ENHANCING AN INTELLECTUAL NEED FOR DEFINING AND PROVING: A CASE OF IMPOSSIBLE OBJECTS

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Consider the two pictures shown below. The first picture (Figure 1) represents the tribar, created by Roger Penrose in 1954, under the influence of Escher's drawings. The second picture (Figure 2) shows a sculpture of the Penrose tribar standing in East Perth, Western Australia. The sculpture was created by Bryan McKay in 1999. Our visual system normally perceives the Penrose tribar as an impossible object [1] and McKay's sculpture as a real object. The existence of the sculpture is perturbing for many people, as is evident from the comments written by visitors to the *Mighty Optical Illusions* website, where a photograph of the sculpture appears. [2] For instance:

"It looks impossible, but it's possible. Cool!"

"I still don't get how the top corner may look as if it was attached…Great illusion!"

"The 'impossible' figure can really be made, no illusions, just real, they are possible to make!"

Such a diversity of perceptions of the photo provides a good opportunity to ask "How can we examine whether the Penrose tribar can exist in 3-D space?" This article is stimulated by my experience of exploring this question with a group of pre-service mathematics teachers. The exploration was a success, in the following sense: the participants felt the need to construct a definition of the Penrose tribar in order to prove its impossibility; the formal proof, in which one of the axioms of solid geometry became useful, was constructed.

A considerable number of recent publications have analyzed why various classroom activities aimed at enhancing in students and teachers the need for defining and proving do not always work as expected (*e.g.,* Brousseau & Gibel, 2005; Ouvrier-Buffet, 2006; Tanguay & Grenier, 2010). I therefore find it interesting and important to present a successful case and reflect on the possible causes of its success. The goal of this article is twofold. Firstly, it is to demonstrate, through the presentation of a two-lesson activity with a group of pre-service mathematics teachers, that some formal definitions and axioms of geometry can be intellectually necessitated for students by means of the exploration of impossible objects. [3] Secondly, it is to characterize the pedagogical choices that made the exploration feasible for the participants and, in Brousseau's (1997) terms, preserved their autonomous learning. The rationale of pursuing these goals in the chosen context relies, in particular, on work by Tanguay and Grenier (2010), who discussed mathematics teachers' proving failures when dealing with diagrams and material representations, which are referred to in their paper as *ostensive objects.* They called for research to explore "under what mathematical and didactical conditions students will recognize their work on ostensive objects as possibly leading to structural explanations or proofs" (p. 41).

Intellectual need and epistemological justifications in geometry

Kleiner (1991) has pointed out that formal definitions and proofs were developed by the Greeks "for good mathematical reasons" (p. 291), including "the desire to decide among contradictory results bequeathed to the Greeks by earlier civilizations" (p. 293). In a more recent paper, Kleiner (2006) has exemplified how new mathematical knowledge

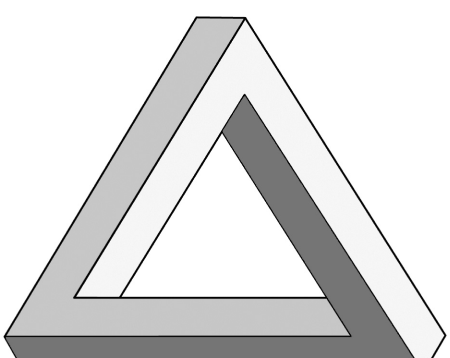

Figure 1. Penrose tribar.

Figure 2. Sculpture of the Penrose tribar (Photograph by Greg O'Beirne, via Wikimedia Commons).

has, throughout the history of mathematics, emerged from mathematical paradoxes, the mistakes of mathematicians and problems that could not be solved by existing knowledge.

Harel (forthcoming) has demonstrated the essential absence of such "good mathematical reasons" in regular mathematical instruction and advocated promoting an *intellectual need* for constructing new mathematical knowledge. He defines an individual's intellectual need as a (subjectively) problematic situation, S, out of which a piece of knowledge, K, can arise. Harel then introduces the notion of *epistemological justification*, which is central to the concerns of this paper. He writes:

> Intellectual need is best observed when we see that (a) one's engagement in a problematic situation S has led him to construct the intended piece of knowledge K and (b) one sees how K resolves S. The latter relation between S and K is crucial, in that it constitutes the genesis of mathematical knowledge, the perceived reason for its birth in the eyes of the learner. We call it epistemological justification.

