Students' Good Reasons

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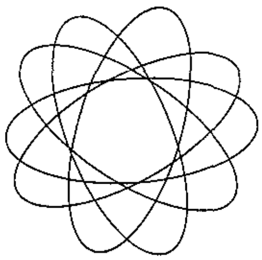
The notion of "students' good reasons" is explored by means of the "Inquiry Cooperation Model" (ICM) which describes a pattern of communicative cooperation between teacher and students The ICM includes: getting in contact, discovering, identifying, thinking aloud, reformulating, challenging, negotiating, and evaluating An ICM is basic to an educational approach in mathematics education, which suggests itself as "progressive" in John Dewey's interpretation of the term, and which tries to develop the students' preconceptions into mathematical competence.



A thought experiment

Imagine that the teacher introduces the following problem to the class: "How many intersections between the ellipses can you find in this figure? You can, for instance, imagine the ellipses to be rubber bands " (Figure 1.) The students start doing their calculations, and after a while the following suggestions are put forward:

Peter: 42
Eva: 36
Tommy: 40
John: I can't find out
Celia: 80
Monica: 12 and more.

Instead of just doing the usual "ticking", the teacher asks the students to explain how they got their results.

Peter: I simply counted the intersections

Teacher: Well, that is a simple and good idea. Counting is a sound mathematical procedure. And you got 42?

Peter: Yes, but it was difficult to make sure I didn't count the same point more than once. Anyhow I did my best

Teacher: And you Eva, you got 36?

Eva: Yes, and I also noticed that it was difficult to make sure I counted a point once and only once

Teacher: How did you do it then?

Eva: I used my pencil and marked the point of intersection as soon as it was counted.

Teacher: An excellent idea, and you got 36?

Eva: Yes, I am sure, but did you notice that in some cases three lines are going through the same point? [The teacher takes another look at the figure.]

Teacher: And you Tommy, you got 40?

Tommy: This was simple, I liked this idea of marking the points with red, so first I marked all the points, and then I counted. Very simple.

Teacher: Well, yes ..and you, John?

John: I couldn't find out You see, I imagined it was rubber bands, and for the instance the one on the top certainly does not touch the bottom one. It was too complicated.

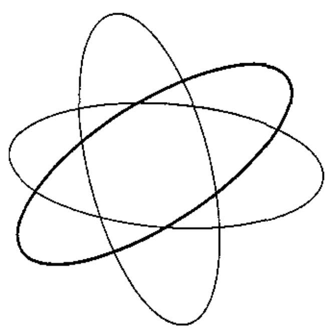
Teacher: And you got 80, Celia?

Celia: I also used the red pencil, but I did it a different way. I did not mark the points, instead I coloured one of the elipses red When I had done this, it was not difficult to see that this red ellipse was intersected in four points by each of the other four ellipses And 4 times 4 equals 16. Because I can choose to colour five different ellipses I have to multiply by 5, and this gives 80.

Teacher: Well done! And you, Monica, your answer is "12 and more". How did you get this?

Monica: I followed Celia's idea, but because it is difficult to study a figure with five ellipses, I made a simpler version I included only 3 ellipses [see Figure 2] I coloured one of the ellipses red. Then it was simple to count the number of intersections this one has with the two other ellipses This was 8. I can choose to colour three different ellipses, and this gives 24. However, it is easy to see that I have counted all the intersections twice, so I have to divide by two. This gives 12. But because I only drew three ellipses, the answer must be "12 and more".

Through this thought experiment we want to emphasise the importance of considering students' spontaneous approaches to the subject. What are their reasons for solving an exercise in a certain way? By focussing only on students' results, which is a common thing for the mathematics teacher to do, he may be ignoring essential ideas produced by the students. In the following we discuss what it could mean to take students' good reasons into account



What do we mean by "students' good reasons"

Let us look at the words one by one. By "students" we mean persons attending school; instead of "students" we could also talk about learners Further, we think of students as communicators. They express themselves in interaction with the teacher or with their classmates "*Good* reasons" is a relative term which calls for some specification. In this context we use the term for reasons that count as *serious* — although they are not necessarily offered *as* reasons. This, however, may not make too much sense to readers until we have had a closer look at the term "reason".

Let us try a negative definition. "Reason" is not synonymous with "opinion" A reason can be related to an opinion, but an argument is needed as well. A good reason therefore cannot be identified with a good solution. A good solution might follow from a good reason, but the terms are still different. A good reason has nothing to do with right or wrong For instance a student might have a good reason for using a certain algorithm, though it turns out to be insufficient for solving the problem.

