THE ERLANGEN PROGRAM REVISITED: A DIDACTIC PERSPECTIVE

CINZIA BONOTTO

One of the ideas which dominates a large part of contemporary mathematics is the notion of group. In 1898 Poincaré attributed an *a priori* nature to this concept. He said that the general concept of group is pre-existent to our minds, at least potentially; it imposes itself on us not as a form of our sensibility but rather of our understanding

The era of the dominance of the group concept in geometry was inaugurated in the second half of the last century, by the German mathematician Felix Klein (1849-1925) with a document that has passed into history under the name of the Erlangen Program. This work is in fact also explicitly cited, as part of the suggested content for the theme *Geometry*, in the Italian experimental programs for mathematics in the final three years of Italian Classical and Technical High Schools (*Licei*, students 16-18 years of age):

> Equations of isometries and similarity transformations. Affinities and their equations Invariant Properties

and in the associated comments:

> From the isometries [ ] one passes to the study of similarity transformations and then to that of affinities, considering the geometric properties invariant with respect to the various transformations. This procedure, which is in the framework of Klein's conception of geometry, will tend to make the student see the progressive widening of the transformation groups and the consequent reduction of the properties of the various figures as one passes from the geometry of congruences to affine geometry.

For Klein, the true object of a geometry on a set S is the study of the properties of the figures of S which are invariant with respect to a group of transformations G on S, in the sense that if a figure A of S enjoys a certain property, then also the transform of A by an element of G enjoys the same property. Such a geometry can therefore be characterized by the group G. Hence, one may speak of the geometry *associated* with a group of transformations.

Roughly speaking we can schematize Klein's work by a triangle (see Figure 1) whose vertices are the set S, the group of transformations G on S, and the set P of properties invariant with respect to G. Thus Klein studies the mutual relations between the three vertices, for example how a change in the vertex S or G changes the other two vertices, or how they can be changed.

The drafters of the Italian Ministerial Program wished to emphasize many of the characteristics of the work of Klein. The program contains not only the concept of a *transformation*

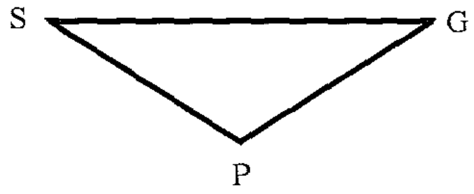

Figure 1: The schematization of Klein's work.

group but also that of geometric properties *invariant* with respect to the action of such a group In this perspective, inclusions are reversed, in the sense that to an enlargement of the group there corresponds a reduction of the corresponding invariant geometric properties. Indeed, if H and G are transformation groups and H is a subgroup of G, then the properties which are invariant for G are also invariant for H, while in general the converse does not hold true. For example, since the group of isometries is a proper subgroup of the group of affinities, *a property invariant under affinities (in brief an affine property) is also invariant under isometries (also called a metric property)*. From the point of view of the geometries associated to the two groups, affine geometry is *properly included* in Euclidean geometry, the latter being the geometry associated with the group of isometries So, all the theorems which hold in affine geometry also hold in Euclidean geometry, but not *vice versa*. For example, Thales's theorem, which is a theorem of affine geometry, also holds in Euclidean geometry, while Pythagoras's theorem, which is strictly metrical in nature, is not a theorem of affine geometry.

Why insist on these aspects referring to work so thoroughly studied, at least in a historical perspective? I often hear the phrase "geometric transformations are in agreement with the spirit of the Erlangen Program" cited as the basis of the choice of didactic paths which, in reality, do not reflect its conceptual framework. In some Italian didactic programs, the underlying group structure (of the isometries or similarity transformations) is not brought out at all, and then "that significant link with geometric invariants which is the essence of the Erlangen program is also missing" (Villani, 1995).

