Zeros and Ones in Advanced Mathematics: Transcending the Intimacy of Number

ELENA NARDI

A first-year mathematics undergraduate [1] is presenting a proof of the Archimedean property:

$\forall x \in \mathbf{R}, \exists n \in \mathbf{N}$ such that $x < n$

In this he is reproducing, from his lecture notes, the definition of a supremum a for the set of natural numbers N:

$n \leq a \ \forall n \in \mathbf{N}$ and if $\varepsilon > 0$, then $\exists\ n_\varepsilon \in N$ such that $a - \varepsilon < n_\varepsilon$

The student points at n_ε and says he does not 'understand this notation'. His tutor (T) replies that the lecturer was trying to 'produce a bit of inflection':

T: You specify ε first and then you are choosing n, therefore your choice will have to depend upon ε. But that doesn't mean there is a function

He then calls this notation a bit 'perverse' and recommends its use for specific values of ε: for instance, for $\varepsilon = 1$, write n_1. The student is puzzled and wonders whether he should avoid using this notation altogether.

The student's puzzlement illustrates his sense of ambiguity relating to the notion of dependence between two numerical quantities - here, mediated by the notation n_ε for this dependence (which, incidentally, in the case of a sequence s_n *does* signify a function). It also illustrates that perhaps students arrive at university mathematics courses with much less clear images of functional dependence than is expected of them

In this article, I examine how components of the concept of function (variable, domain and range) as well as the process-object duality in its nature (Sfard, 1991, 1994) emerge as highly relevant to the students' learning in various mathematical contexts. Starting from the context of analysis, where in fact the main bulk of research on functions has been carried out (a representative item of which is Dubinsky and Harel, 1992), my primary focus is on other, less-researched areas such as linear and abstract algebra. In this sense, here, a more epistemologically global (Malik, 1980), cross-topical approach is attempted: an approach to issues related not only to the learning of the concept of function, but also to how the concept of function permeates a number of mathematical topics.

It is thus hoped that some aspects of the journey of transcendence required of the learner of advanced mathematics - from working with functions as relationships (algebraic, graphical) between numbers through to relationships between an assortment of mathematical objects, including functions themselves - can be charted in its complexity.

The concept of function in the context of calculus

By the time students embark on university mathematics, their experience with the notion of function is at least that of an analytical expression and of a graph (Nardi, 1992). In fact, quite early in the learning process students gradually begin to see a graph holistically as an outcome of the type of rule suggested by the expression $f(x)$. For the students in this study, the existence of sine in $f(x) = x\sin 1/x$ suggested a type of periodicity for the values, for instance, and the graph of f, or the existence of e^x in $f(x) = x^{-1}e^x$ suggested a type of 'bigness'. For these students, then, the 'behaviour' of a function is illustrated in its graph

Problems arise, however, when a function is *identified* with its graph, for instance when a continuous function is identified with a smooth, uninterrupted curve (Even, 1993). In the above example, $f(x) = x^{-1}e^x$, where the students have been asked to produce the inverse for f on a particular interval (this was intended by the question setter as an application of the Inverse Function Theorem), a student (Frances) by-passed using the *IFT* and, when asked by the tutor about her finding f^{-1}, she exclaimed:

Because you can tell from the graph! [*The tutor asks why again and a pause follows*] If . you reflect it on y ...

The difference between the tutor's and the student's approach is of a delicate ontological nature. For the tutor, the *IFT* guarantees the existence of f^{-1}, but for Frances this function already exists: it is there, on the graph, it *is* the graph, and in order to see it, all she has to do is look at the graph in a slightly new way (x becomes y, and so domain becomes range). Frances does not need anything stronger than her own sight to be convinced and she assumes that everyone else shares her sense of conviction. This sense of conviction reflects also the student's state of mind with regard to the rigour with which she invests her mathematical actions, but this issue is elaborated further elsewhere (Nardi, submitted).

In the first encounters with integral and differential calculus (Orton, 1983a, b), a function is simultaneously an object to integrate and differentiate and a means for finding areas under graphs and gradients of tangents. In differential equations, the implicit notion of a function is that of an object, a solution to an equation not an equation itself. This interplay continues in the study of the Fourier series or the Taylor expansion of a function (that these students at least were offered a glimpse of in their first weeks into the course): Fourier series or Taylor expansions are objects we

For the Learning of Mathematics **20**, 3 (November, 2000)
FLM Publishing Association, Kingston, Ontario, Canada

generate by manipulating certain values of the function and this gives the whole enterprise a process-oriented aura.

However, we talk about *the* Fourier series or *the* Taylor expansion of f as characteristics of f, as its properties in an object-like manner. Problems arise, for example, with the Taylor expansion of a function (itself a function) when students attempt to write down the expansion without being concerned for the interval or the point around which the expansion is produced.

