Facilitating Symbolic Understanding of Fractions

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Operations with fractions constitute many of the difficulties elementary school children experience in their study of arithmetic. Behr, Lesh and Post [1981] cite this preponderance as one of their motivations in investigating the teaching of rational numbers in the elementary school setting. Typically, fractions are introduced during the fourth grade and remain a substantial part of the arithmetic curriculum through the seventh and, frequently, eighth grades as well. And yet, difficulties with fractions persist in pervasive numbers, seemingly recalcitrant to the amount of attention given them.

We explore in this paper an alternative approach to the traditional sequence of concepts used in teaching rational number operations. Traditionally, fractions are presented to the learner in the semi-concrete domain of pies, bars and/or unit-length number lines [Larson, 1980]. It is our position that this presentational framework does not generalize to support the concepts necessary to understand the full spectrum of rational number operations. Our position is developed through an analysis of sign and symbol systems and includes anecdotal data derived from four-student protocols [Noddings, et al, 1981].

Symbolic processes

Mathematics is a particularly appropriate field of study to view from the vantage of sign and symbol theorists. Such early theorists as Susanne Langer and Charles Peirce characterize mathematics as a vast system of symbols, whose manipulations, according to Langer, follow the "undisturbed way of pure reason " [Langer, p 13] Langer writes:

> It (the science of mathematics) fell in so nicely with the needs of scientific thought, it fitted the observed world of fact so neatly, that those who learned and used it never stopped to accuse those who invented and evolved if of being mere reasoners, and lacking tangible data. . . . What is the secret power of mathematics, to win hardheaded empiricists, against their most ardent beliefs, to its purely rational speculations and intangible "facts"? . . . The secret lies in the fact that a mathematician does not profess to say anything about the existence, reality or efficacy of *things* at all. His concern is the possibility of *symbolizing things*, and of symbolizing the relations into which they might enter with each other. His "entities" are not "data" but *concepts*. [Langer, pp 13-14]

In his writings, Charles Peirce also underscores the conceptual nature of mathematics:

> Since all the subjects of the mathematician's discourse are figments of the brain, when we speak of the existence of anything in mathematics, we mean its existence as part of the hypothesis. [Eisele, p 56]

In emphasizing the hypothetical, however, Peirce is careful to point out that even though the objects of mathematical reasoning are hypothetical, mathematical reasoning itself is dependent upon and guided by the concrete representations of the problem to be solved. In making this point, Peirce argues that there are two very different tasks necessary to mathematical reasoning: (1) that the mathematician construct a picture of the problem, a "diagrammatization,"the principal purpose of which is "to strip the significant relations of all disguise" [Pierce, p. 339]; and (2) that he submit this "diagram or visual array of characters or lines" [p 339] to the scrutiny of observation, and from that observation and experimentation "new relations are discovered among its parts, not stated in the precept by which it was formed, . . ." [p. 339] "Thus," he writes,

> the necessary reasoning of mathematics is performed by means of observation and experiment, and its necessary character is due simply to the circumstances that the subject of this observation and experiment is a diagram of our own creation, the conditions of whose being we know all about. [p 339]

Peirce's message contains an important juxtaposition; that is, even though the objects of mathematical reasoning are hypothetical, its initial objects are diagrammatic and its processes are guided by the manipulations of the elements in that concrete representation. In a very important sense, then, mathematics is perceptual as well as conceptual.

Exactly what goes on in the head of the learner engaged in mathematical reasoning is, of course, beyond the powers of observation. Yet, the relationships created between the problem to be solved, the diagrammatization of the problem, and the manipulations of the elements of that diagram remain essential steps in the reasoning process. According to Peirce, "All thinking is in signs." [Eisele, p.

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56] He characterized the signing process as the triadic interrelation between the concrete objects involved in the problem, the sign which stands for those objects, and the signification, or meaning, of the sign. The signification serves as a mediator, making "sense" of the object-sign relation The function of the sign is to stand in place of the object, to allude to the object but remain "more easily available than" [Langer p 46] the object In diagrammatic form, the signing process can be presented as in Figure 1.

