CALCULUS FREE OPTIMISATION

SUDAN HANSRAJ

The process of mathematical modeling has attracted a large amount of interest in recent times. Researchers have probed different avenues for the teaching of modeling. For instance, Maaβ (2006) has dealt with the question of how students' mathematical beliefs change during mathematical modeling exercises and how students are empowered to carry out their own modeling. In addition, an attempt is made to investigate the key competencies in modeling and the connections between these competencies and mathematical beliefs. Of equal importance is Blum *et al.*'s (2002) question: What is the role of pure mathematics in developing modeling ability?

I propose in this article that modeling competencies should include mathematical techniques such as utilising the arithmetic-mean-geometric-mean (AM-GM) inequality as well as geometric techniques in optimisaton problems. After all, the modeling exercise may be seen as the first steps in problem solving. It is part of the investigative mindset that can lead to a breakthrough in solving a deep non-intuitive problem. In the survey article of Sriraman, Kaiser and Blomhøj (2006) reference is made to the work of García, Gascón, Higueras and Bosch (2006), who argue that *all* mathematical activity constitutes mathematical modeling. I subscribe to this point of view and encourage the mathematics education community to give special attention to the composition of school syllabi in relation to how they advance the process of mathematical modeling. It is a pointless pursuit to learn mathematical facts such as the AM-GM inequality in isolation or for the purposes of constructing interesting examination questions. It must be realized that these facts are useful in solving specific problems or classes of meaningful problems.

The aforementioned mathematical items are tools to achieve the objectives of many modeling exercises. Here is a typical modeling example. A mango farmer in Cape Town, South Africa sells his produce in France. It is a critical matter for the farmer to decide exactly when to harvest the mangoes so that when the end purchaser in France buys it, it will be ideal for eating. In trying to determine the optimal harvesting date, the farmer may proceed as follows: Record the sugar content of a sample of the fruit on a daily basis (by extracting a small amount of sap and testing with an instrument similar to a glucometer). Continue extracting data until the fruit has begun to rot. Plot the data on a Cartesian system. Now make a conjecture based on the empirical evidence. Of course, it would be useful to find a function of best fit and then use differential calculus to find the optimum sugar level. Naturally, the farmer would not harvest on the day of the optimal sugar level – that would be the day to eat the fruit. The farmer has to take into account transport factors and so on and make a decision based on the model.

Of course, this example may not be solvable by simple algebraic techniques. After all, the model is built up from empirical testing and the use of a sample helps to average out the results and make them more palatable. Clearly, there are also many extraneous factors that come into play in the growth of plants and so the problem is constrained by many parameters in practice. Despite not being able to use simple algebra here, there certainly exists a large class of fascinating problems that could well be solved exactly even after some trial and error in designing the model. For example, find the rectangular box of largest volume with a surface area of 12 square units. This would be completely inaccessible to the average high school student but for using the AM-GM inequality. One should not under-estimate the quality of this problem. The interplay between surface area and volume is believed by some scientists to account for the disappearance of dinosaurs as a species from the planet. A related problem is that of whether dinosaurs were warm or coldblooded. This too is unsolved. Consequently, there is a very important basis not to neglect optimisation exercises of this type. (A useful introduction to this kind of problem may be found at www.sciencebuzz.org.)

The modeling process can take the form of having the class work in groups and taking a set of length, breadth and height combinations that will give a surface area of 12 square units and then computing the volume in each case. It would be more instructive to divide the class into about 5 teams each team with a different starting total surface area. The idea will be to formulate some kind of a general principle to find the maximum volume. Of course, as interesting as the investigation might be, an inductive guess on the optimum solution would be woefully insufficient from a mathematical point of view. Even a million good guesses do not constitute an irrefutable proof. This then calls into question the role of conviction in mathematics – a deep subject that we cannot probe in detail as it falls outside the scope of our discussion. (However, I would draw your attention to the work of de Villiers, 1990 and Hanna, 1995.)

It is a standard exercise in high school mathematics to optimise functions of a single variable. At worst, calculus methods may be invoked. However, depending on the curricular offering, some problems may remain intractable. For example, in many countries students are not exposed to the product, quotient and chain rules for differential calculus at high school and so problems involving square roots of sums of quantities would be inaccessible. Of course, a large class of problems requires the calculus of trigonometric and logarithmic functions and these topics are often deferred to undergraduate courses at university. Nevertheless, many interesting and non-intuitive situations may be effectively optimised with the use of elementary school algebra and geometry.