The above suggest the well-identified finding in research (e.g. Markovits *et al.*, 1986) that when dealing with functions as rules, with little if any emphasis on domain and range, it has the effect that, when these become centrally significant, there is not enough cognitive infrastructure to support their use. Identification of a function with a rule proves evidently insufficient most clearly in the students' confrontation with piece-wise functions such as:

$$f(x) = x/|x| = 1, \text{ if } x > 0$$
$$= -1, \text{ if } x < 0$$

which they tend to consider as two functions, not one (also discussed in Barnes, 1988); or in the students' confrontation with families of functions f_n such as:

$$f_n(x) = x^n\cos(1/x^2)$$

In this instance, a student Camille is providing an eloquent answer to the meaning of the question 'For which n is f_n differentiable at zero?':

> The question is to find for which n to have the limit of this derivative to be zero. [*She points at the derivative in her notes and the proof for the right answer $n = 4$.*]

However, she asks what would happen if $x > 1$ - ignoring that the limits involved in the process are all taken when $x \to 0$. Here, it is possible that Camille's concept image (Vinner, 1983) of the domain of f_n being $\mathbf{R}-\{0\}$ (x can be anything other than zero; therefore it can be > 1) overshadows her concept image of the limiting process which requires x to be taken as close to zero as possible. Even though x can be greater than one, the values of f_n for $x > 1$ are irrelevant to the question posed.

Another instance of this possible conflict between the point-wise/static and the procedural/time-dependent perception of the variable in a function (Dreyfus and Eisenberg, 1983; Vinner and Dreyfus, 1989; Ferrini-Mundy and Graham, 1991; Vinner, 1983) that I encountered in these tutorials is the following: when discussing the limit of a sequence s_n ($lim_{n\to\infty}s_n$), some students ask what the limit is if $n \to 0$. In this case, wondering about the values of s_n as $n \to 0$ is not simply irrelevant, as in the case above, but is also evidence of an incomplete image of the domain of the sequence s_n ($n \in \mathbf{N}$).

In the above, I have sampled a range of problems relating to the students' perceptions of the nature of function in an interplay with their perceptions of domain, range and the notion of variable in the context of analysis. In a sense, these perceptions reflect the mindset with which the students approach the notion of function in other topics. Hence, it is not a surprise that the problems identified above are accentuated so poignantly in the context of linear and abstract algebra.

Conceptual and contextual understanding of functions: applying the subspace test and looking for the zero element of $\mathbf{R}^{\mathbf{R}}$?

The tutorials discussed here are on linear algebra. The tutor explains that a common way to prove that a subset S of a vector space V is a subspace of V ($S < V$) is to use the subspace test, namely proving that:

- $0 \in S$;
- $af + bg \in S$, $\forall f, g \in S$ and for any scalars a and b, i.e. addition and scalar multiplication are closed in S

Applications of the test for two subsets of $M_n(\mathbf{R})$, the set of real-valued n x n matrices, follow.

A third example is to prove that:

> $U = \{f: \mathbf{R} \to \mathbf{R}: f(0) = f(1)\}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$, the set of all real-valued functions from $\mathbf{R}$ to $\mathbf{R}$.

The students [2] look as if they are not familiar with $\mathbf{R}^{\mathbf{R}}$. The tutor realises their unease and asks what is the zero element of $\mathbf{R}^{\mathbf{R}}$. She reminds them that the zero element is an element of U and she asks them what the property is that it has to satisfy. Silence follows.

In Tutorial group 2, Eleanor says about the zero element of $\mathbf{R}^{\mathbf{R}}$ that "It will stay the same". The tutor disagrees, reminds her of the definition of the zero vector in a general vector space ($a + 0 = 0 + a = a$, $\forall a \in V$) and asks for the zero element z in U and $\mathbf{R}^{\mathbf{R}}$ again.

Abidul says $z(x) = x$ and Eleanor says 'nought'. The tutor agrees with Eleanor and adds that this is a function they have been dealing with 'for ages' in analysis 'in a slightly more abstract context'. In Tutorial 4, Patricia says that in a general vector space "if you add zero to any vector, you end up with the same vector" but is unable to apply this to $\mathbf{R}^{\mathbf{R}}$?

In Tutorial group 1, the students simply remain silent in the face of all the tutor's prompts. Finally, in all tutorials, in proving closure (the second condition of the subspace test), the students frequently confuse f with $f(x)$ in evaluating $af + bg$.

The first frictions with regard to the zero element of a vector space appear in the discussion of the first two applications of the subspace test: for instance, when Abidul uses the term 'nought' for the zero vector of $M_n(\mathbf{R})$, the tutor corrects it to 'zero' (meaning "the zero matrix"). Subsequently, Abidul asks whether she can write 'just 0', not the full matrix with zeros everywhere. The tutor says this is fine as long as it is clear what she means. The application of the subspace test then continues almost trouble-free.