For instance, the process described by Lakatos (1976) of gradually refining a definition of a polyhedron in response to the need to neutralize a series of counterexamples is an example of constructing an epistemological justification of that definition. Another example can be based on the well-known question "Why do differently looking triangles have the same sum of angles?" This question may evoke in learners an intellectual need for formulating and proving the theorem about the sum of the angles of a triangle (see Balacheff, 1999). If so, and if the learners can see how the theorem resolves the question, then this question would serve as an epistemological justification of the theorem.

The professional literature suggests miscellaneous ideas about how to construct epistemological justifications of (relatively) advanced geometry knowledge, such as the definition of a polyhedron or a theorem about the sum of the angles of a triangle. The literature is rather silent about how to epistemologically justify for learners such basic knowledge as geometry axioms. Historical justifications of the axiomatic foundations of geometry are, as a rule, too sophisticated to be introduced in a classroom (Harel, forthcoming; Kleiner, 1991, 2006; Tall *et al.,* 2012). This article considers one possible way of dealing with the challenge of creating problematic situations involving ostensive geometry objects whose existence is put into question.

Autonomous learning and teachers' interventions

The idea of helping students to construct epistemological justifications of the knowledge we intend them to learn is in line with Brousseau's (1997) idea of teaching by means of problematic situations that permit students to construct mathematical knowledge autonomously, where "the knowledge will appear as the optimal and discoverable solution to the problems posed" (p. 22). Ideally, argue Brousseau and Gibel (2005), "the teacher must propose problems (whose solutions require knowledge that has not been institutionalized in class yet) to be solved completely autonomously by the students" (p. 23). In practice, classroom problem-solving situations that allow students to construct new knowledge without the teacher's mediating actions barely exist (Brousseau, 1997). The teacher's interventions are often inevitable, for instance, when students face a dead end when coping with a given problem or are uncertain about their ability to solve it.

Brousseau and Gibel (2005) propose three types of teacher interventions in a situation where a problem is open for students (but not for the teacher, who knows its solution): (i) to bring into play the relevant knowledge students had been taught; (ii) to point out information given in the text of the problem; (iii) to refer to conditions which are not included in students' presumed knowledge and which cannot be logically deduced from the text of the problem. Brousseau and Gibel point out that autonomous learning through problem–solving is highly sensitive to the type of teacher intervention. In particular, they show that Type 3 interventions make autonomous learning impossible because "the student can accept the solution only upon his trust in the teacher's authority" (p. 20). However, they do not elaborate on how to resolve the tension between the need to intervene in order to scaffold students' reasoning in time of difficulty and the need not to intervene in order to preserve the autonomy of their learning. In this article, as a possible means for resolving this tension, I introduce an additional type of teacher intervention: the auxiliary problem.

The activity design

The activity presented below occurred during two consecutive 90-minute lessons of the course "Selected mathematical problems" which I taught in 2010. Eight pre-service mathematics teachers participated. They had studied some elements of solid geometry when at high school, *i.e.,* 3-5 years prior to their participation in the course. According to the participants, they had never studied the axiomatic foundations of solid geometry and had no experience of proving in solid geometry contexts.

The activity was designed with the following stages. At the first stage, it was planned to introduce the two pictures shown above (Figures 1 and 2) to the students and ask them to decide whether the sculpture could represent a genuine Penrose tribar. It was expected that the students' opinions would vary so the question "How can we examine if the Penrose tribar is an impossible object?" would naturally enter the discussion.

At the second stage, the students were expected to construct a definition of the Penrose tribar in order to transform the above question into a mathematical problem. One possible definition is as follows:

1. A Penrose tribar is shaped by three sets of parallel segments, namely, $a \parallel b \parallel c \parallel d$, $e \parallel f \parallel g \parallel h$ and $k \parallel l \parallel m \parallel n$ (see Figure 3a, overleaf).

2. The parts of the tribar of different hues (α, β and γ) are plane figures, which belong to three different planes intersecting by pairs.

It was planned that the third stage would concern the question "Does an object that satisfies the constructed definition exist?" A negative answer to this question can be justified by the following proof.

