What is a Line?

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What is a line? If you look at letters from a dot matrix printer, you will see that there are actually a number of discrete dots which compose the lines of the letters. If the dots were very fine, the letters would seem to have no gaps. But if we looked closely, through a microscope, we would see the small dots. Imagine looking very closely at a line you have drawn with a pencil or ink. You have drawn the line *continuously* — without lifting your pen from the paper, unlike the dots of the dot-matrix printer. Yet if you look closely enough in a microscope, you will see that the line is actually little traces of ink, limited practically by the size of the molecules composing the ink, or by the graininess of the paper. Look even more closely, and you see vast spaces of emptiness between the atomic particles in ink or paper. What happened to the continuous nature of the line?

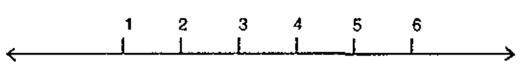
We realize that in physical space all lines must be finite sets of discrete points. Neither the lines nor the points fit our imagined view of what points or lines should be. The points have length and width, and the lines have gaps in them. If we try to think about what we mean by a "line" (not yet a straight line) most of us would say:

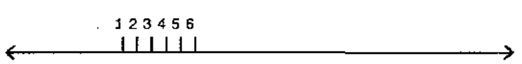
"A line is a set of points" and add "which goes on forever." With more thought we would add: "and having no gaps."

Making a number line

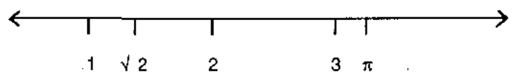
At the beginning of several different years in grade school, an attempt is made to think about the set of points consstituting a line by constructing what we call a number line — a correspondence of points on a line with numbers. There are several presuppositions made (unconsciously) when we correlate numbers with points on the line. Juxtaposing the two very different contexts, the set of real numbers and a geometrical figure, provokes us to ask questions about each of the constituents that we would not have asked separately. For example, the concept of length makes sense for a line but not for real numbers standing alone as a set, while questions about rational vs. irrational numbers raise questions about the "number" of points on a line.

A list is made of the integers — potentially infinite — and they are put in a line. But no matter how close we try to put the integers, there are still gaps between the them. We try to fill in the gaps with more labeled points.





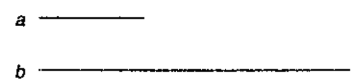
Imagine that we have filled in all the numbers between others — all the *Real* numbers. Will we have filled in all the gaps? We must look more closely. Between any two real numbers a and b there is another real — their average, for example. We can always extend the process in our imagination to fill in more of the "gaps." Yet for this very same reason there cannot be two labeled points "right next to each other."



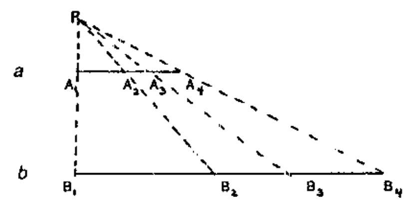
Can we move from one point to another over a bridge of reals, or are the points which correspond to reals an infinite set of islands on the line? Points are discrete, separate items with no length. Can any number of these discrete items ever combine to give a continuous item?

Thinking of a line only as a set of points does not quite fit well with our understanding of length. For example, here is a 1-1 matching of points on two segments with different lengths:

Take any two segments a and b. Put the segments parallel to each other and let P be the intersection of the lines joining the endpoints of a and b.



For each point A on a, draw the unique line AP. Let the point B where AP intersects line b "correspond" to A. Similarly, for any point on b there is a unique point on a.

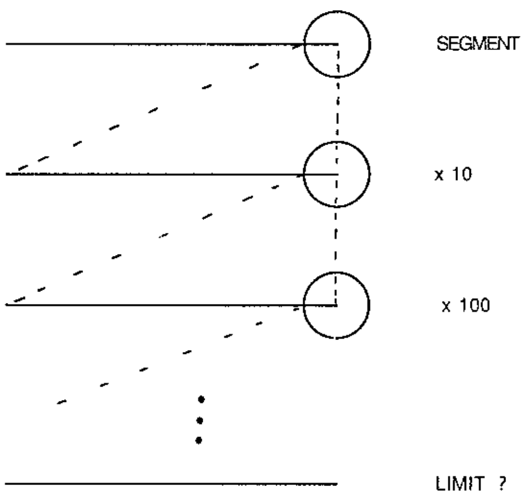


Juxtaposing the idea of a line as a sum of points with the notion of length seems to lead to paradox. Are the partless points on line b "fatter," so that they take up more length? Do they have more space between them? Or do our arithmetic operations just not extend to infinities? Let us look at the line from another perspective.

