as those of Gray and Tall, and Skemp.
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> I was introduced (also by Laurinda) to different parts of the UK and international mathematics education community, such as the British Society for Research in Learning Mathematics, the PME conference, and FLM. In Bristol, I was an enthusiastic participant in seminars, symposia and other intellectual activities. The Graduate School of Education hosted many visitors, including many working in the tradition of sociocultural theory, such as Jim Wertsch, Vera John-Steiner, Barbara Rogoff and Neil Mercer. I observed and participated in vigorous academic debates about these ideas.
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> Throughout this period, I grew, intellectually, with a focus on a couple of key problems. The question of the role of language and of languages in mathematics classrooms opened up more fundamental questions about the nature of language, the nature of learning, and difficulty of interpreting the words of children who were learners of English. I have generally seen these questions as being epistemological in nature. How can we know anything about other people's mathematical thinking? How can we know anything about other people's mathematical thinking when we do not speak the same language or share similar cultural, socioeconomic or linguistic backgrounds?

Time is a feature of my three accounts of learning. There is a difference between the first two accounts and the third. The first two accounts—of learning mathematics, and learning to teach—describe transformational periods of learning. Each account is in two parts. In the first account, the ease and pleasure of learning and doing mathematics is described, followed by a description of university mathematics that is couched in rather different terms. In the second account, a rather functional account of my early teaching of mathematics is followed by a psychologically livelier description of thinking and responding to new circumstances. In each case, there is a before and an after.

Time and space are intimately related in these first two accounts. The temporal division is between an initial 'grounding' period in which a certain pattern of activity is established and a second period in which a transformation occurs. Ease of doing mathematics becomes not knowing where to begin. The new teacher's 'learning curve' is in contrast to a new and deep awareness of something fundamental to the teaching process. This temporal division is inseparable in each account from a spatial shift. In the first, this shift is from a school environment, and more specifically from the sixth-form pure mathematics class, to a university lecture theatre.

The spatial change is apparent in changes in how relations with others are described. In the school environment, there is discussion, working with peers, and a sense of autonomy. In the university space, there is silence, a sense of estrangement from individuals (the ones who could solve the problem sets) and from mathematics (not knowing where to begin). In the second account, the temporal transition is associated with a spatial change, not just to a different institution, but to another country. The account highlights the unfamiliar languages, culture and educational norms. In my account of teaching in the UK, there is an expected 'learning curve' with the implication that most teachers go through a similar process. In my account of teaching in Pakistan, there is difference (relative to the author) and this difference is crucial in the new awareness and shift in practice as a teacher and teacher educator.

In both these accounts, learning has not simply happened 'over time', which could be interpreted to mean that with time learning will happen. Learning has happened 'through space'. The conditions in Pakistan are central to the learning that is described, just as the conditions at university are constructed as central to the change in learning of mathematics. The differences in time and space are constructed in the language of the accounts. In particular, contrasts are drawn (accounts of learning often feature such contrasts, see Barwell, 2009). For example, enjoying solving problems contrasts with not knowing where to begin. A small group of students is contrasted with 'two hundred or more students'. A focus on 'understanding and enjoying *mathematics*' contrasts with a focus on wanting *learners*' 'thinking and understanding to grow'. These two accounts can therefore be seen as similar. They rely on a particular configuration of space and time: an initial 'normal' in terms of the practice of mathematics or mathematics teaching, followed by a shift in time and space to an 'alien world' in which things are different. As a result of this difference, a change happens, a change that is about learning. In the first, the nature of learning mathematics changes: it is about extending ideas previously learned in the previous time-space and even about memorising proofs. In the second account, the nature of learning to teach changes; it is about responding to the alien world to rethink ideas about teaching. There is a loss of agency in the first account; an appearance of agency in the second. These two accounts then seem to have similar chronotopes, which could be characterised as 'transformation in an alien world'.

The third account is chronotopically distinct when compared with the first two. There is no strong 'before and after'. Of course, the account references a before, but it is not central and is not contrasted with a later period. So what is the role of time in this account? In contrast with the first two accounts, time is less important in the third account than place. There is an implicit sense of learning through participation, so that there is an accumulation of experience and a sense of gradual insertion into an academic community (in the manner of a process of socialisation, for example). But no sharp temporal boundary in learning is indicated. Indeed, the learning at the University of Bristol is portrayed as allowing a continuation of previous intellectual activities. Reading, thinking and research in Pakistan is developed in Bristol. There is a sense of continuity, despite another spatial transition. This sense of accumulation is apparent in, for example, the ongoing rounds of seminars and other activities at the Graduate School of Education. A kind of continuity is suggested by the focus on key questions or problems that remain stable over time, as new ideas, researchers or communities are encountered. On the other hand, space is of great importance in the third account: the space is not only the physical location, but that which was experienced in that location, *i.e.* the intellectual community. It encompasses individuals, activities (*e.g.* seminars), academic practices (*e.g.* writing for proceedings, debates), intellectual traditions (sociocultural theory). These features form an academic landscape (in which Laurinda was a guide) into which, according to my account, I entered and found a place to grow. Time and space are again interwoven, but in a different way. Time is cumulative, gradual, lacking in abrupt transitions. Space is stable (though not static), a community with its practices and traditions. The chronotope for this account is different from that of the first two accounts, and could be characterised as 'growth through participation'. On reflection, another difference between the first two and the final account strikes me: in the final account, there is a guide; in the first two, I am feeling my way, largely on my own.