To avoid this distortion of Klein's work, due to the fact that it is perhaps more discussed than studied, I will seek to illustrate some aspects of the Erlangen Program, presenting certain passages taken directly from it [1] I believe that this can be useful in assisting teachers at any level to be able to approach the historical sources directly They could then act as mediators on the basis of their own experience and preparation, naturally bearing in mind the reality of the classes which they face, rather than mediating that which has already been mediated by others [2] In this way, a classroom could become a community of inquiry with students working in the pattern of mathematicians, in accordance with Furinghetti (2000) and as advocated, for example, in Freudenthal (1982) and in Lampert (1990) The idea is to avoid the presentation of a polished theory, rather constructing it together with students along the path of the original work. The aim is to make students reflect on the meaning of mathematical theories through experiencing historical moments of their construction, promoting a deep appreciation of the theories studied [3]

I will also touch on other ideas, perhaps less well known because Klein adds them to the final notes, but of considerable interest for their connection with certain themes such as the relation between intuition and formalization, the use of analytic methods or synthetic methods and the influence (or lack thereof) of different ways of representing space. The quality of Klein's advice is surprisingly high, in view of the young age (23 years old) at which he wrote the work, and his consequent lack of experience in teaching at the time. [4]

Some motivations at the base of the Erlangen Program

What was the condition of geometry in the years when Klein began his mathematical career? The discipline had reached unhoped for goals In the hands of Poncelet, Chasles, and von Staudt the bonds were loosened between metric and projective properties; with the work of Lobacevskij and Bolyai non-Euclidean geometries were created and models found for such new geometries within the realm of Euclidean geometry. It had generalized the traditional concept of coordinates (Möbius, Plücker), begun the path toward abstraction with the introduction of spaces of more than three dimensions (Grassman) and, indeed, it had deprived space of its characteristic as a neutral receptacle and rendered it the object of mathematical investigation (Riemann, Helmholtz). [5]

By the end of the 1860s, geometry appeared to be a discipline that had grown tumultuously, with its various unrelated ramifications lacking a unitary conceptual framework. Klein noted this lack of internal unity in geometry. Indeed, in his introduction he states,

> geometry, which is after all one in substance, has been only too much broken up in the course of its recent rapid development into a series of almost distinct theories, [...] which are advancing in comparative independence of each other (Klein, 1893, p 216)

The most important role is said, in the same introduction, to be that played by projective geometry,

> Among the advances of the last fifty years in the field of geometry, the development of *projective geometry* [...] occupies the first place. Although it seemed at first as if the so-called metrical relations were not accessible to this treatment, as they do not remain unchanged by projection, we have nevertheless learned recently to regard them also from the projective point of view, so that the projective method now embraces the whole of geometry But metrical properties are then to be regarded no longer as characteristics of the geometrical figures *per se*, but as their relations to a fundamental configuration, the imaginary circle at infinity common to all spheres. (Klein, 1893, p 215)

Klein seeks to reconsider these geometric developments in the light of a unifying principle:

> When we compare the conception of geometrical figures gradually obtained in this way with the notions of ordinary (elementary) geometry, we are led to look for a general principle in accordance with which the development of both methods has been possible. (Klein, 1893, p 215)

Klein, however, does not appear to wish to claim undue merit for himself:

> In undertaking in the following pages to establish such a principle, we shall hardly develop an essentially new idea, but rather formulate clearly what has already been more or less definitely conceived by many others. (Klein, 1893, p 216)

The mathematical concept striking Klein for its unifying potential is that of a group He shows how the geometries studied up to then have this in common: they study the properties which are invariant for suitable groups of transformations.