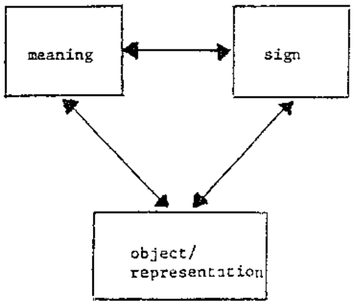


Figure 1
Diagrammatic representation of signing process

There is no linear path that is universally necessary to proceed from one element of the signing process to another. Rather, in most cases, they are coexistent: that is, signs are used in the concrete representation of the problem to capture significant interrelations. In fact, Langer claims that the object and its sign are entirely interchangeable:

> Note, . . . the subject's relation is to the *pair* of other terms, . . one of them the sign and the other the object What is the difference between a sign and its object . . ? (The) two terms . . . associated as a pair, like two socks, two balances of a scale, two ends of a stick, etc , could be interchanged without any harm [Langer, p. 46]

Langer's notion of the signing process is different from that developed by contemporary sign/symbol theorists Janellen Huttenlocher and E Tory Higgins [1978]. They claim that, while there is no necessary path from one element of the signing process to another, there is an order habitually established such that the elements are not equally likely to activate each way The relations are "frequently unidirectional" [p. 104], and are not established in a one-to-one correspondence as Langer indicated. In diagrammatic form, their conception of the signing process can be presented as in Figure 2

Huttenlocher and Higgins proceed to distinguish the signing process from a fully-developed symbolic process. "Intuitively," they write,

> symbolic links differ from other associative links by virtue of the fact that their function is to call to mind the elements to which they are linked. [Huttenlocher and Higgins, pp 102-103]

Langer, too, makes this distinction when she writes:

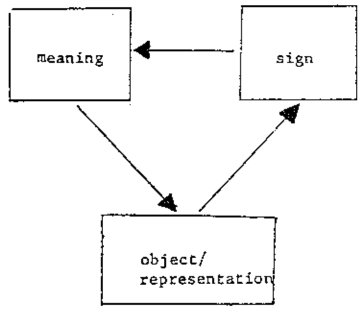


Figure 2
Diagrammatic representation of Huttenlocher and Higgins' conception of the signing process

> Symbols are not proxy for their objects, but are *vehicles for the conception of objects*. To conceive a thing or a situation is not the same thing as to "react toward it" overtly, or to be aware of its presence In talking *about* things we have conceptions of them, not the things themselves; and *it is conceptions, not the things, that symbols directly "mean"* . signs *announce* their objects to (us), whereas symbols *lead (us) to conceive* their objects [Langer, p 49]

Huttenlocher and Higgins go beyond Langer's distinction, however, and characterize the symbolic process as involving a "fixed functional relation" [p. 105] among the elements, hence a bidirectional relation as well. Their conception of a fully-symbolic process is similar, then, to the signing process illustrated in Figure 1.

In important ways, symbolic relations are "fixed": they do not migrate or dissolve once established Further, symbolic processes are not totally unlike signing processes. It is our position that the symbolic process is rooted in and proceeds from the signing process. However, to say a symbolic process is a fixed relation does not capture the fluid nature we feel characteristic of the symbolic level of understanding. Rather, we offer the diagrammatic representation of a symbolic process shown in Figure 3.