A microscope

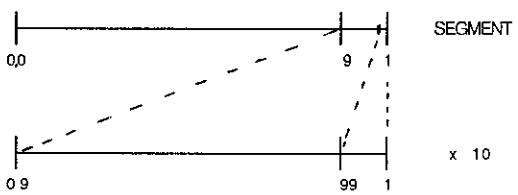
We need an image for a line as a continuous item, not as a static set of points — as something involving intrinsic wholeness. A thought experiment using an infinitely adjustable microscope will allow us to examine a line very closely, to see it as actually continuous no matter how close we look.

Start with a segment, and imagine a microscope with a zoom lens centered at the right endpoint. Increase the power of the microscope 10 times so that you are looking at 1/10 of the segment. Do this again, so you see 1/100 of the original magnified 100 times. Keep going. What happens? The simple answer is: nothing. Even if the process continues indefinitely, we may imagine that the line is indefinitely stretchable, and at the limit of the process we will still see a continuous segment.

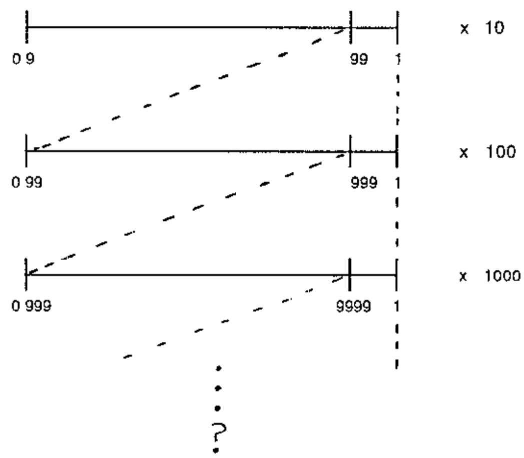


Since at each stage a segment is seen, it is reasonable to say that, at the limit, one will still see a segment — with the same property of continuity. But it will have no ordinarily measureable length — it will be what is technically called infinitesimal.

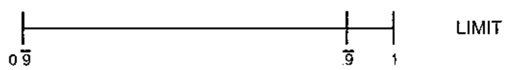
Now label the segment [0,1] for convenience. Imagine to yourself that at some time, say 16 minutes to 12 o'clock, the lens is at 1 power and we are viewing the original segment. At time 8 minutes to 12 the magnification is increased to 10 power, and in the field of vision is now a segment 1/10 as long. If we label the endpoints, at the right will still be 1, and at the left will be 0 9 instead of 0.



At 4 minutes to 12 we increase the power 10 fold, to 100 power. What we see is 1/100 of the original, now having left endpoint 0 99. Continue this process indefinitely, increasing power 10 fold and cutting the field of vision by 1/10 as the time remaining to 12 o'clock is halved.



Now, what will be seen at the limit of this process — at 12 o'clock? What number will represent the left endpoint? We may call it 0 $\overline{9}$.



Though you divide a line indefinitely it essentially remains an infinite continuous item. Or, to put it another way, as we zoom in closer and closer, we always see a line. The attempt to fill up the line with real numbers — even all the infinite decimals we normally use — still leaves a space between the points representing 0 $\overline{9}$ and 1.

If we return for a moment to the previous example of the correspondence between the points of segments *a* and *b*, we may now think about the 1-1 correspondence as not between points but between infinitesimal segments A and B. Since line *a* is 1/3 the length of line *b*, the corresponding infinitesimals A and B also have ratio 1/3, even though neither has finite length. The lines may be thought of geometrically as composed of an infinite number of infinitesimals with comparable measure.