Emphasizing the importance of "the students' good reasons" is not a variation on the assertion: "the students are always right" They might be wrong even though they have good reasons for acting the way they do. But some of our previous investigations also show that the students' good reasons are seldom examined before being rejected by the teacher.[1] In Alrø & Skovsmose [1994] we discuss *bureaucratic absolutism* as a form of teacher-student communication where every mathematical question is treated as absolute and everything can be immediately judged right or wrong. Bureaucratic absolutism considers the committing of mistakes and the correction of mistakes as central parameters in mathematics education In bureaucratic absolutism authority plays an important role: the authority of mathematics, the authority of the textbook, the authority of the answer book, the authority of the teacher Looking for "students' good reasons" in classroom communication will therefore be in vain when "bureaucratic absolutism" prevails.

A "good reason" is not always noticeable in the foreground of the classroom scene. Mostly it is only discoverable by reading between the lines of communication. Therefore it has to be revealed and (re)formulated through a classroom dialogue, where the teacher plays the role of a catalyst in order to get the student's good reasons to the surface for further examination and discussion

There may not always be one single and clearly reconstructible reason. More likely the student's interpretation of a problem is determined by a whole landscape of reasons. In this way there is a close connection between the student's good reasons and his perspective. Discussing good reasons is therefore another way of clarifying and exchanging student and teacher perspectives. Further, this opens up the negotiation of meaning, a way in which the student's competence can be used as a positive resource in the learning process [2]

Types of good reasons

Students' good reasons are not always closely connected to a specific mathematical problem. They are not always rational. Nevertheless we may still consider them good reasons.

Many good reasons are naturally related to mathematical understanding, and in this case we can talk about *mathematical reasons*. The student argues on the basis of his mathematical knowledge and previous experiences of mathematics He might refer to certain algorithms or the results of similar exercises as reasons for acting the way he does. In the thought experiment, for instance, the students gave different mathematical reasons for their problem solving.

The ellipses in the thought experiment belong to an exercise which is first of all situated in an educational practice. But if the ellipses are thought of as rubber bands the exercise becomes contextualised In such a case the students' reasons for how to calculate an answer may refer to the contextualisation itself. In a study by Kirsten Grønbaek Hansen an important phenomenon, "being trapped by the contextualisation", was identified [3] She investigated mathematics lessons given as part of a vocational training The students were prepared for jobs having to do with food production. Some exercises had to do with fish preparation: "If the weight of a fish is ... and 20% of the fish will disappear when it is cleaned, how many fish are then needed if ?" Some students took this contextualisation seriously For instance, they considered what sort of fish were being discussed and how they ought to be prepared If they were plaice and not too big, then each dish should contain one fish, but if they were destined for fish soup it was enough to consider the total weight of the fish used Such considerations bring out the students' *contextual reasons*. However, these students got lost because they did not realise that the contextualisation was only a "virtual reality" [4] Other students who ignored the contextualisation and followed the rituals of mathematics ("use all the information provided and use no other information"), easily solved the exercises.

A third group of good reasons has to do with the situation in which the activity takes place. A suggested contextualisation provides a perspective on how to solve a problem. Maybe the students find that the problem can be solved without using any mathematics at all and that, in "real life", it could easily be solved in a different way. At the same time the students know that they are involved in mathematics, and therefore they feel they must find a mathematical way of solving the problem Such reflections have to do with the school setting, and reasons for doing things in a certain way are based on reflections on this setting These *organisational reasons* refer, for instance, to the student's experience of school discourse: "If I put up my hand, the teacher thinks I know the result" or "If I say that I don't know, the teacher will tell me how to solve the problem" or "I want to appear to be a bright student"

The final type of good reasons we want to mention are *personal reasons*. Students have individual backgrounds and experiences which influence their interests and ways of thinking. They handle new information, experience, and knowledge by relating it to the knowledge they already have. They transfer everyday knowledge to their classroom behaviour in the sense that they draw parallels to experienced situations when solving a mathematical problem. The student is not always conscious of this process, and the teacher cannot possibly know of the individual experiences before examining them Nevertheless, they may be formative of individual perspectives on the process of learning.

We do not try to elaborate a clear-cut classification of the students' good reasons. What is essential to emphasise is the variety of "good reasons" which the students explicitly or implicitly may refer to when they do problem solving. Good reasons are grounded in reflections, and these reflections may concentrate on the (contextualised) mathematical task, but they may also refer to the whole situation of schooling as well as to personal interpretations and priorities.

