DEVELOPING AND USING SYMBOL SENSE IN MATHEMATICS

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The inspiration for the idea of symbol sense has several sources:

- From the work done on number sense in the eighties and early nineties, it seemed natural to think about how to extend the construct *number sense* from the realm of school arithmetic, to the realm of school algebra. Some researchers, like Fey (1990), started to develop the idea.

- Bruner, in his book *Acts of meaning* (1990, p. 20) states that "culture and the quest for meaning within culture are the proper causes of human action".

Sfard (2003) claims that

> the culturally tinged, but essentially universal, need for meaning, and the need to understand ourselves and the world around us, came to be widely recognized as the basic driving force behind all our intellectual activities. (p. 356)

It is my belief that this driving force is not the monopoly of only a few. Thus, in my work with students and teachers in junior and high school mathematics (particularly, in algebra), and with students who are not mathematically inclined (and who have difficulties with mathematics), I felt the need to identify what such a quest for meaning might look like, whether and how it develops, and whether and how it can be integrated, fostered and supported by instruction.

Today there seems to be a wide agreement that meaning is constructed anew in idiosyncratic ways anytime someone learns or handles ideas. With the belief that watching and trying to understand how those processes take place can be insightful, I started to collect interesting behaviors of students and teachers working on algebraic problems. Algebra learning has always interested me and I was involved in research studies of algebra beginners (see, for example, Arcavi, 1995).

Symbol sense in brief

From the examples I have collected, a wide spectrum of interesting ways of sense-making (or lack of it) with symbols emerged. Distilling the core of what I saw led me to suggest a possible 'definition' of symbol sense. Such a definition could become a heuristic device for extending the idea of symbol sense, refining it and making it operational, either as a framework for research on learning, or as a tool for designing instruction or both. Thus, the definition, far from being closed or fixed, is rather a working tool for further reflection.

In the following list, I summarize the main components of symbol sense (for a more detailed account see Arcavi, 1994) [2]:

1. *Friendliness with symbols*: This includes understanding of and an aesthetic feel for the power of symbols – how and when symbols can and should be used in order to display relationships, generalizations and proofs that otherwise are hidden and invisible. Consider, for example, the following problems:

   "Complete the empty cells to obtain a 'magic square' with sum 9."

	3	
2		1

   and

   "Complete the empty cells to obtain a 'magic square' with sum 10."

	4	
2		2

   In the first case, the completion of the square such that the three numbers in each row, column and diagonal have the same sum (9) can be successfully achieved. In the second case, very soon one realises that it is impossible. The question then becomes: when do such number arrangements 'work' in order to yield a magic square, and why? We found that most students, with a substantial background in algebra, do not resort to symbols as a tool to enable them to investigate it in a general way. In spite of having the symbols available as a tool, they are not invoked unless students are prompted to do so. Thus, a first component of symbol sense is to have symbols readily available for understanding situations like the above, and to have the implicit confidence that these are the appropriate tools. Somehow, in opposition to this, symbol sense should also include the feel for when symbols may obscure, or be too costly in terms of the work required and other approaches or other representations should be preferred.

2. *An ability to manipulate and also to 'read through' symbolic expressions as two complimentary aspects in solving algebraic problems*: On the one hand, the detachment of meaning coupled with a global 'gestalt' view of symbolic expressions are needed for the manipulations to be relatively quick and efficient. On the other hand, the reading of and through the symbolic expressions towards meaning adds layers of connections and reasonableness to the results.

   When we observe students performing tasks involving symbols, we mostly witness automatic manipulations. However, consider the following solution process of a student when facing the equation:

   $$\frac{2x+3}{4x+6} = 2.$$

   Instead of 'jumping' at the solution, he paused and made a 'reading' of the symbols. He noticed that, since the numerator is always half the denominator, the fraction on the left hand side could never be equal to 2. However, he wanted to attempt to 'solve' the equation, using the syntactic rules he knew, in order to see how the symbols could tell him what he had found. Unfortunately, technical manipulation yields:

   $$x = 1^1/_2 .$$

   The student was puzzled for a while until he attempted to substitute that value for x into the original equation. Both the *a-priori* inspection of the symbols with the expectancy of gaining a feel for the problem and its meaning, and its *a-posteriori* checking to contrast meaning-making with symbolic manipulations are instances of symbol sense.