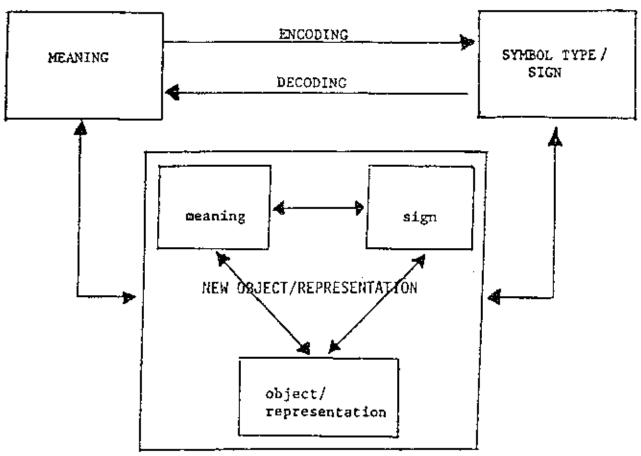


Figure 3
Proposed diagrammatic representation of symbolic process

By characterizing the symbolic process as a nested series of signing processes, several aspects are highlighted: (1) that the core of the symbolic process is a fixed, one-to-one sign object association which is preserved through iterative symbolic processes; (2) that symbolic understanding progresses from signing processes; (3) that symbolic understanding is telescopic in its "unfolding" nature, that is, it grows by repeated extensions of the basic triadic pattern relating sign, object and meaning; (4) that successive iterations over the initial signing process integrate the new symbol into existing symbolic understanding, placing the new symbol into contrast and comparison with existing symbols; (5) that successive iterations over the initial signing process bring nuances to the original meaning achieved at the signing level

Signing and symbolic processes in presentation of rational number concepts

With this characterization of signing and symbolic processes, let us turn our attention to the treatment of rational number operations most commonly developed in mathematics textbooks. A recent article by Carol Novillis Larson begins:

> Geometric regions, sets, and the number line are the most commonly used semi-concrete models for fractions in elementary school textbooks. [Larson, p 423]

The number line model, she points out, is often not fully developed; frequently, the number line presented is of unit length and, hence, represents a one-dimensional version of the geometric regions/part-whole model The metaphor is both clear and familiar: fractions are parts of things Further, the metaphor communicates that fractions, as parts of things, are not numbers.

How, then, might students use this metaphor to make sense of fractions? In the pre-operational, fraction-readiness stage, students might demonstrate the process shown in Figure 4

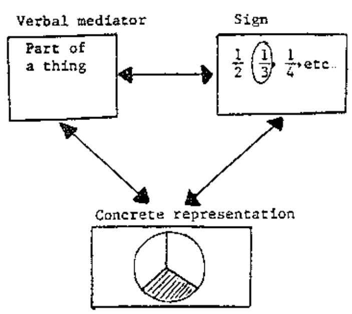


Figure 4
Potential signing process appropriate to pre-operational, fraction-readiness stage

However, when students are introduced to addition and subtraction of fractions, particularly those with unlike denominators, the metaphor offers little assistance (Figure 5).

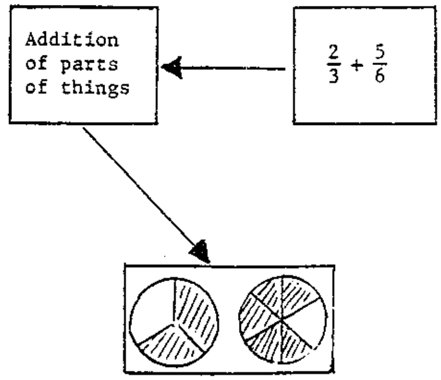


Figure 5
Representation that resists application

In fact, the metaphor diverts the student's attention from potentially helpful avenues of thinking. If students were introduced to fractions as an extension of whole numbers, in particular as numbers whose denominators can be other than one, students might be able to develop some heuristics for operating with fractions by activating what is known about operating with whole numbers Instead, the metaphor remains non-facilitative, and hinders the achievement of symbolic understanding in the number system.