How to handle the student's good reasons in the classroom

One good reason for examining the student's good reasons in the mathematics classroom is that they can be seen as important resources for learning. Good reasons have an influence on the student's way of acting, but perhaps he is not aware of his reasons. Examining the student's good reasons not only helps the teacher to know the student's way of thinking, it also helps the student to an awareness of his way of acting in the classroom.

On examination good reasons can, in the first place, be made visible. Often this examination can reveal a source of misunderstandings and misinterpretations in the classroom communication. Furthermore, the reasons can then be challenged. Checking students' good reasons is not a way to make sure that they learn something. It is a way of making them reflect upon their way of handling the problem. Challenging good reasons, therefore, means making the students reflect upon and widen their perspective and knowledge. Sometimes the result of this process is that the student finds out that his good reasons were bound up in wrong presuppositions.

The important thing is that it is his good reasons and not the teacher's explanation that become the starting point for his reflection. The game takes place on the student's home ground. It is easier to make a score and to win when you are on your home ground. In the same way the learning process is more likely to get somewhere if it is related to the perspective of the student rather than to the perspective of the teacher.

In order to challenge the student's good reasons one first has to discover them. How can it be done? One way is through the process of *active listening*, which means asking questions and giving non-verbal support while finding out exactly what the student is getting at. "It is called "active" because the listener has a very definite responsibility. He does not passively absorb the words which are spoken to him. He actively tries to grasp the facts and the feelings in what he hears, and he tries, by his listening, to help the speaker work out his own problems."[5]

In this phase the teacher should forget about his own explanation and try to adopt the student's perspective. When this perspective is identified the teacher's questions should become more confrontational in order to challenge the student's perspective.[6]

The phases looking at the student's good reasons can be illustrated in a model, which we call the *Inquiry Cooperation Model* (ICM) (see Figure 3).[7] First of all, active listening means that teacher and student *get in contact*. By the term "getting in contact" we understand more than the teacher calling for attention. "Getting in contact" means tuning in to the same channel in order to come to speak the same language. This is the first condition of understanding each other.

After establishing mutual attention the teacher can *discover* the student's good reasons by a questioning strategy. By further active listening the teacher is able to *identify* the student's good reasons for thinking the way he does. While *thinking aloud* the student gets an opportunity to put his ideas and reasons forward in the dialogue. Already in this phase the student will be clarifying the problem — he will be "learning by talking".[8]

The reasons can be *reformulated* by the teacher so that he makes sure that he understands what the student says. Next, the student can be *challenged* on his good reasons. In this phase the teacher plays the role of opponent as well as the role of partner. It is important that the teacher is able to do both in order to strengthen the student's self-confidence while also furthering his learning. In order to stimulate the student's reflections, the challenge should be adjusted to his conceptions — not too much and not too little.[9]

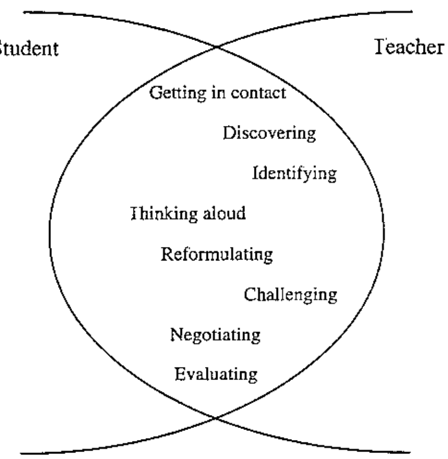


Figure 3
Inquiry Cooperation Model

Challenging the student's good reasons leads to the *negotiation* of teacher and student perspectives. Did they see the same problem? Did they look at the problem from the same point of view? Did they try to solve it the same way? In this phase misunderstandings and other differences may occur explicitly in the teacher-student dialogue. For instance, they may discover that the teacher's reasons relate to a general analysis of the problem while the student thinks of the problem as a concrete, practical one.

On the basis of the negotiation the student and the teacher can *evaluate* their good reasons and might even be able to discuss what the student has learnt in the challenging process.

The ICM describes a pattern of cooperation between teacher and students in which the students' good reasons play an essential role. The ICM therefore refers to a form of teaching where the students' preconceptions and already-established understanding form the basis for the teaching-learning process.[10] In theory at least. But is it possible to observe the ICM in practice in the classroom?

What does the Danish flag look like?