In the third example, however, there is a problem: while the previous two applications of the subspace test involved thinking in terms of the elements of $M_n(\mathbf{R})$, the vector space in question here, $\mathbf{R}^{\mathbf{R}}$, appears less familiar. As a result of this lack of familiarity, the students soon come up against identifying its zero element - even though some of the students can recall the formal definition of the zero element in a vector space. Moreover, even once the hurdle of identifying the zero element of $\mathbf{R}^{\mathbf{R}}$ has been overcome, the students are still uncomfortable with its contents - as their frequent confusion of f with $f(x)$ illustrates.

Considering separate values of f as elements of $\mathbf{R}^{\mathbf{R}}$ is an indication of their difficulty in viewing f as an object contained in a set of similar objects, as an element of the set $\mathbf{R}^{\mathbf{R}}$. The following extracts from Camille (Tutorial group 3) reinforce the evidence of this difficulty: the discussion starts with the student's question about $\mathbf{R}^{\mathbf{R}}$: 'Is it the same as $\mathbf{R}^2$?'

The tutor defines $\mathbf{R}^{\mathbf{R}}$, $2^{\mathbf{R}}$ and $\mathbf{R}^2$ (starting from the definition of A^B as the set of functions from the set B to the set A). Camille (C) asks whether these are 'transposes'. These are the mappings from one set to the other, replies the tutor (T), and are vector spaces over addition of functions and over scalar multiplication, both point-wise. She then asks what is the zero in $\mathbf{R}^{\mathbf{R}}$, but Camille is still trying to understand what $\mathbf{R}^{\mathbf{R}}$ consists of. (In the following transcripts, ellipsis '. ' simply indicates a pause, time passing, and not excision of any text.)

C: It's a mapping from **R** to **R** . . and each element of **R** has a correspondent . . with the mapping … the graph . . it's a mapping from **R** to **R**, so it's

T: No, no, no. Each mapping is a subset of **R**x**R**? It's not . .

C: And U is a subset …

T: Yes. No. No, you are not looking at the individual f. Yes, it's true that f is a subset of $\mathbf{R}^2$. That's true but I'm not looking at the individual f. I'm looking at the set of all fs. [*The tutor returns to the question about the zero element of* $\mathbf{R}^{\mathbf{R}}$.] What's your mapping from **R** to **R** with that property? It's not in there. U is not a subset of $\mathbf{R}^2$

C: Do you have an example of a function f that isn't in $\mathbf{R}^2$?

T: Yeah, I mean any of them . . if you like you could have … $\cos 2\pi x$. this is a function that … I mean to say that something belongs to $\mathbf{R}^2$. what you are saying is that f belongs to $\mathbf{R}^2$. I mean that f belongs to $\mathbf{R}^2$. But it doesn't because that . . to say that f belongs to $\mathbf{R}^2$ is to say that f is an ordered pair. It's a set of ordered pairs.

C: $(x, f(x))$ . . but isn't f an ordered pair?

T: No, f doesn't belong to $\mathbf{R}^2$. f is a subset of $\mathbf{R}^2$. And it belongs to the set $\mathbf{R}^{\mathbf{R}}$. It belongs to a set of functions.

C: It's very hard to imagine that … a set is usually a set of elements or matrices

T: Ah, . . yes, that makes it harder. But I mean you could do it with equations.

[*The tutor stresses that all the things they are talking about here are familiar things in analysis and that their trouble is with the vector space context.*]

C: So **R** is a subset of $\mathbf{R}^{\mathbf{R}}$ . .

T: The elements are …

C: How about $\mathbf{R}^2 \to \mathbf{R}$ … Is it a subset of this?

Camille's remarks illustrate her persistent struggle to construct a meaningful interpretation of $\mathbf{R}^{\mathbf{R}}$ in the course of which she is expressing some risky ideas. Her initial comment here is her first attempt to imbue $\mathbf{R}^{\mathbf{R}}$ with meaning. Her next four turns indicate that despite the definitions given by the tutor, Camille – probably not having found them helpful – insists on trying to relate $\mathbf{R}^{\mathbf{R}}$ to $\mathbf{R}^2$

Knowing that the set of $(x, f(x))$ is a subset of $\mathbf{R}^2$, where f: $\mathbf{R} \to \mathbf{R}$, her thinking gets entangled with f as (a set of) ordered pairs and f as an object-element of a set. In her sixth turn, she seems to have realised that $\mathbf{R}^{\mathbf{R}}$ is a set of functions and explicitly expresses her difficulty in shifting from a set of elements – for instance, matrices – to a set of functions.