3. The awareness that one can *successfully engineer symbolic relationships* that express (given or desired) verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions (see, for example, the sophisticated construction of the symbolic expression for a desired graph described in Arcavi, 1994, p. 28).

4. The ability to *select one possible symbolic representation for a problem* (*e.g.*, assigning a symbol for a certain variable), and, if necessary, having, firstly, the courage to recognize and heed one's dissatisfaction with that choice, and, secondly, the resourcefulness to search for a better one. For example, in the process of solving a problem, to pause in order to consider whether it would be more convenient to represent three consecutive numbers as $n - 1$, n, $n + 1$ as opposed, for example, to n, $n + 1$, $n + 2$.

5. The realization of the need to *check for the symbol meanings during the implementation of a procedure*, the solution of a problem, or, during the inspection of a result, and the comparison and contrasting of those meanings with one's own intuitions about the expected outcome.

6. The realisation that symbols can *play different roles in different contexts* (such as, variables or parameters), and the development of an intuitive feel for those differences.

Some issues for discussion

The construct "symbol sense" opens up several issues for reflection, discussion and research. For example:

- *The characterization of symbol sense is not fully developed*: one possibility would be to enhance the collection of examples for the existing components, to search the space of examples such that we are comfortable in suggesting a comprehensive list of categories, to convert it into a working, operational framework for either research or instruction, or to challenge altogether the very idea from whatever point of view.

- *How do experts develop symbol sense?*: Is it a matter of nature or nurture? If the latter, what may be the roles of instruction, and can we regard symbol sense not only as a competence of experts but also expect it from novices, and to what extent?

- *What is the underlying knowledge required?*: What is the role of the technical manipulations of symbols? Do drill and practice precede, are concurrent with, or impede the development of symbol sense?

I will concentrate on the last two issues, not only to propose some incipient and partial answers but also to sharpen the questions.

How do experts develop symbol sense?

Let us consider first the dichotomy: nature or nurture? Is symbol sense something that only mathematically able people will develop by themselves, through, for instance, practice or insights, or can most (if not all) people develop it at least partially? Can symbol sense be taught?

In my view, being mathematics educators, in part, means designing, implementing and monitoring interventions in order to maximize students potential for learning. Thus we should emphasize 'nurture', rather than surrender to the fatalistic view that we are born with innate mathematical capablities and thus little (if anything at all) can be done in education.

Having said that, I think it can be insightful to look at mathematically competent people and consider how they might develop or display symbol sense, or more widely, the capability of making sense of their mathematical actions and decisions during learning, or while trying to understand or solve problems.

The following is a piece of data, taken from a pilot research project [3] designed to investigate different approaches students have to solve maximum and minimum problems in secondary school calculus. In particular, where, when and how, if at all, different students use informal symbol sense (or sense-making in general) when they solve these kinds of problems. For example, when modeling a situation, do they check the symbolic treatment against the situation itself?

Consider, for example, the following problem:

> P is a point on the graph of the function $f(x) = 1/x$ (in the first quadrant). A tangent line to the graph through P creates (with the axes) a right-angled triangle. What should be the coordinates of P in order for the hypotenuse of that triangle to be maximum/minimum?

In the Israeli curriculum, for those who study calculus (at the most advanced levels), this type of question is quite common. Usually, the traditional version of this question will say, without any explanation, what kind of extreme value one should be looking for and, therefore, does not let the students decide it by themselves. In our view, this deprives students of an opportunity to make sense of the situation. We decided to make a slight change in the problem, wording it as above such that there is an indirect prompt for the students to decide for themselves which extreme value they need to look for and why, providing an opportunity for making sense of the situation.