Interpretation

It is useful to be able to think of a line as both discrete, a set of points, and as continuous, having motion and length. Light provides a good analog to the double nature of a line. Light exhibits a continuous or wave nature and also a discrete or quanta nature. Which aspect is exhibited depends greatly on the context or apparatus used to study the light. Different aspects of light are precipitated by the different ways of examining light. Similarly, if we use one set of conditions to look at a line, such as the microscope, we find that it exhibits its continuous nature to us. If we use another set of axioms, for example if we extend finite arithmetic to infinite decimals in a way consistent with

Dedikind and Cauchy, we do not find the infinitesimal segments — only points. Every point on the line gets assigned a Real number, which in turn is defined as a limit of a sequence. If we were to use the microscope under these conditions, we would probably see it break before we reached 12 o'clock.

In one important sense, the discussion of the line is moving us from the finite to the infinite. We have rules or axioms about finite items, finite operations, and operations on finite numbers. But in order to look at a line we have to move to infinite, or potentially infinite, items. What is true about these new infinite items and their operations? We will find a problem with our term "true" similar to the classical distinction between the consistency and correspondence views of truth.

In one sense, the "truth" about the nature of the line is a question about how we formalize the properties of the line so that they are consistent with the previous properties of mathematics. It turns out that either of the following assumptions is consistent with the axioms of finite mathematics.

(a) A line is a set of points (and equivalently $0.\overline{9} = 1$)
(b) A line is not just a set of points (and $0.9 \neq 1$)

That is, we can take either (a) or (b) as a postulate and get a whole theory which has no logical contradictions.

We may also ask which of these two assumptions is true in a second sense, i.e. which corresponds to the way things "actually" are? It may be that the different properties of a line are modeled by different physical situations. But even to ask about the line, "What is the way things are?" brings into question whether we are dicussing a physical notion of line at all. We have already seen that line cannot mean only the physical line, since "infiniteness" for all physical lines is restricted by the limits of our penetration into the atomic scale; and in any case we have no way of actually performing a physical experiment to divide a line infinitely. So we may also go to our "thought" model and ask which of the postulates corresponds more closely to our image, or notion, of a line. Since both of the possible postulates (a) and (b) come out of thought-models of a line, the line as an Idea is susceptible to both descriptions. Line as an idea, not line as a physical thing, has both of the contrary properties. It is the idea which we are investigating, and it is the quality of an idea that it exhibits multifaceted properties when examined in different perspectives. We now notice how mathematics has caused a subtle shift in our attention from the physical thing to the Idea. For human beings, the Idea is no less important than the physical thing — and no less real. From an inquiry about the nature of the line we move also to an inquiry about the mind which is itself inquiring about the line.

Philosophical interpretations

What seems most interesting is that, in considering the nature of a "real" line, we find that the line as we really think about it is not a thing but an Idea — the Idea-line *is* the line. Moreover, the line, as an Idea, does not seem to conform completely to the normal laws of sensible logic. Any formal theory or concepts about the line are bound by the laws of non-contradiction and the excluded middle, forcing us to take one position or another. But the line obeys or exhibits a logic of complementarity. Discrete and continuous are properties which are precipitated out when we take a viewpoint, when we examine the line. Thus line (the idea) behaves much like a sub-atomic quantum mechanical object — in itself it exists in a state which is indeterminate, while the observer precipitates or determines, by her/his perceiving apparatus, the kind of perception attained. Here in our mathematical model, the set of axioms, or the presuppositional standpoint of the individual, are analogues of the physical device which measures the quantum object. We cannot hold to the view that the individual apparatus *constructs* altogether the properties of the line. Though the properties of the line are not some "things" existing in the "line," nonetheless they are not created out of nothing but are brought into view from the idea we are calling "line" by an act of the observer. The particular viewpoints are carved out of the reality of the line by the observing mind, but there is an essential and unique givenness about "line" which, for example, is quite different from the presence of the idea of "circle."