Her seventh turn then leaves an impression of a second-round effort. Now Camille possibly interprets $\mathbf{R}^{\mathbf{R}}$ as a 'power' of **R**, thus 'containing' **R** – in the same sense that 2^3 'contains' 2? The chain of tentative interpretations of $\mathbf{R}^{\mathbf{R}}$ continues with her final remark which seems to be a compromising combination of her entanglement with $\mathbf{R}^2$ and her newly-acquired idea that $\mathbf{R}^{\mathbf{R}}$ contains functions. In any case, her comments here provide an impressively explicit illustration of a student's striving for a meaningful interpretation of the concepts – and the notation attached – to which she is being introduced.

In contrast to the tutor's constant and steady transmission of information about $\mathbf{R}^{\mathbf{R}}$, Camille's tentative interpretations are unendingly changing. In this sense, Camille seems to be very much alone in this process, even though the steps to the various directions she is taking are obviously fed by the information she receives.

Subsequently, the tutor repeats the definition of $\mathbf{R}^{\mathbf{R}}$ and draws the parallel with A^B, the set of functions from B to A, where A has k elements and B has m. She stresses that $\mathbf{R}^{\mathbf{R}}$ contains the functions they have usually been dealing with in analysis and that it is the vector space context used in the example that makes things look more complicated. She then repeats the question about the zero vector in $\mathbf{R}^{\mathbf{R}}$: it has to be a function in $\mathbf{R}^{\mathbf{R}}$ that satisfies some kind of property.

Then Camille asks:

C: Is the zero vector a function?

T: Zero vector is a function because all of them are functions here … all the elements …

C: They are not vectors?

[*The tutor repeats the definition of the zero element of a vector space and explains that in this case this element is a function that has the properties of the zero element.*]

C: So we are not looking for the zero vector anymore but for the zero function.

[*The tutor accepts that $z(x) = 0$ is the zero element of R^R, Camille quickly explains that $z \in U$ because $z(0) = 0 = z(1)$ and also checks out closure in U. The proof that $U < R^R$ is thus completed.*]

These three turns illustrate Camille's narrow concept image of a vector (also evident in a previous remark) and the expansion of this image that the understanding of $\mathbf{R}^{\mathbf{R}}$ necessitates. This narrowness is revealed in particular in her attempt to combine the information that:

- the zero vector is a function;

and:

- this function has a particular property related to denoting it the 'zero vector';

in order to deduce that the zero vector of $\mathbf{R}^{\mathbf{R}}$ is the zero function. Camille still seems to struggle with understanding how the elements of $\mathbf{R}^{\mathbf{R}}$ can be functions In particular, her final comment highlights a schism [3] between the notion of vector and the notion of function.

The evidence in this study suggests (Nardi 1997, 1999) that the students have acquired a persistent geometric image of a vector through teaching that focuses on examples from the line, the plane and space. Understandably, teachers constantly encourage their students to espouse a geometric approach in order to demystify linear algebra (and especially vector spaces which are deemed as too inaccessible and abstract by the learners and to embed them in already familiar contexts)

This practice unfortunately proves short-sighted because, within weeks of their instigating a geometric approach, the tutors are forced to generalise their discourse on vector spaces to cases where the geometric metaphor not only does not apply but also creates distracting interference. Here, such interference interacts damagingly with the students' weakness in being able to perceive a function as an object-element of a set.

Looking for the 'usual' basis of $P_3(\mathbf{R})$: decontextualised knowledge and the ambiguous nature of 1

The discussion here [4] is on finding a matrix A for $T: V \rightarrow W$, a linear mapping between two vector spaces V and W. As an application, the tutor suggests finding the matrix for:

$$T: P_3(\mathbf{R}) \rightarrow P_3(\mathbf{R}) \text{ where } T[p(x)] = p(x + 1),$$

and where $P_3(\mathbf{R})$ is the space of all polynomials of degree less than or equal to three with real coefficients.

All agree that dim $P_3(\mathbf{R}) = 4$, but some of the students have problems with identifying the 'usual' basis of $P_3(\mathbf{R})$, as the tutor calls $\{1, x, x^2, x^3\}$.

On one occasion, the basis is identified once the students are reminded of the general expression of $p(x)$:

$$p(x) = ax^3 + bx^2 + cx + d$$

Patricia offers 'x^3, x^2, x and the constant' and then explains that the 'constant' is 'just one'. I note here that the students' difficulty in 'seeing' 1 as x^0 results in the phenomenon elaborated later in this section (on T(1)).

On another occasion, Beth (B) gives '1 0 0 0, 0 1 0 0, ...'. The tutor (T) reminds her that they are talking about polynomials, not matrices Beth then recalls the general expression of $p(x)$ and the tutor asks her what is $p(x)$ a linear combination of:

B: Er, to the s from 1 to 3...