We can hypothesize some of the systematic errors children might make while functioning under the influence of the "fractions as things" metaphor. Certainly, students would find difficulty in comprehending improper fractions; we might expect them to say something like, "If fractions are parts of things, then how can we talk about a part of a thing that is larger than the thing we started with?" Addition and subtraction of fractions can be understood using the "parts of things" metaphor, as long as the fractions have a common denominator. As indicated in Figure 5, however, students would find no help from the metaphor in coping with uncommon denominators while adding or subtracting. The metaphor simply does not generalize to give assistance in determining the steps necessary to constructing equivalent fractions with common denominators

Multiplication of fractions cannot technically be treated within the metaphor; rather, it is disguised as operations with whole numbers. Let us pursue an example: (2/3) × (1/5) We might expect students to conceptualize the problem as follows:

> Take a thing and divide it into 5 parts Divide one of those 5 parts into 3 equal pieces. Then take 2 of those "twice-divided" pieces. That will represent (2/3) × (1/5).

In other words, students would return to the primary definition of whole number multiplication and division. This conceptualization of multiplication of fractions might hold until the process is expanded to include mixed numbers. We suspect, however, few students would stumble upon improper fractional equivalents as the necessary intermediate step to multiplying mixed numbers. Nor can division of fractions be treated under the metaphor. We would expect students to memorize the algorithm for how division of fractions is accomplished without being able to explain why the various steps in the algorithm are necessary. Evidence of this might include the case when students remember that one of the fractions gets "flipped over," but not remember which one.

In general, what we would predict from our analysis is that the "fractions as parts of things" metaphor would generate at best a signing-level process and that symbolic iterations would not be forthcoming. Further, we suspect the metaphor directs the student back to the things fractions are supposedly a part of, forging a dependence upon the concrete objects, in which students are searching to find new relations to redirect their thinking. We suspect the ultimate failure of the metaphor to account for the variety of operations on fractions would force the student to learn rotely the algorithms for each of the operations, resulting in a signing-level "know-how" that remains isolated from the understanding of whole numbers previously achieved.

An actual case

How students actually think as they solve word problems, in general, and how they treat word problems involving fractions, in particular, provided motivation for an earlier study [Noddings, et al, 1981]. The focus of this current paper was stimulated by data gathered in that study. Below is a brief description of the study, followed by discussion of relevant student dialogues.

The design of the word-problem research began with the identification of twenty-four students from the fifth and sixth grade student populations of one school. The students were formed into six groups so that between any two group members there was a spread of no more than 1.5 years in the grade level equivalents for either math or reading as measured on the California Test of Basic Skills. The groups were assembled once a week for a total of four sessions to record discussion of the word problems. Each session was divided roughly into two segments: the problem solving segment, and the debriefing discussion. During the problem solving segment, students rotated the responsibility for both reading the problem aloud and recording the agreed-upon solution on a group answer sheet. During the debriefing segment which followed, students were instructed to compare their answers to the solution key and go back to any problems they had missed to figure out how they could get the correct answer. These were the only directions students received. In all other ways, student dialogue remained undirected.

Each group of children worked eight problems in each of their four sessions. The problems were taken from a computer-based curriculum, *Problem Solving*, [Noddings, 1978]. The problems presented to each group over the four sessions were variants on eight "generic forms", and the problems appeared in different orders at each session. Of the eight generic forms used to generate each problem set, the one we are interested in here is:

NAME		OBJECT
[Maria / Sue / Tom / Joseph / James / John / Bob]	ate [1/n] of a	[candy bar / cake / pizza / pie]

How much did [she / he] have left?

Variable: $2 \leqslant n \leqslant 9$, $n \in$ { Natural Numbers }.

Two difficulties appeared in the transcripts of every group. First, although the children drew circular diagrams and marked off the appropriate number of pieces, they had greater difficulty representing fractions with odd denominators; indeed, they seemed to think consistently in terms of "halves". Second, and more to the point of the present concerns, someone in *every* group asked: How big is the pizza (pie, cake, etc.)? In no case did the other students reject this question as irrelevant. We see both difficulties in the following dialogue:

Carol:	"James ate 1/3 of a pizza. How much did he have left?"
Ann:	3/4?
James:	Wait. Oh, yeah
Ann:	OK, 1/3. He had left . . .
James:	He had a half. He had half of it left.