IA, a mathematically able student, a graduate of the most advanced track in a very prestigious school was one of the interviewees. The interview took place about a year after he finished high school, and he had taken no regular mathematics courses since his time at school.

The very first thing he said was "let me see what goes on here". He proceeded to sketch the graph of $f(x) = 1/x$ in the first quadrant, to mark an arbitrary point P on the graph, to draw the tangent line at that point, and to highlight its intersections (with the axes). On the basis of the sketch, IA proceeded, almost flawlessly, to find the function which describes the length of the hypotenuse as a function of the coordinates of $P(x_0, 1/x_0)$.

Firstly, he wrote down the derivative of $f(x)$ $[f'(x) = -1/x^2]$.

He then wrote the equation of the tangent line at P, $y - y_0 = -1/x_0^2(x - x_0)$.

At this point, he said that he needed to express the length of the segment of that line bound by the intersection of the line with the coordinate axes. Therefore, he proceeded to find the coordinates of these intersections – $(2x_0, 0)$ and $(0, 2/x_0)$ – and then to write the length (as the distance) using the Pythagorean formula. Thus, he obtained the (rather complicated) function whose extreme values are sought.

When he started to say that he needed to find the derivative of that function (which involves a square root), he asked himself whether he was looking for a minimum or a maximum, and then turned for the second time to the graph. Thus, the first time IA used the graph as an organizational device, which led him to design a working plan and to carry it out. Only when he needed to make a decision about which extreme value he was looking for did he turn to the graph for the second time, with a different purpose. This time the graph was an operational tool for sense making.

He 'played' with the graph of $f(x) = 1/x$, realising how the length of the tangent segment may change for different points. After a while, he concluded that the extreme value will be minimum at (1,1) because of some kind of 'symmetry':

> I could have done it with common sense, with no calculation whatsoever. Because in that direction [x tending to infinity] it [the length of the segment] will grow all the time, and also here [points to y tending to infinity], and only here [at (1,1)] it will be minimum.

I was the interviewer, and I was curious to see why he turned to the graph for the second time. Being interested in symbol sense, I wanted to know whether he 'saw' something in the symbolic expression of the function, which he wanted to visualize in the graph. He told me that he did not see much in the symbols, but when he wanted to find the derivative and equate the derivative to zero he suddenly realised that he did not know whether he was looking for a maximum or a minimum, and perhaps equating the derivative to zero would yield an equation with more than one solution. He wanted to see which of those would qualify as minimum or a maximum (the way he read the question was that there may be both). When I asked IA why he did not engage in this kind of analysis from the beginning, his response was surprising:

> I have a friend who always does that [playing with the problem and making sense of it], after such an effort, he usually has neither time nor energy to do the symbols, he does not get credit for what he may have done and fails the exams. If I don't have to, I do only the symbols, which is what the teacher and the exam want.

It is interesting that if the problem was given without that slight modification (namely stating right away that one is looking for a minimum), he would have finished the problem after (a lot of) formal work, with a value for the segment and the coordinates of P, possibly without looking for a second time at the graph in order to make sense of what is going on in the situation. Such sense-making activity would not have been rewarded in the classroom culture he came from. As mentioned above, this interview took place about a year after IA had finished high school, while in the meantime, little or no high school mathematics was practised or learned. He was not in an examination situation, and yet IA solved the problem driven by the habits he had developed in his class. These habits seemed to be very deep rooted: he had a secure way – a well designed plan, including symbolic procedures, to follow – and that's what you are supposed to do. The ability to make sense seemed to be there, however, it was not invoked until it was absolutely necessary. Something in the phrasing of the problem forced IA to call upon his 'sense making', which otherwise would have been dormant because he was used to the fact that it may not pay off to bring it in.