[The tutor insists that they are 'looking for some simple looking polynomials, nice simple things', and repeats her question.]

B: Mmm ... it's just made by linear ones ...

T: What do you mean linear ones?

B: It's a product of three . .

The tutor realises that Beth is talking about factorisation and she stresses that multiplication of polynomials is not linear Asking Beth to 'take a step back', she repeats that they are trying to write $p(x)$ as a linear combination of four other polynomials $d_1, \ldots, d_4$, which form a basis, and asks Beth to compare:

$$ad_1 + bd_2 + cd_3 + dd_4$$

with:

$$ax^3 + bx^2 + cx + d$$

Beth then dictates the 'usual basis' $\{1, x, x^2, x^3\}$ for $P_3(\mathbf{R})$.

Beth's case seems to be more complicated than Patricia's: at least Patricia, in her first attempt to find a basis for $P_3(\mathbf{R})$, suggests polynomials Beth starts from 'matrices' as the tutor calls (1 0 0 0), etc. It is likely that Beth was misled by the tutor's persistent use of the term 'usual' basis (not specifying for which vector space) and offered the 'usual' basis for $\mathbf{R}^4$.

Subsequently, Beth mutters her first remark reported here and it seems likely that she was trying to utter 'x^s where $s = 1, 2, 3$', but the tutor rather impatiently interrupts her. Beth's next two turns illustrate her interpretation of the tutor's 'simplicity': she suggests linear polynomials, or in other words polynomials of degree one. Beth actually refers to the decomposition (factorisation) of a polynomial of the third degree into the product of three linear ones.

In that sense, these two remarks reflect how decontextualised Beth's thinking is and how the tutor's responses to her difficulty have failed to address the essential reasons behind this difficulty Beth does not seem so far to have realised the rationale behind looking for a basis; or what a basis is and what kind of expression for the elements of a vector space it provides.

Beth's difficulty in seeing the 'usual basis' behind this linear combination perhaps cannot be attributed to the absence of pre-requisite knowledge: there is evidence that Beth knows what a linear combination is and she also knows that she needs to break $p(x)$ into 'simple' polynomials The fact that she suggests factorisation may reflect the possibility that she has not solidly absorbed the idea that the vector space operation between polynomials in $P_3(\mathbf{R})$ is their addition.

So her difficulty in seeing the basis can be attributed to her lacking the ability to piece together these items of pre-requisite knowledge in a meaningful way and in resonance with the needs of the problem she has been asked to solve. And, even though she finally identifies the 'usual' basis of

$P_3(\mathbf{R})$, there is no evidence as to whether she resolved the mystery of why her previous suggestion (factorisation) was not accepted.

Subsequently, the tutor asks the students to calculate the values of 1, x, x^2, x^3 via T and to start with T(1). The discussion in the four tutorials is as follows:

Tutorial 1	**Tutorial 2**	**Tutorial 3**	**Tutorial 4**
Patricia: Two. *The tutor asks for the definition of T. Patricia says it is x + 1 and the tutor wonders what happens if there is no x to map. To the students silence she notes that* 'in fact there is nothing to do'. *Patricia exclaims* 'Ah!' *and says that then T(1) is* 'just one'.	*Silence* *Then Abidul suggests (1 + x) for which she can however give no reason. The tutor repeats that we have to replace x with x + 1 and asks what* 'problem they have with it' Abidul: *x* doesn't appear One?	Cleo: Two. *The tutor disagrees and asks what* "problem they have with it". "Take your polynomial and wherever *x* appears, replace it with x + 1" *she suggests* Cleo: Just one	Beth: Polynomial 2. *Asked what she means she replies*: Because we are adding one *The tutor repeats the definition of T and notes that to add one you first have to have an x* Is it one?

Almost unanimously, the students interpret T(1) as 2. They appear to be seeing T as an action according to which 'we are adding one'. Actually, in her elliptic expression "Because we are adding one", Beth captures the essence of her and her peers' difficulty: what is missing from her sentence is the object on which the action of the verb is to be carried out, suggesting the question: add one to what?

Underlying the students' response is seeing 1 as the number on which to apply the action suggested by 'adding one', and not as:

$$p(x) = 0x + 0x^2 + 0x^1 + 1x^0 = 1$$

which therefore would help them see T(1) as:

$$T[p(x)] = p(x + 1) = 0(x + 1)^3 + 0(x + 1)^2 + 0(x + 1)^1 + 1(x + 1)^0 = 1$$

Abidul's '$1 + x$' is probably closer to the right answer. She seems to have thought:

I have to make x into $x + 1$

I have no x, I have 1.