James seems to interpret any part as "a half." But notice how easily he convinces the others that such language is appropriate:

Ann:	OK
Harry:	How big a pizza any w . . ?
(This is the universally asked question.)	
James:	No. See it has thirds. This is one . . . watch . circle . that's one, right? This would be two, and this would be three. He ate one of the three so that he has two left
Ann:	One . . two. That's one, two, three.
James:	It's two halves.
Ann:	He had one half.

This dialogue continues for more than 60 acts beyond Ann's statement, but the group finishes with "two halves" as an answer. In the debriefing session, when the answer "2/3" is exhibited, James continues to be confused:

James:	How do you get 2/3 when you only had 1/3?

At this point, Harry takes over and gives a fairly coherent explanation. He and Ann carry on a long two-person dialogue; James remains silent. In their second session, however, James speaks up promptly. The problem is, "John ate 1/9 of a pie. How much did he have left?"

James: It'd be 1/8.

James had drawn a circle with 9 parts and 1 part blackened. He was using the "parts of a thing" metaphor with visual accuracy. After considerable discussion, he explains his answer which is accepted and duly recorded.

James: Pies can have just one piece, or two, or three, or four, or whatever. If you cut it in nine, then you have nine. See, they cut nine out of .
Ann: Why? He ate 9, didn't he?
James: So it'd be 1/8. If there were 9 pieces – he ate only one – it'd be 1/8 .

Apparently, James had learned in the first debriefing session that parts are not always called "halves" and that the proper answer somehow involves the number that represents the pieces left. Hence, now, the part left is "1/8". In the debriefing for this session, the other children seem to catch on, but James is still looking for a rule by which to decide how the answer should be reported. First, he repeats his initial thinking:

James: Yeah, there was 9 pieces, and you took one away so that leaves 1/8 .
Then, after some discussion, he says:
James: There's 9 pieces in the pie and he ate one so there would be 8 left . 9/8.
The other children try to straighten him out on this, but he says:
James: He had 8. Then it would be 8/9? It would be 9/8.

Again, the dialogue moves away from James as the other children take turns explaining how to get 8/9 as an answer. After several pages of dialogue, James agrees that 8/9 must be answer, but just as everyone breathes a sign of relief, he says:

James: But there were . .

The session ended without an opportunity for James to elaborate his "But . . ." In the next session, when "Bob ate 1/6 of a cake," James again was first to answer:

James: Oh, that's easy, 5/6

This time, he successfully led the others through a sound explanation. Everything seems to have fallen into place. Witness what happens, however, in the fourth and final session.

Carol: "Bob ate 1/8 of a pie. How much did he have left?
Carol: 1/7 .
(Jame's earlier error is repeated by Carol.)
Carol: 1/7 .
James: It's 8/7 .

Once again, we listen to a dialogue in which the children cast about for some rule that will help them to connect the picture to an appropriate symbolic form. Harry supplies such a rule, but we shudder to think where this will ultimately lead:

James: . . so 8/7
Harry: 7/8
James: Yeah, 7/8
Harry: The . . . the bigger one can't be over the smaller one

This sequence of dialogues is not different from that of other groups in the conceptual errors it reveals. All kinds of tricks are tried to produce answers. In another group, someone volunteers "1/8" as an answer, someone else "1/5," and a third child writes "(1/5).(1/8) = 1/40." When 1/40 is rejected, he tries, "(1/5).(1/8) = 2/13.

After reviewing page after page of this, one is moved to ask: Is all this confusion inevitable in teaching and learning fractions?

Teaching for symbolic understanding

We propose an alternative to the traditional treatment of rational number operations, an alternative which we feel will facilitate concept formation and integration, hence symbolic understanding across the number system. Our central hypothesis is that fractions are perfectly good numbers and are best taught as numbers. We offer below an outline of our proposed approach.