This example is far from providing an answer to our question about how experts develop symbol sense. However, it provides some insights about what may be involved to support and encourage its development. In my view, this example illustrates that developing the habit of sense making may be strongly related to the classroom culture that supports or suppresses it and is not merely an issue of 'innate mathematical ability'.

A possible conclusion could be that our attention should be redirected to the kinds of practices we support and reward. Thus, developing symbol sense, or sense making in general, is certainly more than a purely cognitive issue. It is connected to what one is expected to produce, to what is valued, to what is accepted as fair game, besides symbol manipulation.

This is hardly ground-breaking news. However, it is worth commenting on as it is consonant with many other research findings, which place classroom culture as a central player in what is learned and what develops. Moreover, it may have some implications for supporting the development of symbol sense, by bringing it to the foreground time and again. For example, asking students to develop the habit not to jump to symbols right away, but to make sense of the problem, to draw a graph or a picture, to encourage them to describe what they see and to reason about it. If students have some initial difficulties in producing informal arguments, we have to let them witness how these arguments may look, how they are produced, and what one may gain from them. If these activities are not experienced by students, or given some seal of approval, then, at best, spontaneous sense making may be relegated to a lower priority, or at worst, it will not happen at all.

As already said, we are still far away from a satisfactory answer to the question, "How does symbol sense develop?". However, we may be one small step further forward when we propose that

1. symbol sense can be nurtured, and

2. one necessary condition for symbol sense to develop is to provide supportive instructional practices in the sense described above.

What is the underlying knowledge required for symbol sense?

The question entails the following: What may be the role of the technical manipulations of symbols? Do drill and practice precede, are concurrent with, or impede the development of symbol sense? These questions would seem to lead to a more cognitivist (purely cognitive) set of answers to the question of what underlying knowledge is required for symbol sense.

One general conclusion, perhaps a trivial one, which I propose to draw from the proposed characterization of symbol sense is that being competent in school algebra would imply, among other things, the opportunistic, flexible back and forth transition from the use of meaningless actions (such as the automatic application of rules and procedures) to sense making (in one of the ways mentioned before, or any other). In other words, competence would include,

1. the timely postponement of meaning in favor of quick and effective applications of procedures, but also, when necessary, desirable or when people 'feel' it,

2. the interruption of an automatic routine in order to question, reflect, conclude, relate ideas or create new meaning – or, in Freudenthal's words, to "unclog an automatism" (Freudenthal, 1983, p. 469).

Meaningful and flexible transitions from meaninglessness to meaning and *vice versa* are at the heart of being competent in school algebra. This is far from being an answer to the questions above, but, in my opinion, it gives us a lead, certainly not a simple one. How does one come to develop and apply such flexibility? Sfard (2000), in her quest to propose explanations to why mathematics seems so difficult to so many, poses the following circularity: if meaning is a function of use, then one has to manipulate a concept in order to understand it (or in our case to manipulate symbols in order to get a feel for them, and what they can do for you), but on the other hand, how can one use something without understanding it (or having a feel for it)? Sfard claims that it is precisely this circularity which can be a serious trap for learners, but it is at the same time what fuels the process of learning. Sfard says,

> In this process, the discursive forms and the meaning, as practised and experienced by interlocutors, are like two legs that make moving forward possible due to the fact that they are never in exactly the same place, and at any given time one of them is ahead of the other. (p. 56)

If we agree with this nice metaphor, then it is quite possible that the development of symbol sense, instead of being mostly linked to cognitive capabilities, could be more an issue of attitude towards knowledge and learning. What it entails, above all, is that one has to develop enough patience towards living in harmony with partial understandings and with the knowledge that sometimes meaning may emerge from meaninglessness (after doing some drill, but within a culture that supports reflection) and at other moments, meaninglessness should be an effective by product of meaning (*e.g.*, developing automatisms to free memory load).

This idea is consonant with findings by other researchers, like Tobias (1990) who investigated why successful scholars in the humanities would not become science scholars. Among other things, she found that part of the answer has to do with the capability, or lack of it, to live with partial understandings for long periods of time, until meanings are connected and a large picture emerges. This seems to be an essential ingredient of successful learning of science.