Characteristics of the Erlangen Program

We now follow, step by step, the exposition of the program through its first paragraphs. What is the concept of transformation group that Klein has in mind? He begins, at the start of the first section, in the following way:

> The most essential idea required in the following discussion is that of a *group* of space-transformations. The combination of any number of transformations of space [...] is always equivalent to a single transformation. If now a given system of transformations has the property that any transformation obtained by combining any transformation of the system [English translation, 1893] belongs to that system, it shall be called a *group of transformations*. (Klein, 1893, p. 217)

The definition of group is not complete, so that a clarifying note is required in later editions. Klein immediately illustrates the notion just introduced with examples and counterexamples The use of concrete examples is typical of Klein, who seizes the earliest opportunity to emphasize how general abstract ideas arise from the requirements of treating concrete cases systematically, and then applying them regularly to concrete cases:

> An example of a group of transformations is afforded by the totality of motions, every motion being regarded as an operation performed on the whole of space A group contained in this group is formed, say, by the rotations about one point [...] On the other hand, a group containing the group of motions is presented by the totality of the collineations. But the totality of the dualistic transformations does not form a group; for the combi-

nation of two dualistic transformations is equivalent to a collineation. A group is, however, formed by adding the totality of the dualistic to that of the collinear transformations. (Klein, 1893, p. 217)

With what groups will Klein be concerned? He, throughout this entire treatise, will distinguish the group of projectivities and its subgroups, from the most general in which he will speak of manifoldness extended several times, without further specification, emphasizing the independence of the way of *representing the space* or of *the nature of the space*. In the case of the projectivities, the subgroup of similarity transformations, which Klein will call the *principal group*, will play a special role. In fact, after having defined *geometric properties* as those

> independent of the position occupied in space by the configuration in question, of its absolute magnitude, and finally of the sense [...] in which its parts are arranged (Klein, 1893, p. 218)

Klein goes on to observe that they remain

> unchanged by any motions of the space, [English translation, 1893] by transformation into similar configurations, by transformation into symmetrical configurations with regard to a plane (reflection), as well as by any combination of these transformations (Klein, 1893, p. 218)

and will call the collection of such transformations the *principal group*. He then states that

> *geometric properties are not changed by the transformations of the principal group*. And, conversely, *geometric properties are characterized by their remaining invariant under the transformations of the principal group* (Klein, 1893, p. 218)

As a good scientist would, Klein immediately extends the problematic

> Let us now dispense with the concrete conception of space, which for the mathematician is not essential, and regard it only as a manifoldness of *n* dimensions, that is to say, of three dimensions, if we hold to the usual idea of the point as space element (Klein, 1893, p. 218)

With this premise he then states the problem in a more general form,

> *Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group* (Klein, 1893, p 218)

or, according to a formulation which he himself states to be more modern

> *Given a manifoldness and a group of transformations of the same; to develop the theory of invariants relating to that group*. (Klein, 1893, p. 219)

For Klein it is "absurd" to study the properties that remain invariant with respect to a subgroup of the principal group considered only by itself. This becomes justified, however, as soon as we investigate the configurations of space in their relation to elements regarded as fixed. Let us, for instance, consider the configurations of space with reference to one particular point, as in spherical trigonometry The problem then is to develop the properties remaining invariant under the transformations of the principal group, not for the configurations taken independently, but for the system consisting of these configurations together with the given point [...] (Klein, 1893, p. 219)

It is particularly interesting that Klein poses the problematic in another form as well:

> to examine configurations in space with regard to such properties as remain unchanged by those transformations of the principal group which can still take place when the point is kept fixed. In other words, it is exactly the same thing whether we investigate the configurations in space taken in connection with [English translation, 1893] the given point from the point of view of the principal group or whether, without any such connection, we replace the principal group by that partial group whose transformations leave the point in question unchanged (Klein, 1893, p. 219)

and again generalizes immediately:

> Given a manifoldness and a group of transformations applying to it. Let it be proposed to examine the configurations contained in the manifoldness with reference to a give configuration. *We may, then, either add the given configuration to the system, and then we have to investigate the properties of the extended system from the point of view of the given group, or we may leave the system unextended, limiting the transformations to be employed to such transformations of the given group as leave the given configuration unchanged. (These transformations necessarily form a group by themselves)*. (Klein, 1893, pp 219-220)

This is the problem posed, in modern dress, at the start of this article It is here and in the succeeding passages that Klein reveals how the remaining two vertices of the triangle (Figure 1) must change when there is a change at the third vertex.