I'll do this once, or leave it the same, so $x + 1$

If this is at all close to her thinking, then at least she has noticed the absence of x which the other students have not. In any case, the students appear weak in interpreting functional information: they understand the action required by the rule for T, but lack the crucial understanding of the objects on which this action is to be applied.

The fragility of this understanding becomes evident in cases where the nature of these objects is ambiguous, such as 1 (a number? a polynomial?). In this case, it seems that the application of T is only ostensibly a routine algorithmic procedure.

In the above, algorithmic procedures, deemed by the tutor to be routine and simple, turn out to be problematic for the students. While, for instance, looking for the 'usual' basis for $P_3(\mathbf{R})$, the students either fail to respond to the task or make suggestions that are severely decontextualised and do not resonate with the needs of the particular task.

Moreover, the students unanimously appear weak in interpreting functional information (e.g. interpreting T(1) as 2). The fragility of their understanding seems to relate closely to the ambiguity of certain mathematical objects (1, for instance, as a number or as the constant function of value one): 'simplicity' then emerges as a contentious notion.

Underlying the students' unease with 0 in $\mathbf{R}^{\mathbf{R}}$ or 1 in $P_3(\mathbf{R})$ is their inflexible concept image of function as well as their perplexed images of domain, all of which were already evident in the earlier examples from analysis. Here, they are merely exacerbated. Analogous incidents occurred in this study in the context of topology and homeomorphisms, and in the context of abstract algebra and a range of elementary group-theoretical concepts. In the following, I exemplify the latter.

Struggling for a meaningful interpretation of the definitions of centraliser and conjugacy class: anchoring to functional language

In group theory, problems arise with the students' resistance to seeing applications of simple functional rules through complicated first encounters with theorems: for instance, the *First Isomorphism Theorems for Groups*.

Let G, G′ be groups and f: G→G′ a homomorphism.

If K = ker(f), then: $G/K \cong Im(f)$.

My analysis of the study concerning group theory (Nardi, in press, 2001) has been organised, loosely following the intentions of the course, around the concepts leading to an understanding of this theorem. Even though the correspondence itself defined in the proof is quite simple, the fact that the sets between which the correspondence is defined are elusive makes the correspondence itself incomprehensible to the students:

- defining a mapping between two groups (or on a group itself);
- defining a new relation between sets of elements of these groups;
- defining a type of morphism:
  - between these sets;
  - between elements of the group and sets of elements.

Unfamiliarity generates anchoring to familiar language, similar to the example of seeing spanning in terms of Im(f) mentioned in [3]. Familiar functional language is what students resort to in the following typical episode in their attempts to make sense of centralisers and conjugacy classes in group theory.

"I don't really understand what they are", says Connie of the centraliser of an element x in a group. The tutor then defines it as:

the set $C(x)$ of elements that commute with x:

$C(x) = \{y \in G: xy = yx\}$

Connie's reaction to this definition is:

C: You can also swap back to y over there, can't you?

T: But this is a different thing though: you can say that the centraliser of y, if y is an element of G, is the set of x in G such that $xy = yx$, but is that what you mean? [*Connie nods.*] It's what I'm saying here I just changed the dummies so to speak.

C: Can you give me an example so that I can understand?

[*The tutor offers an example from permutations.*]

In her first comment, as well as a remark cited later, Connie's use of the words 'swap back' and 'sends it back' possibly reflects an action-dominated concept image of the group operation.

In $xy = yx$, 'swapping back' is the result of the commutativity of the particular x and y: and later, she speaks of the inverse x^{-1} as "sending x back to itself". This latter verbalisation can perhaps be associated with the action-perceived aspect of the inverse function (if f sends x to $f(x)$, then f^{-1} sends $f(x)$ 'back to' x, so f^{-1} is a way of coming 'back' to x). With regard to the definition of centraliser, the symmetry of the expression $xy = yx$ seems to engender the false impression that $C(x)$ and $C(y)$ are the same.

I think Connie here is confused with the quantifiers behind x and y: in the definition of $C(x)$, x is fixed and y runs through G. Some of these ys, the ones that commute with x, are themselves elements of $C(x)$. In his first remark, the tutor (T) illustrates that $C(y)$ is the set of all the elements that commute with y, whereas $C(x)$ is the set of all the elements of the group that commute with x.

In a sense, the verbal presentation of the definition of $C(x)$ seems to be less prone to engendering this false impression, because it puts the emphasis on the fixed nature of x and the variable nature of y.

Subsequently the discussion turns to C_g, the set of conjugates of g:

C: We were introduced to ... We've defined conjugate as x times g times x^{-1}. I mean how do you choose g? g is an element in the group ... ?

[*The tutor defines the set C_g of conjugates of g and says that in an Abelian group an element g is only conjugate to itself.*]

C: So it's for every g belonging to G?

T: For all g belonging to G, you run through them all and you get what you get as a set here.