Accounts and learning

What does the notion of the chronotope offer the understanding of mathematics learning? After all, many theories of learning are already available. In a Vygotskian account, for example, learning mathematics involves internalising concepts first developed on the social plane. Vygotsky's theory has informed many contemporary theorists, including Sfard (2008), for whom learning mathematics is equivalent to learning the discourse of mathematics. The theory of communities of practice (Lave & Wenger, 1991; Wenger, 1998) has been influential in investigating mathematics teacher learning. From this perspective, learning to teach involves becoming an expert member of a community of mathematics teaching. In complexity-based approaches, learning is understood as change in a complex system consisting of learners, teachers and concepts (Davis & Simmt, 2006), while from an enactivist perspective, teacher learning has been characterised as the emergence of effective behaviours (Brown & Coles, 2011).

These various theoretical perspectives and their application have two things in common. First, space-time is present, but largely assumed. In all of these theories, learning involves some kind of change over time. And since all of them emphasise social dimensions of learning in different ways, they all have a spatial dimension, in the sense that learning mathematics or learning to teaching mathematics are understood as taking place in particular contexts. In these theories, then space (*i.e.* place plus context) and time are part of the background; the notion of the chronotope highlights the significance of space-time as a potentially fundamental dimension of learning. Second, the application of these theories relies on accounts of learning. Vygotsky's work includes accounts of children's problem-solving. Lave and Wenger's work includes accounts of learning in tailors' workshops and meat-packing plants. Brown and Coles include excerpts from meetings of a group of mathematics teachers. But these accounts are generally treated unproblematically, as neutral descriptions of learning (although enactivists recognise that they are written by an observer). But such accounts are never neutral, as I have already suggested; they are written for someone and by someone. So what is the relationship between learning and an account of learning?

Bakhtin (1981) discusses the relation between text and reality. He is clear that there is a distinction between what is represented and the 'world outside the text', as well as between the author-creator and the author as human being, although this distinction is not be understood as an impermeable boundary (p. 253). Accounts of mathematical learning are texts, with authors and readers and as such are textual versions of learning and of the situations in which learning arises; they are not direct accounts of learning 'as-it-happens'. Similarly, the author of a text about learning is understood to be different from the learner or observer of learning, inasmuch as the creation (uttering) of the text is a distinct activity from whatever is described. The author of the text is writing for someone, not simply describing. There is, of course, a relationship between text and world, and this relationship is dialogic:

> The work and the world represented in it enter the real world and enrich it, and the real world enters the world and its world as part of the process of its creation, as well as part of its subsequent life, in a continual renewing of the work through the creative perception of listeners and readers. (p. 254)

The role of the author-creator is, in part, one of observer. The observer is working from an 'unresolved' place and time, what Bakhtin calls a 'still evolving contemporaneity', both of the individual author-creator, but also of a particular sociohistorical moment. Accounts of learning reflect prevailing ideas, tropes and genres about learning.

Texts about learning are to some extent always biographical. They describe a little bit of someone's life. In some cases, as in the examples included in this article, they are autobiographical. At the same time, they do not directly represent the actual author-creator; rather, in more contemporary parlance, they construct a version of the author-creator. The self-authored accounts of learning in this article tell you, the reader, something about me, the author, not simply through what is written, but through how it is written.

Bakhtin sees the idea of the chronotope as more widely relevant, almost universally so. He suggests, in a closing remark added at the end of the essay forty years after its composition, that the path to all human meaning is through 'the gates of the chronotope'. What he means is that abstract thought, including mathematics (to which he specifically refers), makes use of signs; and these signs are produced and interpreted in time and space—they can never stand outside of a chronotopic context. Hence by understanding the chronotopes of accounts of learning mathematics, we can get some insight into the process of making sense of learning mathematics (or learning to teach, or learning to be a researcher, *etc*.).

Concluding remarks

This writing is in part a first exploration of the notion of chronotope. Having worked through a first distillation of Bakhtin's ideas, I have applied them to accounts of my own learning. Of course, there is a certain artifice involved. As author of the accounts, I can adjust them to suit my ideas. Nevertheless, they are relatively stable stories that I have more than once told about my learning over the years. The distinction between two chronotopes in these accounts suggests that this kind of analysis is likely to be productive if applied to other accounts of learning. Such an analysis applied to mathematics teachers' accounts of their learning, or children's accounts of learning mathematics, could lead to some valuable insights about how individuals make sense of their learning, at least when expressed to others. I do not believe that direct access to learning, or observation of learning is possible. Often, however, the constructed nature of accounts of learning is not mentioned, or worse, is treated as some sort of bias. There is a need for methods of interpreting or analysing accounts of learning, whether in the form of récits, as here, or in other forms, such as tables and graphs, that treats them as accounts. As Bakhtin points out, the creation and interpretation of any text is a process that has its own chronotopic space. Chronotopic analysis may be a way to examine accounts of learning in relation to the sociohistorical conditions of the author-creator.

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