For the Learning of Mathematics **30**, 2 (July, 2010)
FLM Publishing Association, Edmonton, Alberta, Canada

In many cases, optimisation problems reduce to the study of quadratic functions and with a well-founded quadratic theory at hand, such problems are easy to solve for high school students. For example, physical problems involving motion under the influence of gravity reduce to a study of quadratic functions. Methods such as completion of the square or using symmetry are easy enough for high school students to implement. Many natural processes can be investigated through quadratic type functions. However, there are many interesting classes of problems that are understandable to the high school student, but which involve fairly standard techniques taught in university courses. I look at such problems and investigate how non-calculus approaches could be useful to solve them.

Non-calculus approaches to optimization

I briefly review some techniques that are employed at the level of high school mathematics to solve extrema problems. A detailed treatment of this area may be found in Andreescu, Mushkarov and Stoyanov (2006) where a wide range of problem types is considered. I construct my own set of problems, which I believe can be motivated on physical grounds and that may generate enthusiasm on the part of the learners. These problems in themselves are not novel, but they illustrate how the mathematics may be applied in diverse areas and they argue the case for the inclusion or restoration of such aspects of mathematics in school programmes. The interest in non-calculus methods can be justified by noting that a great deal of calculus at school level is done on an intuitive level. Rigorous proofs such as for the limit theorems are not often dealt with in detail. Of course an epsilon-delta discussion of the meaning of the limit concept can be quite foreboding to a first year student let alone a high school student. Then too, mathematically sound concepts such as differentiability and continuity are also dispensed with intuitively. Consequently non-calculus approaches may have greater mathematical merit at the high school level.

A special class of functions

We now illustrate an approach to solving a class of extrema problems that may be beyond the scope of the high school syllabus for differential calculus – unless of course the chain rule is studied at high school level. This would be unexpected given that proving the chain rule is non-trivial.

It is well known that quadratic functions $y = ax^2 + bx + c$ may be converted to the form $y = a(x - p)^2 + q$, from which it follows that the parabolic function has an extreme value of $y = q$ when $x = p$. However, the question of finding the extreme values of a function of the form

$$f(x) = \sqrt{x^2 + a^2} + \sqrt{(x - c)^2 + b^2} \qquad (1)$$

is non-trivial.

The solution to the problem stems from a novel (and well known) solution to a famous problem type, which I will state in the following form (variations of this are plentiful):

Problem

See Figure 1. Two towns A and B are respectively a km and b km from a long straight electricity cable CD and on the same side of CD. C and D are the feet of the perpendiculars from A and B respectively to the cable CD. It is given that $CD = c$ km. The electricity supplier wants to supply electricity to the two towns from the main supply line CD from a single point E on CD. Where should E be selected so that the cost is least? [Assume that least expensive method means minimising the sum of the distances from A to CD and from B to CD. Of course this agrees with common sense since there will be savings in the cost of the cable as well as in the preparation of trenches for the new cables.]

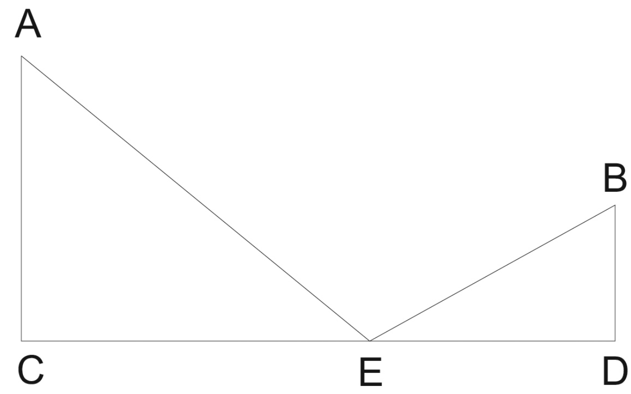


Figure 1. Minimum cable length

No doubt most readers would recognise this as a standard problem in undergraduate calculus. Based on my experiences, first year students, who are immediate products of the high school system, do not consider the possibility of non-calculus solutions when taught the necessary machinery of differential calculus – even when prompted to seek for same. This applies equally well to very capable students and average students – at least in Engineering mathematics where the acceptance criterion for mathematics is very high. The behaviour of students in this regard is under investigation and is the subject of a forthcoming article – but it suffices for now to note that the order of the day is an algorithmic application of calculus amongst those who are taught this. Post-high school students generally have no clue on applying any other techniques. This suggests that their high school training was problematic. In my view, the students were merely coached to work through typical examination questions, as if that were the sole purpose of learning mathematics, and, when faced with actually applying knowledge they already had, could not see the connections. It is an indictment on the quality of teaching they received at the high school level. Again, these issues in mathematical teaching have received attention and are outside the scope of this paper. Let me now revert to the problem at hand.