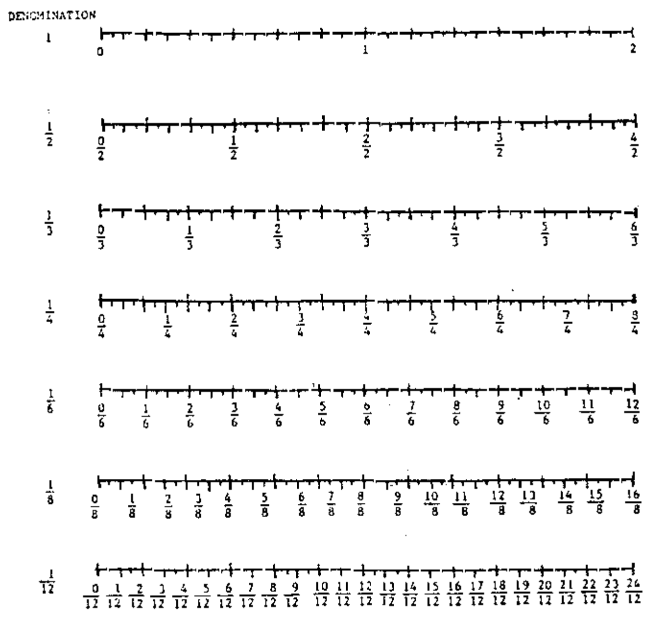


Figure 6
Locating equivalent fractions by counting on carefully scaled number lines

Students are typically introduced to whole numbers through counting. If students are introduced to fractions by counting on number lines, a signing process can be established from which symbolic iterations can both integrate over whole numbers as a subclass of rationals as well as set the stage for the formation of the concept of equivalent fractions. This can be achieved through much work with number lines of greater than one unit in length. We include Figure 6 as an example of an exercise in which students might have been instructed to count in several denominations: 1, 1/2, 1/3, 1/4, 1/6, 1/8, and 1/12. Students can locate several names for a point by locating that point on other lines. For example, we can drop a line from 3/2 through the subsequent lines and visually determine the following equivalences: $3/2 = 6/4 = 9/6 = 12/9 = 18/12$. We anticipate that such work might set up the signing process shown in Figure 7.

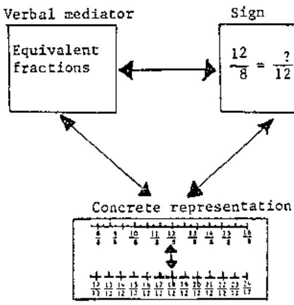


Figure 7
Potential signing process for constructing equivalent fractions

Once students have mastered constructing fractional equivalents, they can be encouraged to operate with equivalent fractions. We envision the following sequence of activities that focus on the properties of the number one. Each is demonstrated in whole numbers first, and is then expanded over the set of rationals (Figure 8).

RULE	EXAMPLES	EXPECTATIONS
A number times one equals itself	$4 \times 1 = 4$ $15 \times 1 = 15$	$\frac{1}{2} \times 1 = ?$ $\frac{1}{3} \times 1 = ?$
A number divided by one equals itself.	$\frac{4}{1} = 4$ or $4 \div 1 = 4$ $\frac{15}{1} = 15$ or $15 \div 1 = 15$	$\frac{\frac{1}{2}}{1} = ?$ or $\frac{1}{2} \div 1 = ?$ $\frac{\frac{1}{3}}{1} = ?$ or $\frac{1}{3} \div 1 = ?$
A number divided by itself equals one.	$\frac{4}{4} = 1$ or $4 \div 4 = 1$ $\frac{15}{15} = 1$ or $15 \div 15 = 1$	$\frac{\frac{1}{2}}{\frac{1}{2}} = ?$ or $\frac{1}{2} \div \frac{1}{2} = ?$

Figure 8
Sequence of activities with "1"