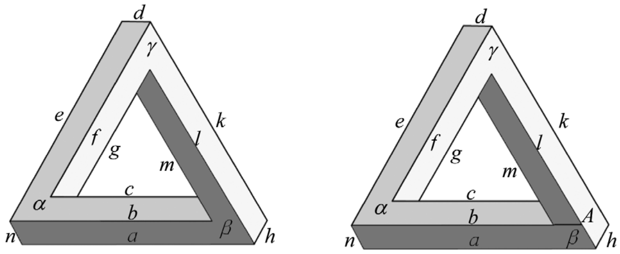

Figure 3. Drawings referred to in (a) the definition of the Penrose tribar and (b) the proof of its impossibility.

> Let A be an intersection of the straight lines b and l: $A=b \cap l$ (see Figure 3b). Point A exists because straight lines a, b and l belong to the same plane β (condition 2 of the above definition), $a \parallel b$ (condition 1) and lines a and l intersect. In addition, $b \subset \alpha$, $l \subset \gamma$ and then $A \in \alpha$ and $A \in \gamma$. Consider now planes α and γ. They intersect by line f and point A, which does not belong to f. Thus, the following solid geometry axiom is violated: "Two planes intersect by a straight line". We obtained a contradiction because we assumed that α, β and γ are *different* planes (condition 2). Consequently, the Penrose triangle cannot be a 3-D object, QED.

In summary, it was expected that the question about the existence of the Penrose tribar as a 3-D object would serve as an epistemological justification of a formal definition of the tribar and of one of the solid geometry axioms. As will be evident shortly, the reality was more complicated than the initial plan.

First stage: a problematic situation emerges

The students' immediate responses to the photo of the sculpture of the Penrose tribar were remarkably similar to the comments made by the visitors to the *Mighty Optical Illusions* website. In brief, they combined surprise and doubt. Some students even became suspicious about the genuineness of the photo. The following utterance by Iris (all the names are pseudonyms) is representative of this stage: [4]

> Iris: It looks like that he [McKay, the sculptor] succeeded to build the Penrose tribar. However, we should think more about it, it is impossible to decide now.

I asked the students individually to express their opinions using a 1-10 scale, where "1" meant "I am certain that the Penrose tribar can be made," and "10" meant the opposite. One student chose "1", two students chose "10", one student chose "9", and four students, including Iris, chose "5," "6" or "7". The students who chose "1" or "10" were given an opportunity to persuade the rest of the class in their rightness. Ahmed, who chose "10", tried as follows:

> Ahmed: The sculpture misses what we clearly see in the picture, namely, interior and exterior parts. If the sculpture would be painted in three colors, like the picture, we would see that it is impossible.

Ronnie, who chose "1", was uncertain not only about the photo, but also about the definition of the impossible object (see [1]), which had been presented to the students along with the two pictures. She said:

> Ronnie: I think that it is possible to make, but not in the way we think about it...Just look at the definition [of an impossible object], it tells us something strange…

The class found these explanations too vague and nobody changed his or her opinion. Moreover, three students, who had been confident in their decisions at the beginning of the discussion, now looked unsatisfied by their own explanations. Thus, the first stage of the activity developed exactly as planned: consideration of the two pictures converged to a problematic situation. As a result of this situation, some students felt the need to remove their doubts about the existence of the presented ostensive object (*i.e.,* the photo of the tribar). For other students, the need was to find ways to persuade others of their rightness. Furthermore, the situation seemed to involve a *potential cognitive conflict* (Zazkis & Chernoff, 2007) for some of the students. At this point, Ahmed made a suggestion:

> Ahmed: We should define the problem.

Second stage: towards a definition

Echoing Ahmed's suggestion, I asked the students to formulate and agree upon a list of properties of the Penrose tribar that could serve as a definition of the object. This request surprised some of the students, who expected that the definition should come from some authority: a textbook or the instructor. Ralph asked:

> Ralph: How can we define it? Should we vote? How can something be defined in mathematics just by an agreement among us?

I returned Ralph's question to the students in the following form:

> Boris: How do you think a definition of, say, a square, was formulated?

This question triggered a brief discussion, which led the class to the conclusion that all definitions in mathematics are, in the words of Ralph, "a sort of agreement among people." In Brousseau and Gibel's (2005) terms, I used a Type 1 intervention. As a result, the defining task was legitimized for the students.