C: x has several conjugates then?

[*The tutor nods and says that if G is Abelian then $C_x = \{x\}$. Connie is confused.*]

C: What does the conjugate actually mean? Because the inverse is actually going to send it back to itself ... what are you using it for ... I mean ... because I am always getting confused with inverses and conjugates

Connie's first two contributions here reflect her preoccupation with the role of g in:

$C_x = \{gxg^{-1}: g \in G\}$

I note that in both of them it is not clear whether she is talking about C_g or C_x. The tutor seems to shift from one to the other adapting each time to Connie's words but never pointing at his shift explicitly. His clearest statement is the process-oriented remark in which he defines C_x by suggesting a way to construct it: run through all $g \in G$ and construct gxg^{-1}. The set of various gxg^{-1} is C_x

Connie's remark "x has several conjugates then" is evidence of the impact that the quasi-algorithmic tutor comment has had: Connie realises that the construction of C_x is an x-centred action and also that running through $g \in G$ may generate more than one conjugate for x.

I note that Connie's confusion with the concept of conjugate seems to rest on the same confusion as the notion of fixed and variable element in the definition of centraliser. "How do you choose g?" and "is it for every g?" possibly reflect an interplay in her mind between the fixed and the variable, between the single- and the multi-valued. She seems to have accepted the idea of the multi-valuedness of the conjugates of x when she enquires: "x has several conjugates then?"

In the above, Connie's concept images of centraliser and conjugate seem to be dominated by a confused perception of the fixed and variable elements in the definitions, as well as with a difficulty in accepting the multi-valuedness of the defined elements (that there maybe more than one element commuting with an element x and that x might have more than one conjugate) There is some evidence here that this persistence of single-valuedness may have been reinforced by the student's association of commutativity and inversion with colloquial expressions such as 'back to itself' and 'swapping back'

In conclusion

In the preceding episodes, the concept of function emerges as a driving force, a 'category of thought' (discussed in Sierpinska, 1994, pp. 133-4) in a diversity of contexts within the first-year course in university mathematics. To make this visible, an abstraction was necessary: the notion of function had to be isolated from a variety of other epistemological features - semantic, logical and conceptual. The purpose was to address the iterative nature of some of the students' difficulties - from analysis through to linear and abstract algebra - and at the same time to acknowledge the complexity inherent in the concept of function, a mathematical structure with such a wide field of application.

From the notion of function as a relationship between variables, a relationship which the learners are accustomed to representing in algebraic and graphical forms, we move to the notion of function as a linear transformation between matrices, polynomials, etc and to the notion of function as a morphism between groups or group-related sets In this move, the notion of fixed and variable quantities, domain and range emerge as ideas urgently demanding visibility.

In the above, objects such as 0 and 1, misleadingly perceived as mythic carriers of simplicity, reappear (disguised in a plain semiotic appearance) in a variety of mathematical contexts and demand of the novice an acute perception of their multiple entity status. But the novice, oblivious and not warned about the possibility of multiplicity, remains a prisoner of a single-minded innocence. Zero can be a number, a function, a vector, a matrix, a polynomial, ... - the list is long and the items are not necessarily distinct from each other.

But this multiplicity, while taken for granted, remains part of the hidden agenda of an utterly unnecessary mathematical mysticism. Similarly, the rules of the game of mathematical formalism, while taken for granted, remain tacit - as if the essence of the game were to discover its rules and not to play it.

In the exacting world of mathematical professionalism, the role of mathematics education then emerges as that of the caring accomplice of mathematics: to demystify the rules of the game for the novice player - because no intellectual accomplishment is possible without either challenge or solicitude If mathematics purveys the former in plenty, then mathematics education ought to provide the latter with agility [5]

Notes

[1] The study (Nardi, 1996), from which all the data presented in this article are drawn, is the author's doctoral dissertation It aimed at identifying and exploring the difficulties in the novice mathematician's encounter with mathematical abstraction For this purpose, twenty first-year mathematics undergraduates were observed in their weekly tutorials in four Oxford colleges during the first two eight-week terms of Year 1 Tutorials were tape-recorded and fieldnotes kept during these observations The students were also interviewed at the end of each term of observation A qualitative analytical framework (Miles and Huberman, 1984) consisting of cognitive (e.g as in Tall, 1991) and sociocultural (e.g as in Sierpinska, 1994) theories of learning was applied to sets of paradigmatic episodes extracted from transcripts of the tutorials and the interviews within five topics in pure mathematics (*foundational analysis, calculus, topology, linear algebra and group theory*). This topic analysis was followed by a cross-topic synthesis of themes that were found to characterise the novices' cognition. The thesis can be found at: http://www uea ac.uk/~m011

[2] In the episode reported here, four pairs of female students are taught in four consecutive tutorials:

Tutorial 1: Abidul and Eleanor
Tutorial 2: Patricia and Frances
Tutorial 3: Camille and Cleo
Tutorial 4: Beth and Cary

[3] However, this schism is not universal. When confronted with abstract new concepts, the students often bring in material they feel familiar with: for example, in an attempt to understand the span of a set A as the set of all linear combinations of the elements of A, a student resorts to "all about the values that thing can take", a description often used for Im(f)

[4] Same as in [2], but in the order 2, 1, 3, 4

[5] Following the psychological concerns of this study, a study aiming at an exploration of the pedagogical awareness of the mathematicians involved in Nardi (1996) was funded by the Harold Hyam Wingate Foundation Also, the UMTP (Undergraduate Mathematics Teaching Project) is currently in progress at the University of Oxford and is funded by the UK Economic and Social Research Council For more information see: http://users.ox.ac.uk/~heg/esrc

References

Barnes, M (1988) 'Understanding the function concept: some results of interviews with secondary and tertiary students', *Research in Mathematics Education in Australia* (May), 24-33.

Dreyfus, T. and Eisenberg, T. (1983) 'The function concept in college students: linearity, smoothness and periodicity', *Focus on Learning Problems in Mathematics* 5(3/4), 119-132.

Dubinsky, E. and Harel, G (eds) (1992) *The Concept of Function. Aspects of Epistemology and Pedagogy*, Washington, DC, Mathematical Association of America, MAA Notes No 25

Even, R. (1993) 'Subject-matter knowledge and pedagogical content knowledge: prospective secondary teachers and the function concept', *Journal for Research in Mathematics Education* **24**(2), 94-116

Ferrini-Mundy, J and Graham, K G (1991) 'An overview of the calculus reform effort: issues for learning, teaching, and curriculum development', *American Mathematical Monthly* **98**(7), 627-635

Malik, M A (1980) 'Historical and pedagogical aspects of the definition of function', *International Journal of Mathematical Education in Science and Technology* **11**(4), 489-492

Markovits, Z., Eylon, B.-S and Bruckheimer, M. (1986) 'Functions today and yesterday', *For the Learning of Mathematics* **6**(2), 18-24

Miles, M. B and Huberman, A M (1984) *Qualitative Data Analysis: a Sourcebook of New Methods*, Beverly Hills, CA, SAGE Publications

Nardi, E. (1992) 'Conceptions of advanced mathematical concepts and some cognitive phenomena relating to them - difficulties, obstacles and errors: the exemplary case of function , in Gagatsis, A. (ed.), *Didactics of Mathematics*, Thessaloniki, Greece, Erasmus Publication ICP-92-G-2011/11, pp 361-389

Nardi, E. (1996) *The Novice Mathematician's Encounter with Mathematical Abstraction: Tensions in Concept-Image Construction and Formalisation*, Unpublished Ph.D Thesis, Oxford, University of Oxford

Nardi, E (1997) 'The novice mathematician's encounter with mathematical abstraction: a concept image of spanning sets in vectorial analysis', *Educación Matemática* **9**(1) 47-60

Nardi, E (1999) 'The novice mathematician's inquiry about new concepts: bestowing meaning through ambivalent uses of geometrical metaphors', in Brown, L (ed.), *Making Meanings in Mathematics: a Collection of Extended and Refereed Papers from BSRLM – the British Society for the Research into Learning Mathematics*, York, QED Publications, pp. 57-68

Nardi, E (in press, 2001) 'Mathematics undergraduates' responses to semantic abbreviations, geometric representations and multi-level abstractions in group theory', to appear in *Educational Studies in Mathematics*.

Nardi, E (submitted) 'Rigour versus intuition: extremities of an unjust war', submitted to *Journal for Research in Mathematics Education*

Orton, A. (1983a) 'Students' understanding of integration', *Educational Studies in Mathematics* **14**(1), 1-18

Orton, A. (1983b) 'Students' understanding of differentiation', *Educational Studies in Mathematics* **14**(3), 235-250

Sfard, A (1991) 'On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics* **22**(1), 1-36.

Sfard, A. (1994) 'Reification as the birth of metaphor', *For the Learning of Mathematics* **14**(1), 44-55

Sierpinska, A. (1994) *Understanding in Mathematics*, London, Falmer Press.

Tall, D (ed.) (1991) *Advanced Mathematical Thinking*, Dordrecht, Kluwer Academic Publishers.

Vinner, S. (1983) 'Concept definition, concept image and the notion of function', *International Journal of Mathematical Education in Science and Technology* **14**(3), 293-305.

Vinner, S and Dreyfus, T. (1989) 'Images and definitions for the concept of function', *Journal for Research in Mathematics Education* **20**(4), 356-366