Communications

Do not fall into the same trap I nearly did: experiences of a mathematics education PhD student researching the triangle proportionality theorem.

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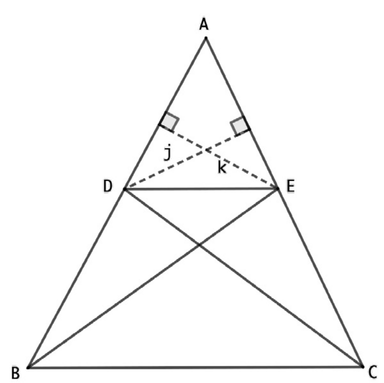
Mathematics in its nature is exploratory, giving learners a chance to view it from different perspectives. However, during most of their schooling, South African learners are rarely exposed to mathematical explorations, either because of the lack of resources or the nature of the curriculum in use. Perhaps, this is due to teachers' inability to design explorative mathematical tasks, or because suitable productive explorations are difficult for high school students and long explorations tend to elicit little interest from learners (Bansilal, Brijlall & Mkhwanazi, 2014; Bizony, 2016). Students are instead taught procedural and factual knowledge where remembering algorithms and facts is deemed significant, and this is evident in their teachers' content knowledge, which is much stronger in knowledge based questions than application questions (Bansilal, Brijlall & Mkhwanazi, 2014). Having matriculated in 2006, during the era where Euclidean geometry was deemed optional by the Department of Basic Education, there is no doubt I am a product of this kind of rote learning. I only encountered Euclidean geometry proof and problem-solving in my fourth year of university, which prepared me insufficiently to teach effectively Euclidean geometry in high school. Sometimes, it would take me days to prepare for a lesson in Euclidean geometry, and sometimes I ended up memorising the answers to different questions without fully understanding them. Given these experiences, I dedicated my life to understanding Euclidean geometry much better, to being an expert in it. Hence, I always tried to look for simpler versions of proofs in geometry. I have learnt that proving is a complicated mathematical activity, one that requires sophisticated mathematical thinking, mathematical knowledge and practice.

As young educational researchers, we seek to understand the methodology and theories applied in teaching and learning, but we also look to enrich ourselves with simpler methods of teaching content that seems difficult for most learners. Given the rapid changes in technology around the globe, new and sophisticated methods of proof are continually being discovered. Hence, old proofs are continually being refined into new yet rigorous proofs. In my quest to understand the triangle proportionality theorem, ('The line drawn parallel to one side of a triangle intersecting the other two sides divides them proportionally.') I encountered a certain predicament. The proof for the proportionality theorem, Figure 1, is one of the general proofs that learners encounter at Grade 12 in the South African curriculum.

I thought this proof is a little bit complicated for Grade 12 learners; especially South African learners who have severe learning deficits from as early as grade 3 (Spaull & Kotze, 2015), and have been shown to fail in attaining the required levels of geometrical thinking (Alex & Mammen, 2014). This proof of the proportionality theorem requires learners to reason proportionally, however, proportional reasoning is difficult and takes time to develop (Frith & Lloyd, 2016). In the Curriculum and Assessment Policy Statement (CAPS), the ability to reason proportionally is deeply rooted in the under-



RTP: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: Construct altitudes j and k from E to AD and D to AE respectively. Join DC and BE.

PROOF

Area of $\Delta ADE = \frac{1}{2}ADk$ and the Area of $\Delta DEB = \frac{1}{2}DBk$

Area of $\Delta ADE = \frac{1}{2}AEj$ and the Area of $\Delta DEC = \frac{1}{2}ECj$

$$\frac{\text{Area of } \Delta ADE = \frac{1}{2}ADk}{\text{Area of } \Delta DEB = \frac{1}{2}DBk} = \frac{AD}{DB} \textbf{ AND } \frac{\text{Area of } \Delta ADE = \frac{1}{2}AEj}{\text{Area of } \Delta DEC = \frac{1}{2}ECj} = \frac{AE}{EC}$$

But Area of ΔDEB = Area of ΔDEC (Area of Δs from same base between parallel lines)

$\therefore \frac{AD}{DB} = \frac{AE}{EC}$

Q. E. D

Figure 1. Proof of the triangle proportionality theorem.