The following sequence is part of the introduction to a course of approximately 12 lessons in a Danish 6th grade mathematics class.[11] The students work in groups of 2–5 participants. They are supposed to make models of European flags, taking care of the proportions of flags, stripes, and crosses. As an introduction to the subject, the students are asked to make a model of the Danish flag (see Figure 4), just as they recall it. Afterwards the groups have to argue and comment on their results and see which model is most similar to the real flag. We follow this discussion with the ICM in mind

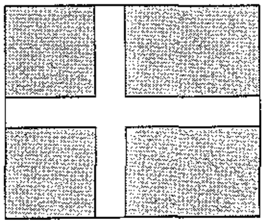


Figure 4
The Danish flag

Alice and Deborah have cut out some white strips to make a cross. But when they are about to place the cross on a piece of red paper, they are not quite sure how to do it. They ask the teacher to help.[12]

Teacher:	Alice, if we decide that it should be *this* broad [the red paper for the flag] . Let us decide that . .
Alice:	Yes.
Teacher:	. . . in order to have something to look at
Alice:	Yes
Teacher:	Then we can estimate it, can't we? How would you place this [the cross] in the middle?
Alice:	I would measure.
Teacher:	. . if you think that it should be placed in the middle. Do you think so? Is this [the white cross] right in the middle or is it a little up or a little down?
Deborah:	It is a little up.
Alice:	It is in the middle.
Teacher:	Okay, and then how would you carefully place it right in the middle?
Alice:	Measure
Teacher:	Yes but. . oh yes. You could to so. How would you measure, then?
Alice:	I borrow a ruler
Teacher:	[laughs] Yes, okay.[13]

First, the teacher suggests that they use the original size of the red paper for the flag, and Alice accepts his suggestion. So far they talk the same language — they are *getting in contact*. The teacher uses the personal pronoun "we" in his opening of the conversation, which indicates that they are working together.

He changes strategy when addressing them as "you" in his first question: "How would you place this in the middle?" The teacher is no longer part of the team. Alice proposes a strategy for the solving of the problem: "I would measure", but before listening to this suggestion the teacher corrects himself and asks again. His first formulation presupposed that the students would place the cross exactly in the middle, but his reformulation questions this presupposition. We interpret this as the teacher's way of trying to *discover* the students' good reasons for their way of placing the cross on the red paper

By asking the students what they think, the teacher tries to *identify* their reasons: "Do you think so?" But Alice and Deborah have different ideas about the placing of the cross, and the teacher uses a kind of selective hearing when ignoring Deborah's proposal and repeating Alice's (identical with his own?). Still, he continues his examination of the students' ideas of solving the problem: "How would you place it carefully right in the middle?" By the words "carefully" and "right" the teacher implicitly claims that the procedure should be mathematically based.

Alice repeats her proposal of measuring, which is accepted by the teacher who then wants to know how Alice would proceed. Here they obviously misunderstand each other. Alice thinks of the tool (the ruler) and the teacher asks for the algorithm she is going to use. The teacher's laughter indicates thai he is aware of this misunderstanding. But he still encourages Alice to continue: "Yes, okay."

When Alice has borrowed a ruler, she goes on:

Alice:	[*ic*][14] this way, and then I would measure that one.
Teacher:	22 1/2, or is it 22 4?
Alice:	22.4
Teacher:	Yes, and this one is . .? [the breadth of the white strip]
Alice:	It is 5 1/2. No! Yes, 5.4.
Teacher:	5 4 okay. What then?
Alice:	What is half of 22.4? That is 11.2 [Deborah interrupts]
Deborah:	. . and then we have to find the middle
Alice:	Yes, then we find the middle
Teacher:	Yes, but when you put [this white cross] on you cannot see your mark in the middle
Alice:	No, but then I just put the dot a little farther out (outside the red paper).
Teacher:	I see, okay
Alice:	It shouldn't be that difficult.

Alice is about to measure both the breadth of the red piece of paper and the breadth of the white strip for the cross. Her indexical expressions "this way" and "that one" are *reformulated* and pointed out by the teacher. The teacher still pursues Alice's ideas for an algorithm as can be seen in his persistent questioning: "and what do you do after that?" and "What then?" During this examination Alice is allowed to *think aloud*: "It is 5 1/2. No! Yes, 5.4" and "What is half of 22.4? That is 11.2." The last example of thinking aloud indicates that Alice has begun, before she has answered the teacher's question, to decide what she would do next. That may be the reason why Deborah interrupts to point out the algorithm: They are about to find the middle of the red piece of paper. Alice confirms this

At this point the teacher *challenges* the students' good reasons by objecting that they will not be able to see their own marking of the middle of the red paper, which means that they cannot place the cross accurately. But still Alice has a solution of the problem: She just puts the mark outside the paper so that she can still see it when placing the cross. She seems quite happy with this solution herself, and the teacher accepts it in the first place: "I see, okay"