Klein does not speak of a group of transformations G and of an arbitrary subgroup H of G given abstractly, but rather seeks to characterize H as that subgroup of the transformations of G which leave something invariant geometrically, or, as he himself says, leave a configuration invariant. And immediately thereafter he takes up the inverse problem, that of determining the properties which are conserved by a group which contains the principal group as a proper subgroup

Extending a group G to a group G* generally requires a modification of the system on which G operates, and the author himself immediately mentions this.

> *If the principal group be replaced by a more comprehensive group, a part only of the geometric properties remain unchanged*. [This is an aspect of the Kleinian problematic which the Italian programs have sought to emphasize - author's note.] *The remainder no longer*

appear as properties of the configurations of space by themselves, but as properties of the system formed by adding to them some particular configuration. This latter is defined, in so far as it is a definite [...] configuration at all, by the following condition: *The assumption that it is fixed must restrict us to those transformations of the given group which belong to the principal group.* In this theorem is to be found the peculiarity of the recent geometrical methods to be discussed here, and their relation to the elementary method (Klein, 1893, p 220)

For example, if, in plain geometry, we wish to consider the group of affinities as a subgroup of the larger group of projectivities, we must extend the Euclidean plane with the set of improper points (which constitute the "particular configuration" in the Kleinian sense). The affine transformations turn out to be those projective transformations that fix a particular configuration, that is they send improper points into improper points.

In the *Concluding remarks* the links between geometric methods, analytic methods, and algebraic methods are made explicit, and this is considered to be the strong point of Klein's program. With regard to algebraic methods Klein underlines the importance of the theory of invariants:

> That this can be done, and how it is to be done, is shown by modern algebra, in which the abstract idea of an invariant that we have here in view has reached its clearest expression (Klein, 1893, p 241)

as well as the link between his theory and group theory. A bit further on he says

> In *Galois* theory, as in ours, [English translation, 1893] the interest centres on groups of transformations. The objects to which the transformations are applied are indeed different; there we have to do with a finite number of discrete elements, here with the infinite number of elements in a continuous manifoldness. (Klein, 1893, p 242)

On the notes

As we have already mentioned, Klein includes some aspects of considerable interest for the teaching of mathematics in the final notes of this paper.

In the first of these notes, *On the antithesis between the synthetic and the analytic method in modern geometry*, there is the observation:

> The distinction between modern synthesis and modern analytic geometry must no longer be regarded as essential, inasmuch as both subject-matter and methods of reasoning have gradually taken a similar form in both. We choose therefore in the text as common designation of them both the term *projective geometry*. Although the synthetic method has more to do with space-perception and thereby imparts a rare charm to its first simple developments, the realm of space-perception is nevertheless not closed to the analytic method, and the formulae of analytic geometry can be looked upon as a precise and perspicuous statement of geometrical relations. (Klein, 1893, p. 243)

The contemporary relevance of these remarks is unquestionable. Valabrega (1989) emphasizes the way in which this "fusionist" aspect was directed to the great debate which developed after the rebirth of synthetic geometry following the work of Monge, and, more significantly for us, notes that Italian mathematical programs for the first two years of high school are still strongly imprinted with the same fusionism present in the Erlangen Program.