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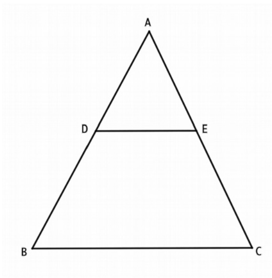

RTP: $\frac{AD}{DB} = \frac{AE}{EC}$

PROOF

In △ADE and △ABC
∠ADE = ∠ABC (Corresponding angles)
∠AED = ∠ACB (Corresponding angles)
∠BAC = ∠DAE (Common)

∴ △ADE III △ABC (A.A.A)

$\Rightarrow \frac{AD}{DB} = \frac{AE}{EC}$ (From similarity, △ADE III △ABC)

Figure 2. Using similarity to prove proportionality.

standing of ratio and proportion from the senior phase (grade 7-9). However, South African learners are insufficiently trained in this regard (Mahlabela, 2012). Furthermore, they have a two year break in grade 10 and 11 where they do not interact with ratio and proportion at a deeper level. Consequently, when they encounter the proportionality theorem in Grade 12, their knowledge and understanding of proportionality have deteriorated. I asked myself, what could be the easier way to prove the theorem that would be more understandable and less complicated for learners? I thought of similarity and I wrote the proof shown in Figure 2.

I was convinced that I had made a small mathematical breakthrough. Upon reviewing literature in mathematics and mathematics education, this proof was missing, and this was of concern to me. I decided to share my 'discovery' with an expert in geometry, Professor Michael de Villiers (my geometry lecturer at university), seeking assurance and confirmation of the correctness of my proof. Even our brightest learners sometimes look up to their teachers for assurance and confirmation of the correctness of their solutions, just as I did. Prof. de Villiers explained to me that the proof is mathematically incorrect, highlighting that it represents a case of 'circular reasoning'.

> The problem is that we use the proportionality theorem to prove the similarity theorems, for example, to prove the result "if the angles of two triangles are correspondingly equal then the sides are in proportion". So basically, 'similarity' is dependent on the proportionality theorem, and we can't therefore use 'similarity' to prove the proportionality theorem as that would be a CIRCULAR argument, which is not allowed in mathematics.

A logical deficit concerning proof structure led to the construction of an incorrect proof (Selden & Selden, 1995), and I was perplexed about the nature of circular reasoning. After reading more, I discovered it was explored already by Aristotle (350 BC) in *Prior Analytics,* as a reasoning fallacy that results from using the conclusion as a premise for a proof. In other words, one proves based on a premise (in the syllogism) which is supposed to be proved. This can be seen in Table 1 and Table 2.

The problem in column 2 of Table 1 is that is that Proposition 2 is supported by B, which is the intended conclusion. Hence, it means we get *'A because B, and B because A'* which then becomes a form of circular reasoning. This is the same as in Aristotle:

> Circular and reciprocal proof means proof by means of the conclusion, *i.e.* by converting one of the premisses simply and inferring the premiss which was assumed in the original syllogism: *e.g.* suppose it has been necessary to prove that A belongs to all C, and it has been proved through B; suppose that A should now be proved to belong to B by assuming that A belongs to C, and C to B—so A belongs to B: but in the first syllogism the converse was assumed, *viz.* that B belongs to C. Or suppose it is necessary to prove that B belongs to C, and A is assumed to belong to C, which was the conclusion of the first syllogism, and B to belong to A but the converse was assumed in the earlier syllogism, *viz.* that A belongs to B.

This is true in my faulty reasoning in Figure 2. I highlight how I came to understand the fallacy in my proof in Table 2.

Paraphrasing the words of Prof. de Villiers, it would be logically redundant to use similarity to prove proportionality. Hanna and de Villiers (2008) question the extent to which learners can identify circular arguments and how teachers can develop an awareness of it. Elsewhere, studies have shown that learners who lack understanding of proof structure usually formulate circular arguments (Heinze & Reiss, 2003; Selden & Selden, 1995). Miyazaki, Fujita & Jones (2017) give an example where a certain learner used the conclusion

Table 1. General logical versus circular reasoning.

Logical reasoning	**Circular reasoning**
Proposition 1: A is true, if and only if B is true Proposition 2: A is true (because C, independent of B) Conclusion: Therefore B	Proposition 1: A is true, if and only if B is true Proposition 2: A is true (because B) Conclusion: Therefore B

Table 2. Proportional theorem logical and circular reasoning.