The defining task revealed that the students perceived the drawing of the Penrose tribar in different ways. For instance, Ronnie opposed Ahmed's suggestion to incorporate into the definition a condition $a \parallel b \parallel c \parallel d$. She argued that though line d *looked* parallel to a, b and c, it was probably *not* parallel to them (see Figure 3a). There was also no agreement among the students about whether the segments that look equal on the picture (*e.g.,* d, n and h) were "really" equal. Another disagreement concerned the angles on the picture, namely, whether $\angle(e,b) = 90^o$ or $\angle(e,b) = 60^o$. These disagreements suggest that the students did not only use empirical considerations when attempting to define the Penrose tribar. In this respect, the presented situation differed from the solid geometry situation described by Tanguay and Grenier (2010), in which empirical and theoretical considerations of the students appeared at different stages.

Agreement among all eight students was eventually achieved for the following statements only:

1. The Penrose tribar looks like a 3-D object;
2. The closed parts of the picture of different hues are plane figures; and
3. $a \parallel b \parallel c$; $e \parallel f \parallel g$; $k \parallel l \parallel m$ (see Figure 3a).

I felt that the discussion lacked shared criteria for deciding whose vision of the Penrose triangle should be accepted by all. A similar phenomenon was described in Zaslavsky and Shir (2005), who studied how small groups of students decide whether particular statements can be accepted as definitions of mathematical objects. Their situations were devoid of any actual use of the definitions so agreements were usually reached based on communicative and social reasons. In our case, a definition of the Penrose tribar should have been constructed for further mathematical use. Thus, I decided to intervene as follows:

> Boris: It looks like we cannot fully agree upon the definition. But is it necessary at this stage? Let us not forget that we wish to agree on the definition in order to examine whether an object satisfying it exists. Which definition is really needed for this purpose? We don't know yet. This may become clearer when you start proving. So, please start proving based on the properties agreed so far, and modify the definition if you feel you need to.

In Brousseau and Gibel's (2005) terms, this was a Type 2 intervention, because it pointed out information already known to the students from the course of the activity.

Third stage: an axiom for the intersection of two planes enters the stage

The students returned to their work in small groups, but showed no progress. In retrospect, the difficulty of this stage for the students could be attributed to three kinds of uncertainty that they experienced at once. The first uncertainty was about the existence of the Penrose tribar; the second one was about the choice of properties of the Penrose tribar that could be taken for granted; and the third one was about how to start proving. Note that the intended proof included elements that had not yet appeared in the discussion: (i) the decision to invest effort in proving that the Penrose tribar is indeed an impossible object; (ii) the idea to look for a proof by contradiction; and (iii) the idea to construct a contradiction using a specific piece of knowledge, namely, an axiom stating that two planes may intersect only by a straight line.

The students came quite close to the first two elements when trying to define the Penrose tribar. For this reason, I felt that advising the students to consider proving the impossibility of the Penrose tribar by contradiction would not deprive them of the pleasure of independently resolving the problematic situation (*i.e.,* a Type 2 intervention was used). As for the third element, a dilemma that could be formulated as a paraphrase of the *learning paradox* emerged for me (*e.g.,* Brousseau, 1997; Sfard, 1998): how could the students autonomously understand that an axiom about the intersection of two planes could help them if they did not know about its existence?

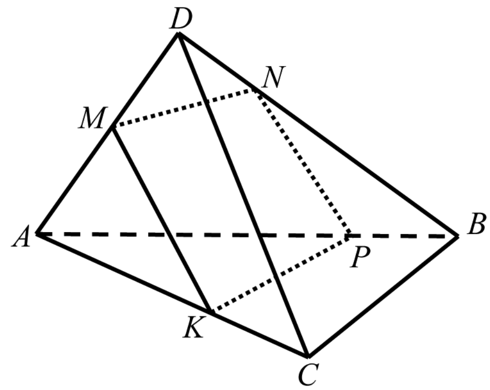

Figure 4. Tetrahedron ABCD and its "impossible" plane section MNPK.

For the reasons discussed above, I did not want to use a Type 3 intervention, that is, to straightforwardly introduce the axiom. I suggested instead:

> Boris: Let us leave for a while the Penrose tribar and consider another object. The drawing on the screen (Figure 4) consists of a tetrahedron *ABCD* and its plane section *MNPK*. In your opinion, does this drawing represent a possible or an impossible object?