Here the story might have ended. The students have good reasons for solving a problem, they found an algorithm after some arguments, and they have got a result So they are done with the exercise. But the teacher changes his strategy

Teacher:	Look here. Why don't you instead. . couldn't you calculate how much that piece should be, I mean the breadth of the red piece above? How much should it be, when the whole piece is 22 point .. [Alice interrupts]
Alice:	This one is 5 1/2.
Teacher:	This one was 5 1/2, and this one was 22 1/2, wasn't it?
Alice:	Yes.
Teacher:	.. approximately
Alice:	How much is that? It is... [3 see]
Teacher:	Yes, it is the math teacher asking
Deborah:	It is 8 point something.
Alice:	No, it certainly isn't 8 point something, it is.
Teacher:	It is the same as 22 minus 5.
Deborah:	Yes, sure..
Alice:	It is 17.
Teacher:	How much is it going to be, the red piece up here?
Alice:	It is going to be. .
Deborah:	Then you just take half of it.
Alice:	Then it is going to be half of 5 1/2.
Deborah:	No, half of 17.
Teacher:	No, half of 17, isn't it?

The teacher introduces another algorithm. He wants the students to calculate in order to be able to place the cross precisely in the middle of the red paper. Instead of following the students' own reasons the teacher suggests his own, and this makes a radical change in the character of the conversation. From being an open dialogue where the teacher was curious about the students' perspective on the problem, it changes to a quizzing strategy, where the students are supposed to guess what the teacher is aiming at[15] The teacher is aware of that change himself, as can be seen in his metastatement: "It is the math teacher asking."

Obviously the teacher wants the students to subtract 5.5 from 22.5 and divide by 2 in order to calculate the size of the red pieces on each side of the white stripe Deborah seems to follow this idea more quickly than Alice, as can be seen in Alice's rejection of Deborah's proposal. "It is 8 point something". So the teacher presents the algorithm step by step: "It is the same as 22 minus 5" and "How much is it going to be, the red piece up here?" Deborah explains the algorithm to Alice: "Then you just take half of it," but Alice does not seem to follow her idea. Instead she wants to take the half of 5 1/2. We do not come to know the students' reasons for their different proposals, and the teacher once again makes use of his selective hearing by repeating Deborah's suggestion. In this part of the conversation the teacher makes less use of active listening than he did in the first Then he changes his strategy once again.

Deborah:	We don't measure in that direction, do we?
Alice:	No sure, it is.. No, what is half of 5 1/2?
Deborah:	What the hell are you doing?
Alice:	It is 2 75.
Teacher:	Yes
Alice:	Then you have to subtract 2 75 from 17. That is, uh,.. 15 point something.
Teacher:	Yes, it is 15 point something, that is right [laughs].
Alice:	But 15 what?
Teacher:	What are you going to use them for, those 15?
Alice:	But I would measure down there ..
Teacher:	. and then down to 15 and then put all of it [the cross] down at the bottom. I will be back in a moment, then you can try to tell me what you have done.

What is conspicuous in the above sequence is that the conversation has changed from a teacher-guided talk into a discussion between the students with only a little interference from the teacher.

This indicates that the students are discussing the problem and the algorithm themselves, and the teacher has given up the leading role. Alice cannot give up her idea of taking half of 5 1/2, and she is challenged by Deborah: "What the hell are you doing?" Alice goes on searching for an answer to her own question, and the teacher lets her. He does not interrupt the inquiry until Alice has come out with a cautious proposal: "15 point something." Then he challenges her: "But what are you going to use them for, those 15?" Alice returns to the algorithm she suggested in the first place-namely, measuring, and the teacher leaves the two girls for a moment, noting that they have to come up with an (his?) algorithm when he returns

We do not know what the two girls have been talking about while the teacher has been away.[16] But as he returns 2 1/2 minutes later, they have got an algorithm and a solution.

Teacher:	Yes Alice Have you found out?
Alice:	Yes.
Teacher:	How did you manage it?
Deborah:	We measured 8 1/2 down and 8 1/2 down.
Alice:	Half of 17.
Teacher:	Yes 17, it was the difference between the red and the white piece, wasn't it?
Alice;	No, half of the red piece.
Teacher:	Yes, of the red piece when you have subtracted the white one, isn't it?
Alice:	Yes, and then half of 17, that is 8 1/2.
Teacher:	Yes.
Alice:	And then we measured 8 1/2 inwards and that is there.
Teacher:	Good.
Alice:	And then we measured 8 1/2 inwards there, too.