Logical reasoning	**Circular reasoning**
Proposition 1: Proportionality is true, if and only if similarity Proposition 2: Proportionality (based on premises not including similarity) Conclusion: Similarity	Proposition 1: Proportionality is true, if and only if similarity Proposition 2: Proportionality (because similarity) Conclusion: Similarity

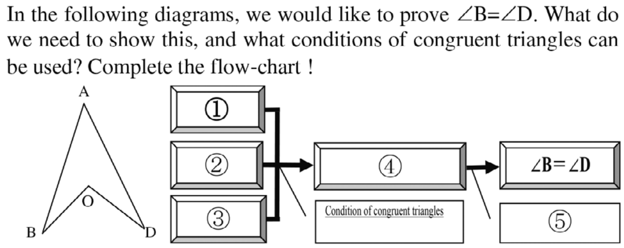

Figure 3: Problem posed to students, from Miyazaki, Fujita, and Jones (2017, p. 232).

as a premise for a proof (see Figure 3). This fallacy is such a pertinent issue such that if it is not addressed appropriately, it occurs even in teachers (see Bansilal, 2012).

Prof. De Villiers noted that it is possible to avoid the circularity through the following reasoning:

> One could use similarity to prove the proportionality theorem if one defines/approaches/proves similarity results completely independently of the proportionality theorem. For example, staying entirely within coordinate geometry one could define similarity as the following transformation: $x' = kx$ and $y' = ky$; and then proceed to prove using only coordinate geometry that angle size is preserved (in other words corresponding angles are also preserved).

But this method is not used in the South African high school curriculum and the textbooks within the CAPS curriculum.

Curriculum considerations

Circular reasoning indicates a lack of understanding of proof structure, which reflects poor understanding of proof in both teachers and learners. In colloquial conversations, it is acceptable to say, 'a black object absorbs more heat because it is black' which is circular, but in a science classroom the circularity is avoided; the reasoning involves light and its properties, instead of the mere fact that the object is black. Circular reasoning presents a threat to understanding mathematics and proof, and despite the fact that logic is not taught as a separate subject in most South African curriculum (even in higher education, Bansilal, 2012), this needs to be addressed in schools and in teacher training colleges. By addressing circular reasoning, we can teach learners how to refute faulty mathematical 'proofs'. By engaging in activities like refuting and exploring circular arguments in mathematics, we can reveal the true value of the activity of proving in geometry. Importantly, during these activities, learners learn that other proofs can seem correct but can be mathematically unacceptable because of the structure of the proof.

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Cardinals versus ordinals: which come first?

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The aim of this short communication is to remind readers that natural numbers may be introduced as ordinal numbers or cardinal numbers and that there is an ongoing discussion about which come first. In addition, through several examples, we demonstrate that in the process of answering the question "How many?" one may, if convenient, use both ordinals and cardinals and seamlessly switch from one to the other.

A substantial body of cognitive science studies on the relative order of acquisition of ordinal and cardinal concepts shows that cardinality precedes ordinality (*e.g.,* Colomé & Noël, 2012; Wiese, 2003). In mathematics education research, more studies focused on cardinal knowledge in children than ordinal competence (*e.g.,* Brannon & Van de Walle, 2001; Coles & Sinclair, 2017). This commentary is inspired by Maffia and Mariotti's recent article 'Intuitive and formal models of whole number multiplication: relations and emerging structures' (2018). In it the authors build their *formal repeated sum model* (FRSM) of multiplication by defining whole numbers as sets, through the construction that was introduced by von Neumann (1923), and then define the operations of addition and multiplication accordingly.

The authors then introduce multiplication of two whole numbers, m and n, where m and n are sets obtained through von Neumann's construction, as the number $|m \times n|$, the *cardinality* of the set $m \times n$. How do we obtain $|m \times n|$? The authors suggest that one should count the cells of an array, *i.e.,* a "double-entry table in which each column corresponds to an element of one set, while each row corresponds to an element of another one" (Maffia & Mariotti, 2018, p. 31). This leads to a second model of multiplication, the *formal array model* (FARM).

If the reader accepts that whole numbers are defined as sets, what does *counting* then mean [1]? And if one uses the

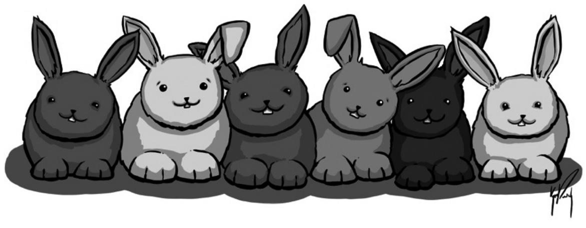

Figure 1: How many rabbits?

phrase "set with n elements" is n still a set or does it represent something else?