Solutions

One solution technique – the classic undergraduate method of attack – is to write $AE + EB$ as $\sqrt{x^2 + a^2} + \sqrt{(c - x)^2 + b^2}$ using the theorem of Pythagoras. Now we try to minimise this function. Finding the derivative and setting it equal to 0 may not be worthwhile if one is not familiar with the chain rule. It is worth noting at this point that the minimum value occurs at $x = \frac{ac}{a+b}$ when a, b and c are all positive. In the case where $c < 0$, the minimum is attained at $x = \frac{ac}{a-b}$. Of course this latter case is not worthy of our attention given the context of the problem.

An alternate (geometric) solution (see fig. 2) would be to reflect one of A or B in the line CD to a new point A' or B' and then join either $A'B$ or $B'A$. Where these lines intersect the line CD identifies the point E which minimises the sum of the distances $AE + EB$. Why? The argument is simple and follows from the congruency of triangles and the fact that a straight line minimises the distance between two points in Euclidean geometry. Incidentally, it should be clear that the requirement that A and B be on the same side of CD is superfluous. Now to actually locate E we note that the triangles ACE and BDE are similar. Thus we have

$$\frac{a}{x} = \frac{b}{c-x}$$

which is readily solved to give $x = \frac{ac}{a+b}$. The remaining length EC works out to $\frac{bc}{a+b}$. Of course the solutions are neat rational numbers, which is quite difficult to anticipate if one uses the method of differential calculus. Applying the chain rule and then squaring results in a biquadratic function and fortunately many of the terms do cancel off to reduce to the simple rational solution, given that a, b and c are rational from the outset.

Aligned to this method is the subtle observation that one could solve the problem by asking: What path would a light ray follow if it is shone from A onto a mirror CD in such a way that the reflected beam passes through B? Clearly the path of the light will minimise the required distance. It is well known from elementary high school physics (optics) that the light will obey Snell's Law that the angle of incidence on the mirror will be equal to the angle of reflection off the mirror. This again leads to the case of both triangles ACE and BDE being similar and the solution proceeds as in the second method.

If we call the angle the light ray makes with the line AC the *departure angle* and denote it as α, we find that

$$\tan\alpha = \frac{c}{a+b}$$

where a, b and c are as above.

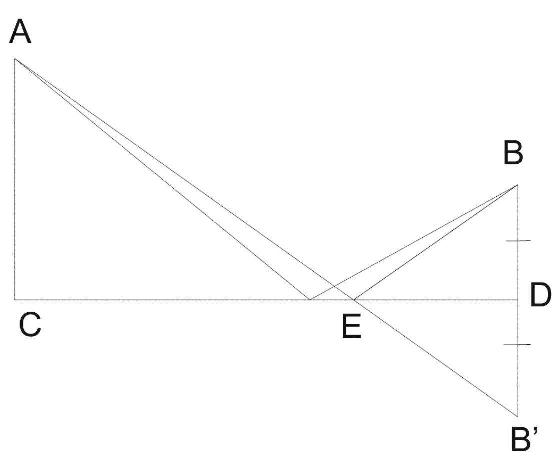


Figure 2. Geometric solution using the Reflection Principle

Deductions and new problems

An important observation now is that the geometric approach above gives us a method to minimise functions of the form

$$f(x) = \sqrt{x^2 + a^2} + \sqrt{(x-c)^2 + b^2}.$$

We know, from routine application of calculus, that the minimum occurs at $x = \frac{ac}{a+b}$ for a, b and c positive. Note that the coefficients of x^2 need not be 1; however, for the success of this method they should be equal in which case they can be extracted as a common factor.