With regard to synthetic and analytic geometry, Prodi writes,

> In our mental habits Euclidean geometry and analytic geometry are two quite distinct themes rather than two different ways to treat the same object. At bottom, we are still caught in the net cast by the fascination of the pure geometry of the 1900s, that typical of the classic treatises on projective geometry, in which the use of coordinates was seen as a stylistic lapse, and the only formulas which one read were those involving cross-ratios. Today we consider it important to know how to pass from a synthetic description to an analytic one, and vice versa, according to the difficulties of the moment, without purist preconceptions; and then, as to aesthetics, we may note that frequently the formulas have their own beauty which echoes that of the geometric object they represent. (Prodi, 1996, p. 636)

In that same first note Klein proceeds to emphasize once again the efficiency and merits of formalization:

> On the other hand, the advantage to original research of a well formulated analysis should not be under-estimated, - an advantage due to its moving, so to speak, in advance of the thought. But it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aid of analysis is only a first, though a very important, step. (Klein, 1893, p 243)

Moreover, in the *Concluding remarks* after having emphasized the clarifying role played by the formal concept of invariant, he underlines the potentiality embodied in the dual roles of an adequate formalization:

> For the analytical formulation should, after all, be congruent with the conceptions whether it be our purpose to use it only as a precise and perspicuous expression of the conceptions, or to penetrate by its aid into still unexplored regions. (Klein, 1893, p 241)

It is very probably that these ideas, in a more or less conscious form, were spreading in that period. I do not know if one can state that Klein enjoyed priority, but one must certainly recognize that he has the merit of having succeeded in expressing them with clarity and a notable degree of consciousness.

Naturally, space-conception (*räumlicher Auschauung* in the original Klein) too plays an important part, so much so that Klein dedicates all the *Note III*, *On the value of space-perception*, to its role in education:

> Space-perception has then only the value of illustration [for the purely mathematical contents - author's note], which is to be estimated very highly from the

pedagogical stand-point, it is true. A geometric model, for instance, is from this point of view very instructive and interesting (Klein, 1893, p 244)

from which more generally:

> But the question of the value of space-perception in itself is quite another matter I regard it as an independent question. There is a true geometry which is not, like the investigations discussed in the text, intended to be merely an illustrative form of more abstract investigations. Its problem is to grasp the fully reality of the figures of space, and to interpret – and this is the mathematical side of the question – the relations holding for them as evident results of the axioms of space-perception. A model, whether constructed and observed or only vividly imagined, is for this geometry not a means to an end, but the subject itself (Klein, 1893, p. 244)

Thus one notes in Klein's writing both the necessity of bringing all the various geometries under a unique unifying concept, and also the consciousness that it is advantageous to use different formulations from time to time, whether they be the analytic versus the synthetic method, the use of intuition versus formalization, or the different ways of conceiving and representing a space of more than three dimensions. Hence Klein appears to be a mathematician with an open and flexible mentality, attentive to bringing out the peculiarities and potentials offered by various methods he discusses

In *Note V, On the so-called non-Euclidean geometry*, Klein starts by saying,

> The projective metrical geometry alluded to in the text is essentially coincident, as recent investigations have shown, with the metrical geometry which can be developed under non-acceptance of the axiom of parallels, and is to-day under the name of non-Euclidean geometry widely treated and discussed. (Klein, 1893, p. 245)

He concludes by emphasizing his lack of interest, at that time, for the discussions on the role and the nature of the axioms, questions which he himself defines "philosophical", which can be of no interest to a "mathematician, as such":

> Quite independent of the views set forth is the question, what reasons support the axiom of parallels, i.e., whether we should regard it as absolutely given, as some claim, or only as approximately proved by experience, as others say. [ ] But the inquiry is evidently a philosophical one and concerns the most general foundations of our understanding. The mathematician as such is not concerned with this inquiry, and does not wish his investigations to be regarded as dependent on the answer given to the question from the one or the other point of view (Klein, 1893, p. 246)

Later, however, it seems that questions of a foundational nature did interest Klein. For example, in his *Lectures on mathematics* (1911) he characterized the three principal categories into which the mathematicians were divided at the end of the nineteenth century, namely the logicists, the formalists, and the intuitionists, a division that later became popular.