Teacher:	Yes, that is right. You dropped the idea about 15?
Deborah:	Yes, because it turned out to be wrong.
Teacher:	Okay.

The teacher questions the students' methods. "How did you manage it?" and he *evaluates* them by responding: "Good" and "That is right". Finally he wants to know what came out of the proposal of 15. "You dropped the idea about 15?" and Deborah concludes: "Yes, because it turned out to be wrong." It would have been interesting to listen to Alice's reasons against Deborah's in order to come to know how they found the algorithm in the end. But the point is that they actually found out by reflecting and arguing on the basis of their own perspectives and reasons.

The empirical investigation

An interesting phenomenon in our empirical investigation of students' good reasons has to do with the "argument of silence". This argument has its origins in the science of archeology. If a thesis is put forward stating that a certain culture has been influenced by another culture during a certain historical period, then it should be possible to observe some archeological evidence as, for instance, similarity in pottery. If nothing can be observed, the argument of silence becomes a strong argument against the thesis.

Our evidence is of similar nature. If a thesis is put forward that teachers in their normal practice pay attention to students' good reasons then it should be possible to find some evidence of this in an empirical investigation. Our material is, however, "rather silent" on this point. The absolutely dominated structure of communication between students and teacher (as well as between students) is that of explaining the right of algorithm and of correcting mistakes.

As our example with the Danish flag shows, our empirical material is not completely silent, but it has certainly been very difficult to identify ICMs. It must be emphasised that we interpret an ICM very liberally. We have outlined it as a sequence of communicative cooperation, but we do not suggest that all the elements have to be present, or that they should take place in exactly the described order. Instead, an ICM can be seen as a characteristic of a communicative cooperation in which (some of) these elements, explicitly or implicitly, are brought together in some cluster.

In fact only in a very few cases have we identified a fully developed ICM. More often we can identify several *mini*-ICMs; in particular the phenomenon of ICM-deformation is interesting and possible to observe. We find situations in which an ICM seems to be initiated, yet it is soon broken up and eliminated. Therefore our conclusion is: the examination of students' good reasons is all too often absent in classroom practice.

ICM obstructions

We have identified different forms of degeneration of the ICM, and we shall characterise some of them.

First, an ICM can degenerate into "quizzing". This was initiated in the above example when the teacher wanted the students to try an alternative algorithm to measuring.

Secondly, an ICM can be initiated by the teacher, but when he finds that the student is not on the right track, he stops the examination immediately. Let us give an example, still having to do with flag construction:

Tommy:	Could it be those two lines which part right there? [points at the blackboard].
Teacher:	I didn't quite catch that. Would you try again, please?
Tommy:	Okay.
Mike:	Go up and show it.
Tommy:	These two lines, the white ones, and then the cross on the left. [Tommy points at the blackboard while sitting at his desk. The teacher goes to the blackboard in order to point out what Tommy is talking about.]
Teacher:	You mean here?
Tommy:	Yes.
Teacher:	No. No, that is not what I meant.

It is not clear what Tommy is getting at, but instead of ignoring Tommy, the teacher asks him to explain: "I didn't quite catch that. Would you try again, please." Thereby, he signals that he wants to discover Tommy's good reasons. Later he asks a question of clarification: "You mean here?" But as Tommy confirms, his proposal is rejected. He did not hit the teacher's point, and there is no further examination of Tommy's reasons.

Thirdly, an ICM can be defeated by the time schedule. "Sorry we do not have time, do something like this and this." The ICM becomes transformed into the language of management. This can be caused by the fact that the teacher has an obligation to teach what is needed for the students to proceed to the next class. If the students have to pass an examination by the end of the year, then the teacher is obliged to make sure the students have acquired the mathematical skills which constitute the basis object for examination. Whether these techniques, from some philosophical point of view, in fact define the genuine nature of mathematical thinking is not essential. To introduce students to mathematics (in some genuine sense) is not essential compared to the task of ensuring that the students get the best possible preparation for passing the examination.

Fourthly, there is self-censorship. The student may have an idea how to handle a certain problem, but he does not want to articulate this when the teacher is present. The student does not want to reveal himself by making a (maybe silly) suggestion, which could spoil the teacher's good impression. instead the student replaces the initiated ICM by the official classroom discourse.[17]

We can think of several reasons for the teacher *not* to try to discover the students' good reasons and *not* to use them as a resource for learning purposes. One is the fact that it takes time to explore the individual student's reasons. To pay much attention to the students' good reasons demands use of time taken from other classroom activities. Therefore, the teacher, who is responsible not only for the students who might be the most eager to present their ideas, but for the whole classroom community, chooses to ignore students' good reasons.

