Mathmatisation as a Pedagogical Tool [1]

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At a meeting about mathematics education within a congress of mathematicians, I think it is appropriate to begin with a question: 'What can mathematicians do for mathematics education?'. They can:

(i) tease out, unravel, the genetic development of mathematical ideas - i e. help us understand how the mathematics we have *came into being*;

(ii) objectify, describe the mental processes that produce mathematics - i e. help us understand how any mathematics *comes into being*.

In one sense, all mathematicians know a lot about mathematical activity. But they do not necessarily know what they know. Even those who know what they know may not find the words, the appropriate language, to communicate it

Mathematicians who are genuinely interested in mathematics education would make a more useful contribution by helping with these difficult questions than by advising teachers what they should teach

1.

I begin my remarks about mathematisation by admitting that I have no solid conclusions to offer. This talk is not a presentation of my discoveries It is rather a sign of the importance I attach to the topic and an invitation to join in thinking about mathematisation and exploring it. My view is that a complete, effective, pedagogy of mathematics awaits a deeper and more subtle understanding of the processes that produce mathematics.

I cannot in a short space of time offer many examples of mathematisation at work. Indeed, perhaps the best strategy, since it is not possible to give an infinite number of examples, is to give none. I will compromise and give one

(The example supposes the presence of suitable material - for example, the Gattegno Prisms and Cubes or the Dienes Multibase Arithmetic Blocks) A cube equivalent to a large cube can be constructed with ten flat tiles Suppose one subjects this arrangement to a physical transformation, pushing each tile at right angles to a vertical face until each tile overlaps the one below by a small amount The shape is no longer cubical, but since no material has been added or taken away, the resulting shape is in some sense still equivalent to a large cube What attributes remain? Which have changed?

One may notice that the surface area has increased.

A 'dialogue' with the material might take the following direction:

> How do we know the area has increased?
>
> Can it be increased even more?
>
> Can it be increased indefinitely?

No, for it has an upper bound - the total surface area of the ten tiles.

> Does the fact that the area increases depend on the number of tiles?
>
> What could be said if the 'cube' were made of twenty tiles, each half the thickness of the original ones?

These tiles do not exist, except in the mind. It may be discovered that the maximum possible area available to 'stepping' the tiles has increased.

> What would happen if the tiles were halved in thickness again? And again?
>
> Does this increase continue indefinitely?

The argument may lead us inexorably to the conclusion that a finite volume can be associated with a surface area that increases without limit - a non-intuitive conclusion from a thoroughly concrete beginning.

I invite you to consider what are the key processes in the mathematisation of this situation.

2.

In common with many others, I adopt the word 'mathematisation' to refer to the mental processes which produce mathematics. I wish to suspend judgement at present about the precise extension of the word - does all mathematising lead to mathematics? is all mathematics produced by mathematising? - and about its possible associations with other words meant to describe similar phenomena. However, I am sure that mathematisation should not be equated with 'making a mathematical model', although it may include it. 'Modelling' is too suggestive of completeness and intentionality to serve as a description of the basic ingredient of mathematical activity.

An adequate construction of the concept of mathematisation would have useful consequences:

- it would locate mathematical activity within the sphere of general mental operations, while acknowledging that mathematical thinking is a *sui generis* activity;

- it would make it easier to discuss the dynamics, the movements, of mathematical thought: at present, we have only an impoverished language to refer to the *experience* of mathematical thinking as opposed to its results.

On the assumption, which I hold, that there is no difference *in principle* between the mental activity required to learn mathematics and to make mathematics, it would make it easier for educators to know 'what goes on inside the learner's head'.

For the Learning of Mathematics **21**, 2 (July, 2001)
FLM Publishing Association, Kingston, Ontario, Canada

3.

Although mathematisation must be presumed present in all cases of 'doing' mathematics or 'thinking' mathematically, it can be detected most easily in situations where something not obviously mathematical is being converted into something which obviously is We may think of a young child playing with blocks and using them to express awareness of symmetry, of an older child experimenting with a geoboard and becoming interested in the relationships between the areas of the triangles he can make, an adult noticing a building under construction and asking himself questions about the design, etc.

We notice that mathematisation has taken place by the signs of organisation, of form, of additional structure, given to a situation I use these tenuous clues to suggest that:

> mathematisation is the act of *putting a structure onto a structure*

Consider the experience of solving a problem, or mastering a new game. In each case, there are moments when the whole situation or a part of it is suddenly *seen differently*; the perceptual difference marks a new stage in the mental structuration of the situation.

The game of 'Nim', for example, has a certain structure defined by its rules (or by the way the players perceive the rules) Playing the game may lead a player to become aware of the 'end game' – that particular dispositions of the piles are unfavourable or favourable to him. This is a structure put onto the structure of the game. He may become aware that certain positions are reducible to others. This is a new structure He may see that there is an ensurable sequence of 'safe' positions This is a new structure. He may devise an algorithm for identifying safe positions. This has structured the situation still further – and if the player is only concerned with knowing how to win, his activity may stop here. (This example was elaborated in discussions with Dr. Weinzweig and others)

Not all structurations are as hierarchical as in this hypothetical example Even in goal-directed problems and games, the explorer may have to 'cast about', trying out various structurings, until he finds the 'thought-vector' that will take him towards the goal. 'Casting about' in some systematic way is, of course, the role of heuristics

4.