All the students quickly observed that "something is wrong with the picture." Ronnie was the most specific and said that point *N* was problematic. She explained:

> Ronnie: *PNM* belongs to a face of a tetrahedron and to the cutting plane. But if two planes intersect, it should be a straight line!

I reformulated her explanation as follows: the object is impossible because it violates the fact that two planes intersect by a straight line. I then informed the students that this fact has the status of a solid geometry axiom. Thus, the axiom of the intersection of two planes entered the stage as a tool for solving an auxiliary problem, or, in Harel's terms, an auxiliary problem served as an epistemological justification of the axiom. The point is that the above intervention, shaped as an auxiliary problem, allowed me to avoid a Type 3 intervention.

The lesson time was over and I asked the students to keep thinking about possible proofs for the impossibility of the Penrose tribar until the following week's lesson.

Fourth stage: a proof-generated definition is constructed

Two students, Iris and Ronnie, announced their success at the beginning of the next lesson. Their proofs were first discussed and polished in small groups, and then presented to the whole class. The proofs appeared to be similar in two ways.

First, they both relied on definitions of the Penrose tribar that were fully compatible with the intended definition

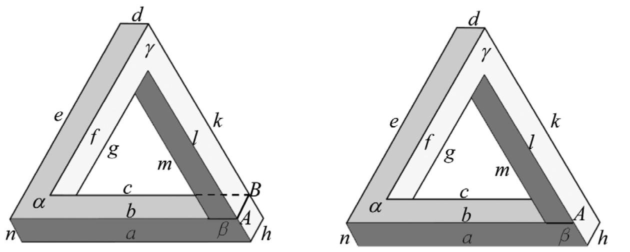

Figure 5. Drawings related to (a) Iris's proof and (b) Ronnie's proof.

(described above). This meant, in particular, that both students decided to include the property "all lines that look parallel on the Penrose tribar's drawing are parallel" in their definitions. This property had not been agreed a week earlier, but Iris and Ronnie managed to convince the rest of the students by simply pointing out that without this property their proofs would not work. Thus, Iris and Ronnie succeeded in constructing a *proof-generated definition* (Lakatos, 1976), which, by itself, is a prominent accomplishment (see Ouvrier-Buffet, 2006, for elaboration on the difficulties in creating situations with the potential to lead students to construct proof-generated definitions). The point here is that the constructed definition became, for the students, knowledge whose emergence was stipulated not only for communicative or social reasons (Zaslavsky & Shir, 2005), but also for a reason of an epistemological nature.

Second, both proofs included a final step of constructing a contradiction with an axiom of the intersection of two planes. Thus, the axiom also became, for the students, epistemologically justified knowledge.

The difference between the proofs resided in different auxiliary constructions (see Figure 5). Specifically, Iris constructed line *AB,* where $A=b\cap l$ and $B=c\cap k$ (Figure 5a). In this case, the contradiction with the axiom was that planes α and γ intersect by f and *AB*. Ronnie constructed point *A* (Figure 5b) and, importantly, justified its existence. In her proof, the contradiction with the axiom was that planes α and γ intersect by f and *A*. In fact, Ronnie's proof was fully compatible with the proof presented earlier in this article.

Ronnie's proof was the most parsimonious of the two, which was noticed by several students. In addition, Ronnie observed that the proof by Iris included an implicit assumption about the existence of point *B*, which belonged to the "invisible" part of the drawing (Figure 5a) and wondered whether this assumption must be incorporated into the definition. Ronnie's question was difficult to understand for some students and I decided to elaborate on it through an additional auxiliary problem. I drew on the blackboard a sketch (Figure 6) and asked the class: "What may point *M* represent?"

Those who perceived the drawing as a plane figure suggested that point *M* may represent the intersection of *NP* and *KL*, and those who perceived the drawing as a drawing of a 3-D object, said that point *M* "does not exist" or may belong to *NP* or *KL*. The students then concluded that it was impossible to answer the question without additional information about the picture. More importantly, the students understood

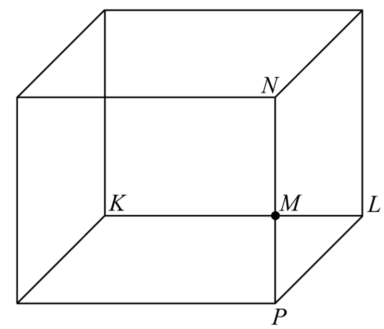

Figure 6. A drawing for the question "What does point M represents?"

the connection between the auxiliary problem and the question asked by Ronnie. As a result, the class decided that Iris's proof did indeed require incorporating a condition into the definition of the Penrose tribar that would ensure that point *B* exists (Figure 5a). The point is that the intervention by means of an auxiliary problem allowed the students to arrive autonomously at the intended conclusion.