A different reason for not taking notice of the students' good reasons is not to believe they exist. Or, what is slightly different, that the students' good reasons are not considered worth discussing by the teacher Maybe the students' good reasons are considered to be fixed ideas leading the students on the wrong track and, therefore, as an obstruction rather than a resource for the learning process

Obstructions to the ICM cannot, however, simply be interpreted as an obstruction caused by the teacher acting in a traditional way. It is important to realize that students are brought up within a certain school discourse, which influences their preunderstandings and expectations of classroom activities For instance, the students often expect the teacher to present explicitly the knowledge he wants them to gain They do not insist on their own ideas because they expect to be controlled and evaluated by the teacher. In this way they do not have to be responsible for their own contributions. The teacher will always give the right answer or come out with the right algorithm in the end Such preunderstanding of teacher and student roles prevents the teacher from practicing ICM, because this demands active participation of the students in the classroom communication

The final reservation about the ICM that we want to point out is that following the speech acts of the model makes some demands on students' verbal abilities The strategy might favour those students who express themselves willingly and easily and disfavour others: for instance, serious, but quiet students who are just as interested in learning mathematics.

This shows some of the difficulties in an inquiry-based mathematics education. It is not a simple task to realise an ICM.

Inquiry-based mathematics education

The notions of communication and cooperation have different values within different educational theories. Structuralism suggests a strong and well-elaborated learning material reflecting the basic structures of the architecture of mathematics In this theory the teacher must take the role of an interpreter, while the students must try to grasp the logic of the curriculum as presented by the text and interpreted by the teacher. According to structuralism, the preunderstandings of the students can obstruct the learning process The students may cling to some habits and some interpretations of mathematical concepts and operations which might hinder genuine understanding of mathematics (the students might be trapped by a certain contextualisation). The cooperation and communication between teacher and students here are similar to that of a priest and his congregation

Other approaches, such as constructivism and ethnomathematics, pay special attention to the experiences of the students. The preunderstandings of the students are seen as resources for further epistemic development. In this paper we refer to such interpretations as examples of an *inquiry-based interpretation of mathematics education*. For the sense of the term "inquiry" we refer to the work of John Dewey.[18]

According to Dewey, the notion of "truth" is not the primary notion in epistemology. The key term is *inquiry*. It is essential for the learner to be involved in a process of "finding out" The sort of inquiry which is relevant to education is similar to that of a scientific investigation because the way of learning is similar to the way of studying any phenomenon. An inquiry-based education is completely dissociated from the idea of transferring knowledge Knowledge cannot be delivered, it must be developed. A process of inquiry must start from where the students are: "Anything which can be called a study, whether arithmetic, history, geography, or one of the natural sciences, must be derived from materials which at the outset fall within the scope of ordinary life-experience " And, Dewey adds: "It is a cardinal precept of the newer school of education that the beginning of instruction shall be made with the experience learners already have; that this experience and the capacities that have been developed during its course provide the starting point for all further learning "[19]

It seems difficult to observe to what extent an educational approach is inquiry-based To be "inquiry-based" is not a characteristic which manifests a direct empirical appearance at the surface of an educational practice. Instead, we may talk about *empirical indicators* of inquiry-based education. We can study the curriculum, the textbook, etc. We can, however, also observe what is happening in the classroom. *We suggest that an ICM is an empirical phenomen which indicates whether or not the educational practice is, in fact, organised with reference to the notion of inquiry.* If the educational process is to be inquiry-based, the students must be invited into this process The students cannot be spectators, they must be actors And an ICM serves as such an invitation.

Thus, we have suggested a characteristic of inquiry-based education which refers to the nature of the communication, i.e. of the cooperation, between teacher and students In particular, an ICM refers to the teacher's communication strategy.

The notion of inquiry-based mathematics education is very broad: it seems to include a wide variety of approaches to education. However, if it is not possible to observe ICMs during the educational process, it is unlikely that the observed practice does belong to this variety

Qualifications, however, are needed The ICM is a characteristic of the communicative practice of the teacher It might be the case the students, even though the teacher does not try to discover their good reasons, are involved in a process of inquiry. Furthermore, even if the teacher follows an ICM, then the actual teaching-learning process need not be inquiry-based mathematics education, e g. if the content of the communication has nothing to do with mathematics. In short, observations of ICMs only serve as empirical indications.