A well known 'process' word frequently applied to mathematical activity is *generalisation* It seems to me that most uses of the word are *clichés*, as if most users have not recently stopped to listen to what they are saying.

When I look at generalisation introspectively, I find several cases

(i) I encounter, or look for, or investigate, several situations and observe a regularity or pattern

(ii) I hear someone else describe something in a different way from the way that had occurred to me Because I trust what he says, I know that our stories should be compatible and I believe there must be a 'cover story' that includes both.

(iii) I can put a situation into words, or symbols, or express it in a diagram; this mediates the conviction of generalisability because of the inherent generality of these forms of represen-tation

(iv) I know while I act on the situation that some elements of it are arbitrary and that I could deal with other 'values of the variable' just as easily.

(i) is usually taken as the paradigm for becoming aware of generalisability. On the contrary, I think (iv) is the paradigmatic case – where the thinker has noticed the freedom available even while engaged in working on the details of a particular case This choice is supported by the fact that hasty or premature generalisations are much more common than failures to generalise at all

Generalisation is an instance of putting a structure onto a structure (even though the general is less highly structured than the particular). The generaliser structures the situation by becoming aware of those relationships that define the 'essence' of the situation and emphasising them to the exclusion of all others The generalisation is then only the formal expression of the class of situations that share this common essence, this structure

5.

In analysing acts of mathematisation, it is not particularly difficult to describe the structures that the mathematiser imposes, but it seems much more difficult to get close to the 'reasons why', to the source of the particular decisions that the mathematiser takes It seems clear that the energy source that powers the structuring activity is awareness – awareness of some feature of the situation that suggests the structure to be imposed.

But why 'this' awareness rather than 'that'? 'What' triggered this awareness? These and similar questions are probably intrinsically unanswerable – in the way that, say, the question of how we recognise objects and situations is unanswerable. We only know that we do and that everyone else does too

At the moment, I can only say that experience must be given a place as one of the determinants of the awarenesses one will achieve; but that statement is so obvious and expected as to be of very little use. A narrower and potentially more fruitful hypothesis concerns the way is which *language, notation, graphical representation* and *imagery* appear to facilitate awareness. They seem like carriers of awarenessess, or at least like media which facilitate the achievement of awarenesses because *they mediate between* the mathematiser and the situation he is mathematising. Aspects of their functioning are well known, but few people have studied these media with the aim of seeing how they bring mathematisation nearer to hand.

6.

It is worth trying to elucidate 'mathematisation', because it will help in talking about what mathematicians do – and I think it may be agreed that there is a strong case for attempting to demystify and deglamourise mathematical activity. But mathematics educators are obviously more concerned to discover what an understanding of mathematisation can do to influence the pedagogy of mathematics. This is an even more difficult area of study, for as soon as we begin to consider pedagogy we can no longer be satisfied with clarifying meanings, formulating descriptions, making hypotheses and constructing theories; we must take into account practical effects. A pedagogy has to work or it is useless.

At the moment, I can only say that the pedagogical consequences of looking at mathematisation rather than mathematics have been worked out by a few people. Gattegno's work seems to me to be the most notable example There are probably many reasons why his achievement has not been generally acknowledged nor widely followed, but among them I would certainly include his failure to depict in reasonable and recognisably attainable terms the activity of a good teacher. His writings conceal much of his pedagogy, and when one observes him teach, it is difficult to avoid the feeling on coming away that only a thoroughly exceptional person could teach as he does. But it may be to the point to say that no doubt we would have felt the same if we had been privileged to attend a seminar conducted by Socrates.

One of the difficulties in devising a pedagogy based on mathematisation is to give a picture of what a teacher should be able to do. A pedagogy based on mathematics – especially 'ready made' mathematics, to use Freudenthal's term – shows us a teacher who is mainly an expositor, someone who displays to his students selected parts of the mathematics that is known. This is not a trivial function and requires considerable skill, but at least it provides everyone with a simple image of what a teacher does.

But a pedagogy based on mathematisation *must* yield much more responsibility to the students. It is they who, by the exercise of their mathematising powers, will cause the mathematics to be 'acted out' (Freudenthal's phrase again) – i.e. reinvented, rediscovered, recreated, 'brought into being'. The image of a teacher in such a situation is much less simple, more shadowy. What is he actually doing there in the classroom? He is not teaching mathematics, in the customary sense; but nor is he teaching the students how to mathematise, since they already know how.

I find an analogy helpful here. Everyone is born with an innate ability to perceive, but particular perceptions cannot be directly taught (or only with extreme difficulty). Nevertheless, the ability to perceive is clearly educable and is influenced by experience and the interventions, intentional and unintentional, of a great many people. Mathematisation seems to me to be an entirely similar case. The teacher is one of the interventionists – a professional interventionist, perhaps – who brings to the students those situations which will educate their mathematising capacities.