In fact more powerfully we observe that the method solves all problems of the type

$$f(x) = \sqrt{x^2 + bx + c} + \sqrt{x^2 + dx + e}$$

since we can rescale as follows. Both arguments of the square root can be converted to the form $(x - m)^2 + n$ and $(x - p)^2 + q$ in that order, where we make the change of coordinates

$$n = \frac{4c - b^2}{4} \qquad m = -\frac{b}{2} \qquad p = \frac{4e - d^2}{4} \qquad q = -\frac{d}{2} \quad (2)$$

and then we simply generate new coordinates $X = x - m$ and so $x - p$ becomes $X + m - p$. Then we have the transformed function

$$f(X) = \sqrt{X^2 + n} + \sqrt{(X - (p-m))^2 + q} \quad (3)$$

and it easily follows that the minimum of (3) occurs at $X = \frac{\sqrt{n}(p-m)}{\sqrt{n} + \sqrt{q}}$, where for practical purposes we take $p > m$. Now it follows trivially that the minimum of $f(x)$ occurs at

$$x = \frac{\sqrt{n}(p-m)}{\sqrt{n}+\sqrt{q}} + m$$

where m, n, p, q are related to b, c, d, e via (2).

With the aid of *Mathematica* we give the complete solution

$$x = \frac{b^2d - 4cd - bd^2 - \sqrt{b^2-4c}(b-d)\sqrt{d^2-4e} + 4be}{2\left(-b^2+4c+d^2-4e\right)}$$
$$x = \frac{b^2d - 4cd - bd^2 + \sqrt{b^2-4c}(b-d)\sqrt{d^2-4e} + 4be}{2\left(-b^2+4c+d^2-4e\right)}$$

for the equation $f'(x) = 0$.

The most general form of the problem that of minimising functions of the form $f(x) = \sqrt{ax^2 + bx + c} + \sqrt{dx^2 + ex + f}$ is non-trivial and does not appear to follow directly from the methods discussed here. The solution is very complicated in form and after some computing time *Mathematica* was able to generate the x values. However, the final solution occupies several pages and so is not included here.

Returning to the original simplified version above, what is interesting is that if we substitute the values of x as above, the minimum path length turns out to be $AE + EB = \sqrt{(a+b)^2 + c^2}$. We immediately interpret this to be the diagonal of a rectangle with sides $a + b$ and c. This of course also follows simply from the geometric discussion using the theorem of Pythagoras. Note that for the harmless version $f(x) = \sqrt{ax^2 + b} + \sqrt{cx^2 + d}$ the roots of the equation $f'(x) = 0$ are given by

$$\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{bc^2 - a^2d}}{\sqrt{a^2c - ac^2}}\right\}, \left\{x \to \frac{\sqrt{bc^2 - a^2d}}{\sqrt{a^2c - ac^2}}\right\}\right\}$$

$$\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{bc^2 - a^2 d}}{\sqrt{a^2 c - ac^2}}\right\}, \left\{x \to \frac{\sqrt{bc^2 - a^2 d}}{\sqrt{a^2 c - ac^2}}\right\}\right\}$$

which allow us to establish the maximum and minimum values of *f*.

The discussion above leads us to pose and solve the following problem.

Problem

ABCD is a rectangle with side lengths *a* and *b*. Assume the inside of the rectangle's sides are mirrors. A ray of light is shone from a point *P* on *AC* and returns to *P* after touching each of *CD*, *DB* and *BA* once only. Find the minimum length of the light ray's path.

Solution

By the above discussion, the minimum path length is $2\sqrt{a^2 + b^2}$. Of course there is a unique path for the purposes of this problem. However, it is possible that for certain departure angles, the light ray may not strike all the sides once before returning to *P*. To see what departure angles are suitable, a geometrical approach is helpful. It is easy to see from this discussion that we can argue that there are an infinite number of departure angles that will result in the light ray returning to *P*. However, of all of these the one with path length twice the diagonal length is the shortest.