Thus, it would seem that competence may include intellectual patience towards partial understandings, confidence that further actions (not totally clear at the beginning) will advance you. This implies having a very different image of learning from the one that regards (and popularizes) learning as an effortless enterprise. Regarding understanding as 'either you have it or you don't' and learning as a 'quick road to achieve that' may hinder progress. Such a belief would not allow enough time, space, and experience for symbol sense, or any other sense making habits, to develop.

What can we do?

Certainly, the awareness of the two issues discussed so far is in itself some progress. How can these be translated into instructional practices that support and encourage the development of symbol sense? What can we do with beginners and non-mathematically oriented students? And how can we support the patience needed?

In a pilot study we have done recently in the lowest track of an 8th grade (13-14 years old) in a school which streams students, we designed a lesson to introduce the idea of, and notations for, simple inequalities. The lesson was designed in the spirit of the Dutch *Realistic Mathematics Education* (RME). The lesson starts with a photograph showing a vehicle at the entrance to a tunnel on top of which there is a sign: "2.90". Students were requested to interpret what the number might mean. They proposed that the number refers to the weight of a passing vehicle, its width, its height and also discussed relevant units. When students agreed that it meant 'maximum height for a vehicle to pass', some proposed that they estimate heights of different kinds of trucks they knew to see which one would be able to enter.

The teacher asked for examples of heights that would be allowed and those that would not, in order to implicitly stress the idea that it is a matter of a range of numbers. Again the discussion was lively, and some of it centered on what happens with heights that are close to 2.90. At a certain moment a student said: "all heights below 2.90". The teacher seized the opportunity and asked, "Can you write it down in a mathematical way?"

Since the students were somewhat familiar with algebraic notations, one student suggested $x < 2.90$. The teacher asked "What does x stand for?", and students said "The height". The teacher asked whether there is another mathematical way to express this. The teacher had in mind the number line, with which she knew the students were familiar. However, a student suggested $y < 2.90$. For experts, "x " and "y" are the same type of representation, it is irrelevant that the letter is different. For these students, these were two different ways to express the same idea, and there is no reason why the students who are not into mathematical habits, culture and ways of representing would think otherwise.

What a proper teacher intervention would be in a situation like this may be debatable, possibly nothing more than an agreement and the request for yet other ways there could be. A teacher 'mini-lecture' about why using x or using y are in essence the same type of representation may not be effective in this context. Yet, this type of 'lecture' may often be heard. And thus, without being fully aware, even very good and responsible teachers may involuntarily contribute to student impatience. In conversations with many teachers, there is a sense of a pressing need for closure within a lesson and some consider that leaving partially unresolved issues 'hanging in the air' may even be immoral. They feel they have to clear up what they think are uncertainties. This shortness of breath, or better, this need for immediate closure (even when based on the best of intentions), may be fuelling students' impatience, and unconsciously strengthening their conviction that, without the timely rescue prompts from the teacher, they will never be able to sort things out by themselves.

Let us return to the lesson. Since students did not come up with another representation, but the teacher knew they were familiar with the number line, she suggested it. When she did so, students were able to mark the origin and 2.90 on the line and highlight all the numbers in between. The teacher asked whether they thought that $x < 2.90$ and the number line "say the same thing"? This kind of question is, in our view, an appropriate instructional intervention: on the one hand, considering mathematics as a way of expressing an idea and thus focussing and discussing these ways explicitly, and on the other hand, providing an opportunity for students to express freely how they see and sense symbols versus other representations.

This kind of meta-mathematical talk and reflection can be done very early (even before you do procedures), and if wisely repeated could be a support for developing aspects of symbol sense. One of the issues the teacher had in mind when she asked this question is that the number line stressed the place of zero whereas $x < 2.90$ omitted that x (representing a height) should stand for a positive number.