After much work with the above activities, students might then be asked the value of the following examples:

$$(4/1) \times (2/2) = ? \quad 4/(8/8) = ?$$

The transition to multiplication and division of fractions will, like the plot of a good novel, have been foreshadowed, in this instance by ample work with the properties of the number one. At this point, we offer only a sketch of further activities in multiplication and division. Again, the framing for the rules for multiplication and division of rationals can first be cast in whole numbers, and then followed into rationals:

$$8 \times 7 = 56 \qquad 14 \div 2 = 7$$

and

$$(16/2) \times (21/3) = ? \qquad (28/2) \div (6/3) = ?$$

Following students' mastery of multiplication and division of fractions, we suggest a return to work with number lines, where addition and subtraction of fractions with common denominators can be treated with the counting algorithm. Mixed number operations, including borrowing during subtraction, can be graphically demonstrated as well.

Continuing with number lines, we propose two methods for graphically determining the lowest common denominator/multiple (LCD). The first, and simplest, might be used to introduce students to the use of equivalent fractions with a common denominator. Consider the following example:

$$(1/3) + (1/4) =$$

We anticipate that students might produce the symbolic understanding of the problem shown in Figure 9.

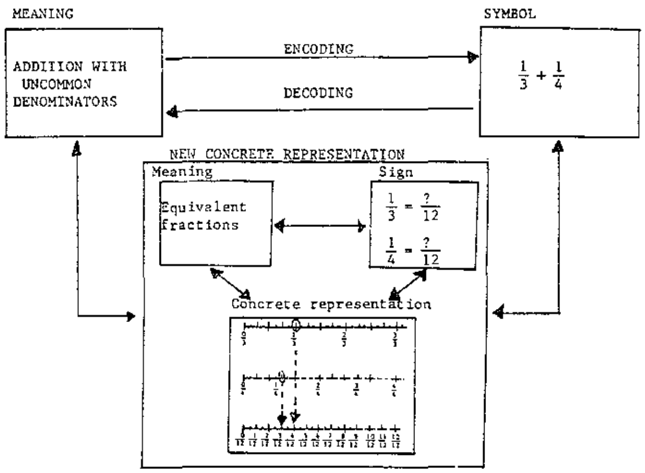
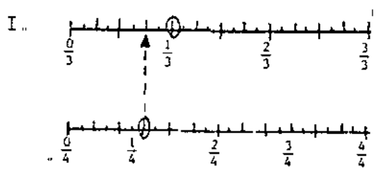


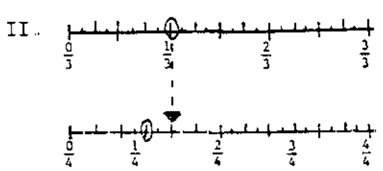
Figure 9
Potential symbolic process for addition of fractions with uncommon denominators

This method, however, is dependent upon the presence of the third number line, whose denomination is that of the lowest common denominator/multiple. A stronger algorithm can be presented to graphically determine the LCD. Considering the same example as above, we project one unit of one denomination onto the other number line; it is not important which is projected. We show both (Figure 10).

Once students have determined the lowest common denominator, they can proceed to draw a solution to the problem, utilizing the concept of equivalent fractions. The symbolic processes used will be similar to those in Figure 9 above.



Note this projection is $\frac{1}{4}$ of the distance less than 1/3. Thus,
$LCD = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$.



Note this projection is $\frac{1}{3}$ of the distance between 1/4 and 2/4. Thus,
$LCD = \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$.

Figure 10
Graphic method for determining LCD

Conclusion

In essence what we have proposed is a sequence of concepts which, we feel, facilitates a transition between whole number and fractional number operations, demonstrating in the process that the operations are consistent across the class of rational numbers It encourages students who have achieved symbolic integration of whole number concepts to use that understanding to mediate their understanding of the larger realm of rational numbers. Underlying our proposal is our belief that it makes sense to consider a guiding theory while constructing curricula; we have embedded our approach to the instruction of rational number concepts in a consideration of signing and symbolic processes

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Note

For ease of setting, fractions in the text have been printed in horizontal form Readers will understand that this is not how the authors, or the children they quote, actually wrote them – Ed

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