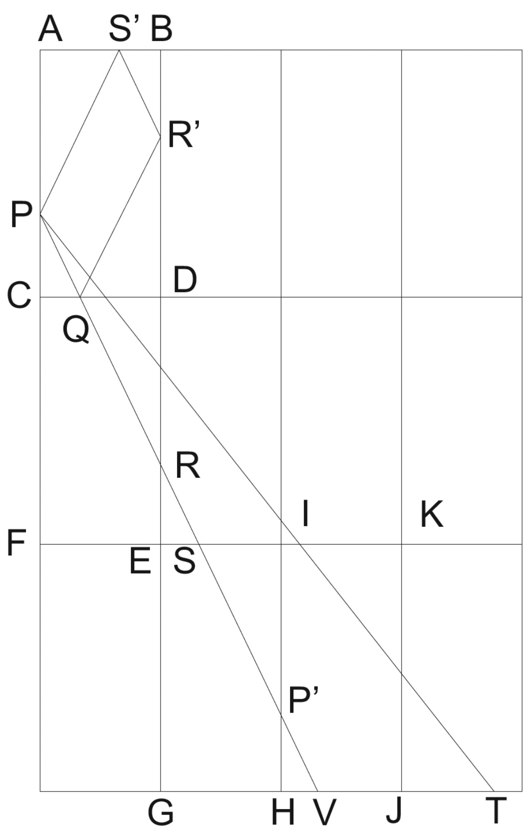


Figure 3. Minimal path length of light ray in a mirror rectangle

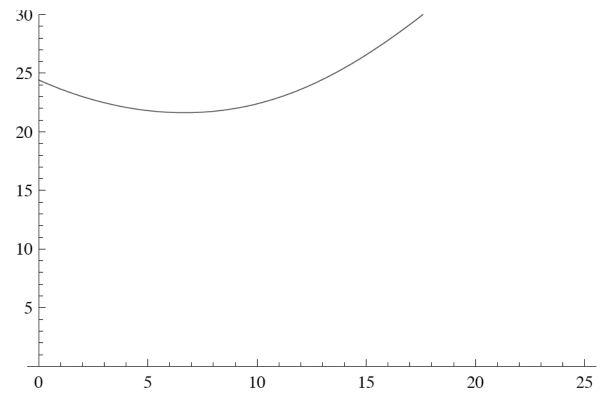


Figure 4. Graph of the Distance Function

To see this consider a rectangular grid where each rectangle has sides *a* and *b* (see fig. 3). Suppose that the departure angle from *P* on *AC* towards *Q* on *CD* is θ. Now we produce *PQ* to intersect three other rectangles in *R, S* and *P′* where *PC* = *P′H*. The images of *R* and *S* namely *R′* and *S′* on *ABCD* are on the path of the light ray. Note *DR* = *DR′* and *ES* = *BS′*. The straight-line distance *PP′* is obviously the shortest path and the corresponding path on the rectangle is *PQR′S′P*. If the departure angle is too large – as for *PT* – it is still possible that a triangle like *P′HV* would arise elsewhere in the grid. Copying this path onto the original rectangle will give the required path.

Naturally, the above problems have extensions. For example we pose the problem: ABCDEF is a regular hexagon with side length a. Assume that the inside of the hexagon's sides are mirrors. From a point P on AB a light ray leaves and reflects off each of the sides of the hexagon and returns to P. Find the minimum path length of the light ray. Give necessary and sufficient conditions for the light ray to actually reflect off each side once only.

In addition to spawning new problem types, this problem yields an instructive exercise for high school learners to make use of a computer software programme such as *Mathematica* to generate graphs of functions of the form $f(x) = \sqrt{x^2 + bx + c} + \sqrt{x^2 + dx + e}$ for a variety of combinations of *b, c, d* and *e*. Clearly the function will be unreal for particular choices of these variables. An interesting observation is to note that for very large values of *x* (for suitable

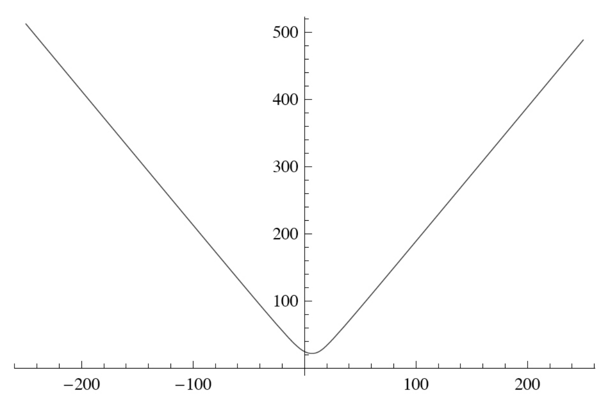


Figure 5. Graph of Distance Function with large domain

choices of b, c, d, e), the function approximates an absolute value graph – obviously because of the dominance of the square term for large values of x. For illustration purposes, we provide a sketch of the function $f(x) = \sqrt{x^2 + 10^2} + \sqrt{(12 - x)^2 + 8^2}$ (fig. 4) first for a standard domain and then for a very large domain (fig. 5) to show the dominance of the x^2 terms in making the graph appear linear.