Practitioners have often interpreted "paying attention to the students' background knowledge" as a principle for organising some forms of teaching material: It should include examples of students' everyday experiences Some forms of mathematics education which claim to be oriented towards the students' background knowledge and the students' experiences do however, make a false claim. If the students' good reasons are ignored in actual classroom practice, then it is unlikely that we are in the presence of an inquiry-based educational practice. This means that the study of ICMs can help to formulate a critique of educational practices which claim to pay special attention to the students' pre-understandings. In cases where we cannot identify examples of ICMs it becomes difficult to describe the educational practice as inquiry-based.

Many descriptions of progressive education protect themselves from criticism.[20] If they claim to illustrate an inquiry-based education, there should be some detailed presentation or some empirical observations that make it possible to evaluate the practice. It becomes difficult to criticise a description for not being in accordance with a suggested educational framework if it does not refer to empirical observations that make it possible to identify the existence (or non-existences) of ICMs

Thus, the existence of ICMs not only indicates whether or not an educational practice is organised with reference to the notion of inquiry. It also becomes a strategy for determining whether a certain educational theory is realised in practice

Notes

1 See Alrø & Skovsmose]1994] Our studies are based on an observation corpus, which Helle Alrø audio- and videotaped in the spring of 1993 in one 5th and two 6th grade mathematics classes in Denmark. Three weeks of mathematics lessons, about 12 lessons, were videotaped in each class The observations are part of her research project on "Communication in the mathematics classroom" which is related to the research initiative: "Mathematics education and democracy" financed by the Danish Council for Research in the Humanities (1990-1993).

The purpose of the survey of communication in the mathematics classroom is to examine the teacher-student communication strategies and discuss how they influence the students' learning of mathematics

The observed mathematics lessons were part of the normal teaching programme; the observer had no influence on the planning or teaching whatsoever

2 For a discussion of the term "negotiation of meaning", see Voigt [1994] and Alrø & Skovsmore [1994]

3 Kirsten Grønbæk Hansen has described some of her observations to us Here we only summarise a main point, not giving the actual details

4 Iben Maj Christiansen]1995] has used the expression "virtual reality" in her discussion of students' interpretations of different forms of contextualised mathematics.

5 Rogers & Farson [1969] p 481

6 Sigel & Kelly [1988] present a similar way of thinking in their "Spiral learning cycle" that contains elements of focussing, exploring, restructuring, and refocussing to describe the patterns of teacher questioning (distancing strategies) that challenge the students' mental operations in the learning process.

7 We have developed the model as an alternative strategy to bureaucratic absolutism, where student proposals are stated as right or wrong without any consideration of the reasons for their way of thinking and acting. These reasons are at the core of the ICM. In that way it reflects the interpretation of a communication in which the communicators try to understand and respect each other as equal human beings. This means that the model might be applicable to other contexts than mathematics education.

8 We use the term as a metaphor for the point that knowledge develops not only in the students' mind but through interaction with others

9 This refers to Bateson's [1972] notion of "The difference that makes a difference".

10 As described here, the ICM designates a teaching strategy Naturally it is possible to interpret the communication between teacher and students from the perspective of the learner and then develop the ICM as a learning strategy

11 The sequence is taken from the observation corpus mentioned in note 1

12 This is an unbroken sequence, but we divide it into smaller parts in order to be able to study carefully what is going on

13 In order to understand the transcript it is necessary to add some information about the indexicality of the spoken dialogue, i e. a description of, for instance, the paralanguage, the body language, and the deixis of persons, time, and place, all of which the teacher and students use and understand quite well in the shared context of communication, but which we as analysts and readers of communication have to interpret in order to reformulate the meaning of the words outside the original context

The English transcript is a translation from Danish, which may naturally be an important source of inaccuracies. But as it has to be so, we shall only use the example as an illustration of the model.

14 "*Ic*" means "incomprehensible talk".

15 For a discussion of the function of the "quizzing" strategy see: Stubbs [1983], Lemke [1990] and Alrø & Skovsmore [1993].

16 The microphone for the audiotape was carried by the teacher and the video sound was not good enough to catch the dialogue between Alice and Deborah

17 See Christiansen [1995].

18 A general overview of Dewey's educational ideas can be found in Archambault [1964]. See also Dewey [1938, 1966]

19 Dewey [1963] p. 73-74.

20 In their study Lene Nielsen and Susanne Simoni [1994] discuss the possibility of criticising an educational theory by referring to a description of an educational practice. They emphasise that many educational perspectives have been presented in a great number of papers without being accompanied by detailed descriptions or empirical observations of educational practices. By ignoring descriptions and observations of classroom situations, the theoretical and general educational perspective protects itself from criticism

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