Maffia and Mariotti's (2018) work touches on a question that has been a source of debate among logicians and philosophers since the last quarter of the nineteenth century. Linnebo (2018) formulates the question in the following way:

> How are natural numbers individuated? That is, what is our most basic way of singling out a natural number for reference in language or in thought? (p. 176)

In other words, the question is how to define/construct a set of (mathematical) notions or objects that would satisfy the Peano axioms, but at the same time would fit into the basic human experience with natural numbers: the processes of counting and answering the question: "How many?"

In simple terms, the divide among participating logicians and philosophers comes from the two different ways natural numbers are used:

- as ordinal numbers to count objects in a certain order
- as cardinal numbers to answer the question "how many?"

For example, imagine that you ask a child to count how many rabbits there are in Figure 1.

The child counts, "One, two, three, four, five, six!" At a young age children learn that, to avoid double counting or skipping objects, systematic ordering is helpful, for example, counting from left to right. In other words, the child would perform the process of counting by seeing the rabbits as a sequence and then labeling every element of the sequence by an appropriate *ordinal number.*

Next, imagine that you ask the child: "How many rabbit tails are there?" The child would answer "Six!" "How do you know?" you ask. "Because there are six rabbits!" In other words, the child would establish the one-to-one ('1–1') correspondence [2] between the set of rabbits and the set of rabbit tails and answer the question by saying the *cardinal number* of the latter set [3].

A child, or an adult, could be confused by the question of what comes first: Is the cardinal number of the set of rabbits equal to six because we labeled each rabbit by one of the (ordinal) numbers 1, 2, 3, 4, 5, 6? Or were we able to label the rabbits by the elements of the sequence 1, 2, 3, 4, 5, 6 because the cardinal number of the set of rabbits was six?

To illustrate how both distinct and interrelated these two questions are, let us ask the child from the above example: "How many rabbit legs are there?"

The child starts with the far-left rabbit and points with a finger to the front legs: 1, 2, and then points to the place where the back legs are supposed to be: 3, 4. Next, the child moves to the second rabbit: 5, 6 for the front legs and 7, 8 for the back legs, and so on, all the way to 24.

In this example, the child has established that the cardinal number of the set of the rabbit legs is 24 by counting legs in a certain order, *i.e.,* by using the ordinal numbers. But in this process, when counting the back legs, the child used the fact that the cardinal number of the set "back legs of a rabbit" was equal to 2; hence the labels 3, 4; 7, 8 and so on.

Cardinal and ordinal numbers

The essence of the idea of cardinal numbers is given by Hume's Principle (HP):

> Let "$\#F$" abbreviate "the number of objects falling under the concept F" and "$F \approx G$" express the existence of a one-to-one correspondence between the objects falling under F and those falling under G. Then Hume's Principle may be written: $\forall F \forall G(\#F = \#G \Leftrightarrow F \approx G)$. (Boolos, 1998, p. 209)

Boolos (1998) comments:

> According to HP, for any concepts F and G, there are certain *objects*, namely, the number x belonging to F and the number y belonging to G, such that x is identical to y if and only if F and G are equinumerous. (p. 209)

The child in our example has established a 1–1 correspondence between the objects falling under F = *Rabbits* and those falling under G = *Rabbit Tails* and thus established that the same number 'six' belongs to both concepts.

Over the years the idea of cardinal numbers has evolved, or as Assadin and Buijsman (2018) claim: "different philosophies of mathematics give us different notions of cardinal numbers" (p. 9).

The notion of natural numbers as ordinal numbers comes from the notion of a discrete linear ordering with an initial object (Linnebo, 2018). We consider a set S and a relation $R \subseteq S \times S$ that is anti-reflexive, antisymmetric, and transitive and such that the following conditions hold:

1. For all $x, y \in S$, if $x \neq y$ then $(x, y) \in R$ or $(y, x) \in R$.
2. There is $i \in S$, such that for all $x \in S \setminus \{i\}$, $(i, x) \in R$. (We say that i is the *initial object*.)
3. Each $x \in S$ except the greatest [4] (if any) has a *successor, i.e.,* there is $y \in S \setminus \{x\}$, such that $(x, y) \in R$ and such that for any $z \in S \setminus \{x, y\}$, $(z, x) \in R$ or $(y, z) \in R$.