It is also worthwhile to note that, as is amply shown by the passages cited here, throughout the work Klein's methodological-expository choice is that of going from the particular to the general. Indeed, as soon as he can he emphasizes how general abstract ideas arise from the need to systematize concrete cases, a choice which he himself had already justified in the introduction:

> Abstractly speaking, it would in what follows be sufficient to speak throughout of manifoldness of n dimensions simply; but it will render the exposition simpler and more intelligible to make use of the more familiar space-perceptions. In proceeding from the consideration of geometric objects and developing the general ideas by using these as an example, we follow the path which our science has taken in its development and which it is generally best to pursue in its presentation. (Klein, 1893, p. 216)

One can hardly avoid making a connection with what Poincaré (1899) will state a few years later:

> La tâche de l'educateur est de faire repasser l'esprit de l'enfant par où a passé celui de ces pères, en passant rapidement par certaines étapes mais en n'en supprimant aucune. A ce compte, l'histoire de la science doit être notre guide. (Poincaré, 1899, p. 159)

The Erlangen Program and the teaching of geometry

With regard to the Italian mathematics programs for the experimental lyceums, the most highly contested theme is that of the teaching of geometry and this divergence of views is reflected in the differing didactic paths that are proposed Indeed, an official program is one thing, while its interpretation on the part of working educators is quite another. In particular authors of textbooks and teachers may well have differing expectations regarding the acquisition (or non-acquisition) of certain knowledge on the part of the students.

For these reasons, it is not a simple task to discover just what are the didactic paths which in some sense, that is at least in part, reflect the Erlangen Program, nor to know how widespread is their use. In the Italian context there are some pedagogical suggestions like those of Mammana, Speranza, and Villani, which are formulated or at least bear in mind Klein's views. However, we are unaware of their present level of implementation and of the results of possible experiments within the framework of their proposals.

What it does seem to us possible to assert is that the didactic paths most frequently used to teach geometric transformations in Italian schools do not completely respect the spirit of Klein's Program, either because of fundamental cultural choices or for objective reasons of space and time. For example, a presentation of geometric transformations that is based exclusively on symmetries in order to construct all isometries or one based on the composition of isometries and homotheties for the construction of all similarity transformations hides the group structure of the isometries or similarity transformations, respectively, and lacks that significant link with the geometric invariants which is the essence of the Erlangen Program In the course of their studies high school students encounter only the group of isometries and that of similarity transformations, and, if things work out well, only in passing, that of the affine transformations, too little to understand and appreciate, in its real importance, the meaning of the Erlangen Program in view of the fact

> there is not, after all, a great difference between the invariants of geometry of isometries and that of the geometry of similarity transformations [...] On the other hand it does not seem to be realistic to weigh down scholastic programs even further by introducing a systematic study of the group of projectivities (which, furthermore presupposes a widening of the domain in which one works, with the adjunction of the 'improper points') (Villani, 1995, p. 683)

Even the textbooks do not seem to take up these subjects in the spirit of Klein's program:

> Analytic geometry is developed almost exclusively on the basis of monometric Cartesian (orthogonal) coordinate systems which makes it impossible to appreciate the conceptual difference between the relation of parallelism (which is also invariant for coordinate changes which alter the unit of measure along the axes by differing proportionality factors) and the relation of perpendicularity (which is invariant only if the units of measure along the coordinate axes are altered by the same proportionality factor)" (Villani, 1995, p 683)

If difficulties like those discussed here are real and do not allow the formulation of didactic paths which adequately reflect the strictly mathematical content of Klein's program, there are also aspects of his educational and expository philosophy that I have attempted to bring out in this article, and which are still highly relevant and which it would be well to keep in mind in the teaching of geometry at any level Indeed, Klein demonstrated a precocious sensibility regarding a typically didactic problematic which he discussed with an open and flexible mentality, bringing out peculiarities and potentialities of different formulations of geometry. He suggests using both synthetic and analytic methods to show the role of intuition and that of formalization. He considers several ways of conceiving of space, its "nature" and its "representation", and suggests starting from examples in order to arrive at principles or general theorems by treading "the path which our science has taken in its development", and which it then pays to re-apply to concrete cases [6] Who among us can succeed in bearing in mind all these aspects in our teaching and will be able to transmit them to our own students?