However, it was interesting to hear the following answer, which was totally unexpected. A girl said, very convinced: "$x < 2.90$ shows something, one something less than 2.90, but the line shows all the numbers at once" [4]. This was an explicit verbalization about the way this student sensed the symbols. Such a comment emerged because students were allowed to voice what they see and sense. This girl had an opportunity to voice her sensing of symbols, including 'explanation' of her preference for the number line. Possibly, this incident (if followed by others) would provide her with better chances to develop aspects of symbol sense, and talk about them. Since the activity allowed her another representation to display generality, she (and certainly we as teachers) can coexist with her partial view of what "x" (or for all x) may imply for her. The student knew that the issue here is that there is more than one number that you can plug in to the x (because of the previous talk, and because this type of substitution was practised before this lesson), but nevertheless she said she could not "see" more than one number at a time. The number line representation was better for her because it visually displayed all the numbers at once.

The morals from this example may be over-stretched, but provide an illustration that there is hope for instruction when it

1. recognizes issues related to symbol sense
2. respects and encourages partially developed ideas
3. allows and supports the talk about them, and
4. is not necessarily based on rushing towards immediate closure.

A sense for symbol purpose

Project 2061 recently evaluated many "reform oriented" (as they are called in the USA) algebra textbooks from different curriculum projects. They established a set of criteria for that evaluation, the first of which was "identifying a sense of purpose". [5]

Most books did poorly on this. One way of interpreting this criterion is to evaluate how instructional materials engage students with activities in which they can identify a sense of purpose for the tools they are learning.

In particular, what is the purpose of symbols? One way to share with students the purpose of symbols is that, in their activities, symbols make them feel that they gain understanding (and thus power) over a situation.

The following example may be illustrative (see also, Arcavi, 2002). A high school student came back home from school with the story that her mathematics teacher was upset by the grades of the class in an examination on functions and that she had decided that the questions had been too hard. The teacher had therefore adjusted the grades: if x was the original grade it now would become $10\sqrt{x}$. Apparently, this correction is common among Israeli teachers. A host of interesting questions to explore (using as a tool the concept function and its representations) arise from this situation. For example, are there students who get the same grade before and after the correction? Does everybody increase their grades? Why? Who gets the greatest increase? Is this correction factor fair? To whom? How does this correction compare to others such as increasing all grades by 10 points or by 20%? Can you construct a fair correction factor of your choice? and explain the reasons for your choice.

However, the class missed a golden opportunity to use their knowledge of functions in order to sense the power of the mathematical tools they were just learning (symbols, graphs) as a way to gain insights into a situation that is interesting and affects their lives, and which is very difficult to explore without these tools.

There are many instructional activities in which such a sense of purpose can be nurtured, and it is an interesting challenge for curriculum developers to embark on designing and implementing them even from the first encounters with school algebra (see, for example, the research reported in Arcavi, 1995).

A brief (and certainly partial) conclusion

This article is only a modest beginning to attempt an answer to how one may support the development of symbol sense, and to suggest the issues that may be at stake. One of them, for example, refers to learning materials and classroom practices that:

- nurture the search for symbol meaning alongside, and after solving routine or non-routine problems, and also before one automatically starts using symbols

- support the building of the patience needed for learning in general, and more precisely the capability of accepting partial understandings

- nourish a sense of purpose and empowerment gained by using symbols.

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Notes

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[2] If you do not have access to this issue of FLM, you will find a copy of the article on the FLM website: http://flm.educ.ualberta.ca.

[3] Ita Naftalis, a master's degree student supervised by Tommy Dreyfus and myself, can be contacted directly at Mnaftalis@hotmail.com.

[4] I suggest (although I have no data to support it) that a more complete mathematical expression (such as 'for all $x > 0$, $x < 2.90$') would have not made much of a difference at this point.

[5] See, for example, http://www.project2061.org/newsinfo/research/textbook/hsalg/criteria.htm (accessed, April 26th, 2005).

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