AM-GM Inequality

The arithmetic mean – geometric mean (AM-GM) inequality discussed by Hardy, Littlewood and Pòlya (1959), is also a useful mechanism to solve a wide variety of optimisation problems where the sum of a set of variables is known and a product is to be maximised or minimised. The theorem states that

$$\left(\frac{x_1 + x_2 + ... + x_n}{n}\right)^n \geq x_1.x_2...x_n$$

for all $x_k \geq 0$. Moreover equality is achieved for $x_1 = x_2 = \ldots = x_n$. There are many ways to prove the AM-GM inequality and methods due to legendary mathematicians George Pòlya and Augustin Cauchy are well known. For a simple general proof, see Hirschhorn (2007). These proofs often rely on the principle of mathematical induction. It should also be noted that the AM-GM inequality is a natural consequence of Jensen's inequality and the rearrangement inequalities – both devices should be well known to mathematics olympiad enthusiasts. A proof for two variables is very easy and is found in most standard high school textbooks.

It is instructive here to review a proof for three variables in the AM-GM inequality. The factorisation employed here is not widely known although the technique is usually in the arsenal of students preparing for contests such as the International Mathematical Olympiad or similar local events. We wish to prove that

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc}$$

for three positive numbers a, b and c.

Proof: Put $a^{1/3} = x$, $b^{1/3} = y$ and $c^{1/3} = z$. Then we want to show that

$$\frac{x^3 + y^3 + z^3}{3} \geq xyz$$

that is,

$$x^3 + y^3 + z^3 - 3xyz \geq 0$$

At this stage we use a highly inspired and sinister factorisation of the left hand side of the inequality to obtain

$$(x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz) \geq 0$$

Now we observe that the second bracket can be rearranged neatly into the form

$$\tfrac{1}{2}(x + y + z)((x - y)^2 + (y - z)^2 + (x - z)^2)$$

from which it immediately follows the left hand side is positive. As a bonus, it is immediately transparent that equality to zero can only be achieved if $x = y = z$ that is $a = b = c$.

AM-GM Inequality versus Lagrange Multipliers

Since the geometry most familiar is 3 dimensional, the result above is extremely useful in a wide variety of physically relevant situations. We give a few problems that could be approached simply with the *AM-GM* inequality – although other methods may work.

Example 1. Prove that the sum of a positive number and its reciprocal is always greater than or equal to 2.

Solution. The AM-GM inequality easily leads to this conclusion: Take a positive number denoted by x and its reciprocal 1/x . Then by AM-GM we have

$$\frac{x + \frac{1}{x}}{2} \geq \sqrt{x \cdot \frac{1}{x}} = 1$$

and it readily follows that $x + 1/x \geq 2$.

Example 2. If a, b and c are positive real numbers, find the minimum value of $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$.

Solution. Once again by the AM-GM inequality we have

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 1$$

from which we easily deduce that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$. In fact it is now trivial to generalize the result: For positive real numbers $a_1, a_2, \ldots$, an we have

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + ... + \frac{a_n}{a_1} \geq n$$

Example 3. A rectangular box has a total surface area of 2 square units. Find the dimensions of such a box of largest volume.

Solution: If the sides of the box are x, y and z, then we are given that $2xy + 2yz + 2xz = 2$ that is

$$xy + yz + xz = 1 \qquad (4)$$

We must find the maximum volume $V = xyz$ subject to the above condition. Now by the AM-GM inequalty we have

$$\frac{xy + yz + xz}{3} \geq \sqrt[3]{xy.yz.xz} = \sqrt[3]{x^2y^2z^2} \qquad (5)$$

Using the given condition (4), inequality (5) becomes

$$\frac{1}{3} \geq \sqrt[3]{x^2y^2z^2}$$

which may be rearranged as

$$x^2y^2z^2 \leq \frac{1}{27}$$

which implies

$$xyz \leq \sqrt{\frac{1}{27}}$$

From this it is clear that the maximum value of the volume $V = xyz$ is $\frac{1}{\sqrt{27}}$ and is achieved when $x = y = z = \frac{1}{\sqrt{3}}$.

Note that a problem of this type could only be handled by undergraduate students perhaps in their second year of mathematics. The Lagrange multiplier technique can be used to solve such problems; however, some problems using this method lead to very messy algebraic manipulations. A foundation in partial derivatives and some other topics on surfaces are required. However, with a knowledge of the arithmetic-mean geometric mean inequality, the problem is readily accessible to high school learners.