Just by glancing at the image, the child was able to establish the binary relation R = *"to be to the right of"* on the set S of the rabbits displayed on Figure 1. So, if x and y are two different rabbits that belong to the set S, then $(x, y) \in R$ means that rabbit y is to the right of rabbit x from a viewer's point of view. Clearly, R is a discrete linear ordering of the set S with the rabbit to the far left being the initial object. Each rabbit, except the rabbit to the far right, has a successor, the rabbit who is immediately to its right.

Next, we label the initial object in the given discrete ordering of set S by 1 (or by I; or by *one*; or by *un*; or by 一; or in any other desirable way.) This is followed by labelling the successor of the initial object by 2 (or by II; or by *two*; or by *deux*; or by 二; …), and so on [5].

Probably the best-known model of the idea above is von Neumann's (1923) construction of (finite) ordinals: 0 = ∅, 1 = {∅}, 2 = {∅, {∅}}, 3 = {∅,{ ∅}, {∅,{∅}}}, ...

Cardinals versus ordinals

The child in our example was able to answer questions "How many rabbits?" and "How many rabbit tails?" through the process of counting. Does this mean that the notion of cardinality is dependent of the process of counting?

Richard Heck Jr. provides a list of objections that "are directed against the view that ties cardinality to counting" (Heck, 2000, p. 194). His objections include the following:

> Children, and adults, seem to be able to understand some attributions of cardinality quite independently of any connection with counting. (p. 200)

It seems that there is an agreement among cognitive scientists that "humans and many nonhuman species do have biologically endowed abilities for perceiving and discriminating quantities, at least in an imprecise manner and in particular formats" (Núñez, 2017, p. 421).

> There seems to be an important distinction between the sort of knowledge one has when one knows that there are as many Fs as numerals from 'one' to 'nine' and when one knows that there are nine Fs. (Heck, 2000, p. 200)

For example, this semester the first author is teaching a class with, according to the Registrar's Office, 504 students enrolled in it. Since students had enrolled in the class through an online enrollment process, and the class list was generated by software, it seems reasonable to assume that a human eye saw the (total) number of students in the class for the first time on a document that was produced by software.

Cardinals and ordinals

One may wonder what happened between the start of enrollment and the moment when the software produced the number 504. How does the machine collect, store, and manipulate [6] all that *raw* data; *i.e.,* how was the presumably large set of sequences of 0s and 1s in the end, so nicely put in a 1-1 correspondence with the cells in the spreadsheet that contained the class list? Recently, a participant of the Math Catcher workshop answered the question 'How many rabbit legs are there in total?' by counting the front legs, 1, 2, …, 11, 12, and then stating that there were the same number of legs in the back. Did the software establish the fact that 'the number of objects falling under the concept of being a student in the class' was 504 in the same way? In other words, did the machine *count*, for example, the elements of the set of student numbers [7] and then, by using the 1-1 correspondence between the set of students and the set of student numbers, establish that there were 504 people in the class?

Did the child and the machine do the *same* thing: count the elements of one (convenient) set one-by-one, observe a 1-1 correspondence, and then establish the cardinal number of the other set?

Finally, we illustrate the complexity of the dynamics between the two concepts by comparing the roles that two special numbers, zero and one, play in supporting the opposite views in two recent articles on the topic of the individuation of natural numbers.

Linnebo (2018) observes:

> If our basic conception of natural numbers had the form *#F*, then zero would have been a very obvious number. By contrast, the view that our most basic conception of the natural numbers is as ordinals predicts that zero should be just as non-obvious and problematic as the negative numbers; for every sequence of numerals has a first but no zeroth element. (p. 181)

And the author concludes that, "our most basic conception of the natural numbers is ordinal-based rather than cardinal-based" (p. 181).

Assadian and Bujisman (2018) write:

> Note that being able to evaluate this quantifier [$\exists!x$] is sufficient to acquire the cardinal concept ONE. There is no need for (further) ordinal notions to make the step from the quantifier to the cardinal number concept. (p. 13)

Hence, the authors argue, the fact that the concept of the cardinal number 'one' is autonomous with respect to the notion of ordinal number "is sufficient as an argument that we should have a philosophical account that allows for such [*i.e.,* autonomous with respect to the notion of ordinal number] a notion of cardinal number" (Assadian & Bujisman, 2018, p. 11).

Conclusion

The child in our examples answered multiple "How many?" questions by counting objects in a certain order, observing 1-1 correspondences, and using already known cardinal numbers. We believe that this, more or less, summarizes what "educated members of contemporary Western culture" (Linnebo, 2018, p. 180) would do when asked the same question.