We cannot disclaim the intrinsic nature of "line" without falling into complete relativism, even though our limited mental apparatus may only be able to grasp this nature in a relativized view of the whole. To a more powerful, intuitive perception, the simultaneity of the discrete and continuous natures of the line may be an available experience. We can even make the claim that at the core of any of the determined, finite, partial or relative expressions or views of line, is the presence or apprehension of the intrinsic nature of line, though this is ordinarily unavailable. Every time we actually "see" something about a line we are experiencing the line, but in a determinate mode. We may think of the analogy of electricity running through a toaster, light bulb, motor. In all these appliances where we generate a "mode" of energy, we do not experience the raw energy itself, but the heat, light, or motion. Yet all of these determinable characteristics are "line." We do not say the motor creates the motion, or the light bulb creates the light, though if we are careful we may say that the electricity is susceptible of expression in the forms of light and heat. Yet we have to be careful not to *reduce* heat to light or motion.

Similarly, knowledge itself is susceptible of formulation according to the modes of individual minds. Nonetheless, knowledge itself is not created by these minds, only modelized by them. Knowledge is not a thing, it is susceptible to or capable of immediate usage in formulation and determination when an individual apparatus — the human mind — precipitates an instance of knowledge as an experience. Unlike the material apparatus of science, the wonderful human mind not only transforms or precipitates a form of knowledge, but the mind is itself transformed by the knowledge. Our minds are capable of refinement and growth to deeper and deeper levels of perception or understanding. A powerful idea, for example, first grasped in an inspired moment, may develop the individual's mind over a period of years as it is more and more completely expressed. It is important to stress that the idea transforms the individual's mind — for this is as much the case as the reverse notion that the individual's mind must transform or give form to the inspirational idea.

If this description of processes which we think are well known sounds too "mystical," we would argue for returning some of the mystical to mathematics. Terms like "mind," when they are not trivialized to an association with quantitative analysis, or reduced to material secretions of the brain by some of our contemporaries, have a grandeur which is quite sufficient to humble even the Einsteins among us. Mind is a completely open term, capable of reaching to depths of awareness and experience only hinted at in most experiences, but more fully known in a few. There is a deeper source of wisdom, which we would capitalize as Mind rather than mind since it is not our individual possession. Indeed, if we take the meaning of the word "mystical" to indicate a relation to mystery, the mind is the greatest and most intimate of mysteries that we know. If a mystic is one who explores mysteries, then all who are interested in wisdom are mystics.

In any case, every time we "see" something we are experiencing the revelation of some "mystery." For although logic can trace with great care all the links by which we verify or make consistent or draw out our discoveries, logic itself cannot explain how the moment of "seeing" occurs. Empirical logic cannot explain away the mystery of awareness itself, nor the way a previous sequence of attentions of the mind gives rise to the moment of seeing, nor how something new is grasped, nor how two disjoint items are suddenly related, nor the feeling of certainty, of peace, and of luminosity which comes with a sense of "seeing into" an idea. Yet these familiar experiences are at the heart of knowledge itself.

Conclusion

Returning now to the line, we see we have come quite far from our original question. We can see that both the nature of the line, and the processes we use in discovering the nature of the line, give glimpses into the nature of experience and knowledge in general. The double nature of the line can be taken as a symbol for the dual nature of the mind — momentarily discrete yet also continuous. By the word "symbol" we mean some form or image which is a means of perception for an otherwise intangible, formless idea. So we move from the physical to the idea to the mind which has the idea. In the best of cases, the symbol allows us to "see into" a deeper part of the mind — it becomes a transparent window through which we can glimpse something of the infinite, whereas ordinarily an image or form can only present the finite and delimited. Our attempts to understand the nature of the line are revelatory of the nature of understanding itself. There are deeper levels of meaning and experience which are available if we can learn to think more deeply, and also to put thinking aside — to go beyond logic. It would require a genuine discipline or transformation of the mind in order to make explicitly available the experience of "seeing" which is the core of inquiry.

ITEMS

— Please note that with this volume of FLM the individual and institutional subscriptions have been increased (see inside front cover). Back volumes are available at the old rates.

— The address of the publication has changed (see inside front cover). Both editorial and subscription matters are dealt with from the White Rock address.

— The next issue, Volume 11 Number 2, is a special issue edited by John Fauvel containing papers from a conference on history in mathematics education.