Conclusion

I have discussed some limitations of high school calculus in tackling certain classes of problems, but which could be solved using algebraic and geometric approaches. The methods used involved a very interesting interplay between algebra and geometry – which is seldom appreciated by the average high school learner. In particular I have shown how problems involving a sum of square roots of quadratic functions (with equal leading coefficients) can be solved geometrically. Additionally I have explored the *AM-GM* inequality – most importantly the case involving three variables. This case is bound to be the most useful from a physical point of view. I propose that this result, though well known to mathematical enthusiasts, should be dealt with in the regular curriculum as it solves in an elegant manner a large class of problems involving volumes and surface areas

My main lament is that, judging from the product coming out of high school and into programmes at the university, I gain the distinct impression that students have been merely trained, as in cram classes, to answer typical examination type problems. It is my belief that they are casualties of a system that has failed to show them the immense power of mathematics as a subject. I have argued that exposure to problems of the kind mentioned above should feature regularly in classroom mathematics. There is a compelling case to be made, I believe, for the teaching of algebraic and geometric techniques as problem solving methods that may be extremely useful in the process of mathematical modeling. These ideas should not be relegated to the realm of the so-called advanced student, but should be taught as essential tools in the mathematical process. I believe that even if the student imbibes the main idea of the result it would be a valuable life skill. For example, if a student understands that if a set of numbers has a fixed sum then their maximum product is obtained when the numbers are all equal, that is an extremely useful fact. Or at least, its beauty cannot be denied.

References

Andreescu, T., Mushkarov, O. and Stoyanov, L. (2006) *Geometric problems on maxima and minima,* Boston, MA, Birkhauser Verlag.

Blum, W. *et al.* (2002) 'ICMI Study 14: Application and modeling in mathematics education – discussion document', *Journal für Mathematik-Didaktik* **23**(3/4), 262–280.

de Villiers, M. (1990) 'The role and function of proof in mathematics', *Pythagoras* **24**, 17–24.

García, J., Gascón, J., Higueras, L. R. and Bosch, M. (2006) 'Mathematical modelling as a tool for the connection of school mathematics', *Zentralblatt für Didaktik der Mathematik* **38**(3), 226–246.

Hanna, G. (1995) 'Challenges to the importance of proof', *For the Learning of Mathematics* **15**(3), 42–49.

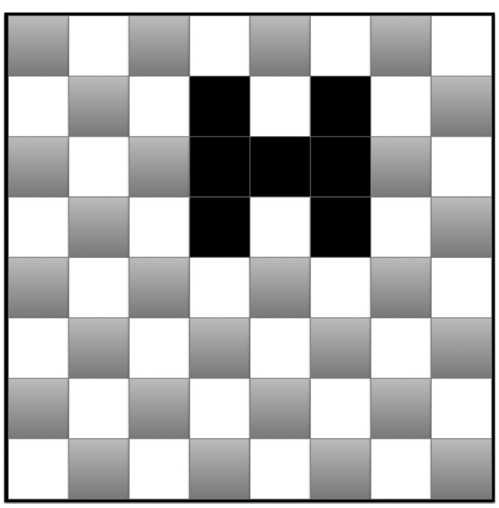
Hardy, G. H., Littlewood, J. E. and Pòlya, G. (1959) *Inequalities,* Cambridge, UK, Cambridge University Press.

Hirschhorn M. D. (2007) 'The AM-GM inequality', *The Mathematical Intelligencer* **29**(4), 7.

Maaβ, K. [2006] What are modelling competencies? *Zentralblatt für Didaktik der Mathematik* **38**(2), 113–142.

Sriraman, B., Kaiser, G. and Blomhøj, M. A. (2006) 'A brief survey of the state of mathematical modeling around the world', *Zentralblatt für Didaktik der Mathematik* **38**(3), 212–213.

Wolfram S. (1988) *Mathematica,* www.wolfram.com



An H-figure exactly covers 7 standard squares on an 8 × 8 chessboard, as shown. What is the maximum number of H-figures that can be placed on the board?

(originally published in Booth, P., Shawyer, B., and Grant McLoughlin, J. (1995) *Shaking hands in Corner Brook and other math problems,* Waterloo, ON, CA, Waterloo Mathematics Foundation; selected by John Grant McLoughlin)