Of course, the child, or her teacher as a matter of fact, may not be aware that her answer "One!" to the question "How many *black* rabbits are there on Figure 1?" is the evaluation of the quantifier $\exists!x$ and thus denotes the corresponding cardinal number. Similarly, the child may not be aware of her "One!" in counting the rabbits had the same meaning as von Neumann's ordinal {∅}.

What would our advice be to the child and our readers? Maybe a quote by von Neumann [8]:

> Young man, in mathematics you don't understand things. You just get used to them.

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Notes

[1] For a definition of the function *count* see von Neumann (1923).

[2] A mathematician would add, "If we assume that each rabbit has exactly one tail", but, as we all know, a child has much more common sense than a mathematician.

[3] Examples in this short communication are based on our interactions with Grades K-1 students during the Math Catcher Outreach Program workshops: http://www.sfu.ca/mathcatcher.html.

[4] An element $g \in S$ such that $(x, g) \in R$ for all $x \in R$.
[5] We are just scratching the surface of some complex logic and philosophic issues. For details, see Linnebo (2018).
[6] This is a complex mathematical and computational problem itself. For example, see Codd (1970).
[7] In the process in storing and manipulating the relevant data, each student gets a primary identifier, called a unique key in relational database language. It is reasonable to assume that the student number plays that role.
[8] Wikiquote, https://en.wikiquote.org/wiki/John_von_Neumann

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From the Archives

The following observations are excerpted from Acoustic Counting and Quantity Counting *by Jan van den Brink, which appeared in **4**(2).*

Counting

Counting is not limited to the counting of quantities. It is used by children to accompany movement, games and singing, and just to pass the time. This kind of counting is first and foremost of an acoustic nature. The counting of quantities is an application thereof.

Acoustic counting is the recital of a sequence of numbers in a certain order. This is a broad definition as the sequence does not necessarily need to be the natural or "expected" sequence nor does it need to begin with "one". Neither are quantities of objects necessary in order to give the answer to the question: "Can you count?" [...]

Let us now look into a few differences between acoustic counting and quantity counting.

A The order of numerals does not need to be the natural one of 1, 2, 3, … Children's games actually ascribe a function to the omission of objects, even if this is contrary to the official counting sequence 1, 2, 3, … Consider the game of hopscotch or skipping songs. In Holland there is a skipping song "In *de* tent stond *een* vent" (In *the* tent stood *a* man) and you twirl the rope more at the stressed words and once at the other words. This is one form of omission. Another form is in counting 10, 20, 30, … where the numerals 1–9 are omitted. Note that the omission of articles, places or numerals varies. In acoustic counting the omission of objects appears more often than the omission of numerals.

On the other hand, in counting quantities, omission is strictly forbidden. The sequence of numerals is fixed and each object must be counted. This is in contrast to the rhythmic acoustic counting which precedes it.

B Note that it is not necessary to start acoustic counting with the sound "one". *E.g.* Paul is three and counts while the other children go and hide: 4, 12, 18, 6, 4, 12, 18, 6, 4, 12, 18, 6, 4, … He starts his counting sequence of sounds "4, 12, 18, 6". And here we notice another phenomenon of acoustic counting.

C The final sound is not fixed. When one is playing "Hide and Seek", one needs a terminal sound, but Paul just goes on. For him there is no fixed final sound.

When one is counting quantities, the most important thing is the final sound. When counting acoustically, the final sound is far less important. So it is not difficult to understand that children will have difficulties with the two forms of counting. There are therefore many researchers who have engaged themselves in the study of the establishment of the final sound as number. In our acoustic investigation we found that a pause in between times was very useful to accentuate the number.

D Double counting (counting one number twice) in counting quantities is not allowed. In acoustic counting double countings are in a certain sense even necessary. However, children have continually to repeat the basic sequence of sounds from 1 to 9, of course with the sounds 10, 20, 30, 40, *etc.,* in between (19, 29, 39, 49, *etc.)*

E Moreover, it is not necessary that there be quantities of visible objects around while the child counts acoustically. We have already said that no visible objects are necessary for the child to answer the question "Can you count?" Here again acoustic counting differs from quantity counting.

Acoustic counting offers great freedom in the choice of sequence, initial sound and final sound. In this respect, quantity counting is a limited particular form of acoustic counting.

Summarizing:

- Sounds may not be omitted in quantity counting. After '10' comes '11' and not '20'.
- No numeral may be said twice.
- You must begin with the sound 'one'.
- You must go on counting to the end of the row of visible objects.

Acoustic counting differs in these respects.