A pedagogy based on mathematisation must include at least the following categories of activity

1. The educator must be able to reorganise substantial portions of the content of the mathematics curriculum so that they can be mathematised. Anyone who has tried to make a mathematical film knows how the mathematical content has to be detached from its normal context and considered afresh to see how it might be apprehended 'from scratch'.

2. The educator must be able to select suitable 'proxy' experiences which indicate to the students how certain situations have been or could be mathematised. The example of the cube given earlier in this paper is an instance. History provides another resource – the teacher might reconstruct, for example, how Leibniz mathematised the problem of finding the sum of an infinite geometric series.

3. The teacher must be able to take advantage of the spontaneous events of the classroom that will occur when students are given the freedom to employ their own mathematising abilities. Indeed, this is a function that *only* the classroom teacher can perform. One of the best descriptions of the process by an aware teacher is given in *Mathématiques sur mesure* by Madeleine Goutard.

In conclusion, I state my hope that a serious, subtle, study of mathematisation will help all of us in two respects: it will help connect the inner mathematical experience with its outer objective form; and, in the classroom, it will help us to handle the individuality and spontaneity of students who are coming to terms with the most impersonal subject we ask them to learn

Additional remarks

Among the points raised in response to this presentation were:

(i) the difficulty of getting teachers to see their work as that of facilitating, or educating, children's powers of mathematisation;

(ii) the relevance of these kinds of consideration to the education of all students;

(iii) the need to engage students in mathematising real problems. All of these points deserve lengthy discussion, but I add an observation or two on each

(i) I doubt if it is actually more difficult for teachers to act as educators rather than instructors. Indeed, the logic of the situation says it must be easier: they would be working *with* the students' powers rather than against them. But I am aware that it is not easy to convince all teachers that this is the case. The central problem, it seems to me, is that they do not know the personal resources that they possess – just as many of their students do not know that they possess the capacity to mathematise. The main task of teacher education is to bring these personal resources to the level of awareness.

But there are technical matters to deal with, too. We need to develop techniques for creating situations which impel a mathematical response, for working on students' awarenesses in various ways, and for monitoring and evaluating what students do when they are encouraged to mathematise.

And, not least important, we have to try to recast the image of a good mathematics teacher in the eyes of administrators, employers, parents and the general public, so that teachers are not prevented from acting as educators by institutional constraints and inappropriate expectations

(ii) It may be necessary to consider as separate questions whether 'mathematics for all' is a feasible goal and a desirable goal. Neither question can be answered without taking into consideration particular social, economic and ideological contexts

The starting point for an answer is that the ability to mathematise, like the ability to speak, comes with the fact of being human. But just as speech does not develop in an environment which does not display it, the power to mathematise does not become properly functional in an environment which fails to show that mathematisation is a possibility.

I would like to take the position that everyone is *entitled* to be shown that he has the ability to mathematise and can exercise it if he chooses; but the reality is far from the ideal, even in countries whose educational systems appear to promise mathematics for all I believe we should try to move closer to the ideal not because the world needs more mathematicians - I think it has quite enough in total, though they could be distributed more fairly - but because the experience of being able to mathematise can bring a valuable increment of self-awareness to each individual, and because societies would benefit from an increase in the number of people who are not paralysed by mathematics and who know it well enough to be aware of what it can and cannot do

(iii) The value for a mathematical apprenticeship of seeing how mathematics can solve 'real problems' is a topic that needs more careful scrutiny than it generally gets It is an attractive idea, like the idea that students should 'understand' mathematics rather than perform it by rote. Who could possibly be against either?

I am concerned about two points in particular. What is meant by the word 'real'? Too often this seems to be taken to mean exclusively the physical, tangible world or (as it is often called) the world of 'everyday life' I find this an objectionable qualification To exclude from reality, including the students' reality, the life of the human intellect and the life of the imagination seems to me a potentially more oppressive dehmanisation of education than any of which traditional education has been guilty.

Secondly, we do not as yet have the pedagogy to tell us *how* students can be brought to the stage of being able to mathematise real problems. Most 'real' problems are intractable; the amenable ones often connect only with trivial mathematics; and the problems we know how to solve are generally adults' problems, not students' problems.

None of these difficulties is insuperable, and the real pedagogical problem does not lie here. It lies in our relative ignorance about 'what goes on inside the head' when a real problem is the occasion for some mathematical activity. A 'model' is like a 'theorem' - the end result of a complex mental activity which tells us very little about the nature of the activity itself. My interest in mathematisation stems from a desire to 'get inside' the mental activity, if possible, in order to know it a great deal better. If we are preoccupied with 'problems' and 'models', we will remain outside, unable to penetrate the characteristic processes of mathematical thinking

Acknowledgement

The editor is very grateful to the Institut für Didaktik der Mathematik of the University of Bielefeld, Germany for permission to reprint this piece here.

Note

[1] This piece first appeared as 'Mathematization as a pedagogic tool', in Steiner, H.-G (ed), *The Education of Mathematics Teachers: What Knowledge, Experience and Understanding Should a Mathematics Teacher Have?*, Materialen und Studien Band 15, Bielefeld, IDM Universität Bielefeld, pp 147-158.