Notes

[1] I will follow the outline of lectures that I give on this subject to the students, future mathematics teachers, of the course *Elementary mathematics from an advanced standpoint* (a name originated by Klein himself) The aim of this course, which the students find both interesting and particularly useful for their future profession, is to aid the students in reflecting on the evolution of mathematical theories, in particular of geometry, by reconsidering the key historical moments in their construction; original sources, taken directly from the works of Euclid, Saccheri, Klein and Hilbert, are used.

[2] For a discussion on the use of the original sources for the roots of concepts, see Arcavi *et al* (1982), or for a more general discussion, see Tzanakis and Thomaidis (2000)

[3] In the Italian experimental programs there is the phrase:

> At the end of the three years the student should possess, in a conceptual manner, the contents specified by the program, and should be capable [...] of framing the historical evolution of the fundamental ideas

[4] It is not, however, surprising if one knows Klein's personal history; he was, for example, the first president of ICMI (see Bass, 2005, for further information).

[5] Sourced from M. Emaldi's Mathesis Patavina Lecture on *Programma di Erlangen*, May 3, 1991, Padova, Italy

[6] Klein, even later in his life, continues to have the same ideas and attitude (see Bass, 2005)

References

Arcavi, A , Bruckheimer, M and Ben-Zvi, R (1982) 'Maybe a mathematics teacher can profit from the study of the history of mathematics', *For the Learning of Mathematics* 3(1), 30-37.

Bass, H (2005) 'Mathematics, mathematicians and mathematics education', *Bulletin of the American Mathematical Society* **42**(4), 417-430

Freudenthal, H. (1982) 'Should a mathematics teacher know something about the history of mathematics?', *For the Learning of Mathematics* **2**(1), 30-33.

Furinghetti, F. (2000) 'The history of mathematics as a coupling ring between secondary and university teaching', *International Journal of Mathematical Education in Science and Technology* **31**(1), 43-51.

Klein, F. (1893, English translation of 1872 German original) 'A comparative review of recent researches in geometry', *Bulletin of the New York Mathematical Society* **II**, 215-249.

Klein, F (1911) 'Lectures on mathematics , New York, NY, American Mathematical Society.

Lampert, M. (1990) 'When the problem is not the question and the solution is not the answer: mathematical knowing and teaching', *American Educational Research Journal* **27**(2), 9-63.

Poincaré, H. (1898) 'On the foundations of geometry', *The Monist* **9**, 1-43

Poincaré, H (1899) 'La logique et l'intuition dans la science mathematique dans l'enseignement', *L'enseignement mathematique* **I**, 157-162.

Prodi, G (1996) 'Poligoni e algoritmi', *L'Insegnamento della Matematica e delle Scienze Integrate* **19A-19B**(6), 633-650.

Tzanakis, C and Thomaidis, Y (2000) 'Integrating the close historical development of mathematics and physics in mathematics education: some methodological and epistemological remarks', *For the Learning of Mathematics* **20**(1), 44-55

Valabrega, E (1989) 'Le trasformazioni geometriche nell'insegnamento alla luce della storia della geometria', *L'Educazione Matematica* **4**(2), 135-141

Villani, V (1995) 'Le transformazioni geometriche nella scuola secondaria superiore , *L'Insegnamento della Matematica e delle Scienze Integrate* **18A-18B**(6), 669-688.

> Myths – and mathematics is also a myth – have their own clarity but like other human artefacts they disguise how they were interpreted by their makers.
>
> (Dick Tahta (1980) 'About geometry', FLM1(1).)