At the end of the lesson, we came back to the question about whether the Penrose tribar can be made as a 3-D object. All the students but Ronnie confidently stated that the Penrose tribar cannot be made "because we've just proved it." Ronnie said:

> Ronnie: It is still difficult for me to decide. Perhaps, it can be made, but not by our definition…

Though it is beyond the scope of this article, it is worth mentioning here that Ronnie's intuition can be mathematically backed up (*e.g.,* Sugihara, 2011).

Discussion

Identification of mathematical and didactical conditions under which learners can successfully progress from experimenting with ostensive objects to defining and proving is recognized to be a challenging research enterprise (*e.g.,* Ouvrier-Buffet, 2006; Tanguay & Grenier, 2010). The classroom experiment presented in this paper was successful because the intended knowledge—a definition of the Penrose tribar and a proof of its impossibility involving one of the axioms of solid geometry—were eventually constructed by the students and epistemologically justified.

The success of the experiment can be attributed to a particular configuration of conditions related to the initial problematic situation and the way deviations from the "ideal" scenario were handled. Some of these conditions have been pointed out in previous research, but, to my knowledge, the entire configuration has not. I will summarize this configuration as follows:

- The initial situation involved an ostensive object, whose existence was put into question. Consequently, the strength of empirical considerations based on mere observation of the object was essentially suppressed. This is in line with Tanguay and Grenier's (2010) suggestion that "in students' minds, the material existence of the object might warrant its theoretical existence" (p. 37). Thus, uncertainty regarding the material existence of the ostensive object opened a window for introducing theoretical considerations.

- In line with the recommendations of Brousseau and Gibel (2005), the situation of uncertainty was set up so that the students started proving not just because somebody asked them to do so, or to demonstrate mathematical validity in itself, but in order to remove their own real doubts. In particular, the students, who strongly expressed opposing opinions, were given a chance to persuade their peers, used non-mathematical arguments and failed. As a result, the idea of using mathematical tools for resolving the problematic situation entered the stage as a remedy.

- It is important that the participants already possessed a deductive proof scheme (Harel & Sowder, 1998), so they were ready to accept an argument in the form of a formal proof as persuasive.

- Defining and proving interlaced and epistemologically supported each other during the activity. Unlike some previous studies (*e.g.,* Zaslavsky & Shir, 2005; Ouvrier-Buffet, 2006), the situation encouraged the students to look for a proof-generated definition.

- When the students could not make progress autonomously, I attempted to help them without using Type 3 interventions. That is, I did not simply introduce relevant knowledge that the students had not yet encountered (Brousseau & Gibel, 2005). I instead utilized interventions in the form of auxiliary problems, with the intention that the students would discover the connection to the main question. Each auxiliary problem was quite easy (though not trivial) for the students and helped them to autonomously construct an intended piece of knowledge.

- Such a creative undertaking as constructing a new mathematical proof requires an incubation period, even when all relevant knowledge is available (Wallas, 1926, cited in Sriraman, 2004). Consequently, it was lucky that in the described case, all relevant knowledge had been put forward by the end of the first lesson, so that the students had a week until the next lesson in order to work on the proof.

In closing, it is clear that the case presented in this article does not allow me to make generalizations. Nevertheless, I have formulated the above list of mathematical and didactical conditions in a somewhat decontextualized way. This is because I believe that the identified configuration of conditions can be adapted to different contexts, including regular school classrooms, and hope that future research will empirically substantiate them.

Notes

[1] Wikipedia refers to an impossible object as a type of optical illusion consisting of a 2-D figure which is instantly and subconsciously interpreted by the visual system as representing a projection of a 3-D object, although it is not actually possible for such an object to exist, at least not in the form interpreted by the visual system.
[2] The comments are retrieved from the *Mighty Optical Illusions* website http://www.moillusions.com/2007/04/perth-impossible-triangle-optical.html on November 19, 2011.
[3] Additional examples are presented elsewhere (Koichu, 2011).
[4] All the quotes from the lessons are translations from Hebrew.

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