FUNCTION SAMENESS: BRINGING COHERENCE TO IMPLICIT DIFFERENTIATION

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John is doing well in his introductory calculus class. He has learned how to generate a new function, a derivative, from a given function. He has completed several implicit differentiation problems and can apply differentiation rules fluently. He is excited to participate in a mathematics education interview. After he successfully differentiates an equation, the interviewer asks him "Why was it OK to take the derivative of both sides?". John thinks for a moment back to his experience in secondary school algebra and replies "If I have $x = 1$, I multiply by 2 and get $2x = 2$; it would be the same thing". Next, the interviewer asks John what happens if he differentiates both sides of $x = 1$. John writes "$1 = 0$" on his paper and feels confused. Later in the interview, John differentiates an equation without hesitation. When asked why it was OK for him to perform this differentiation, he says "because math teacher said so" (with a chuckle).

The vignette centers around (a lack of) what Star (2005) calls *deep procedural knowledge*. This involves knowledge of when and why a procedure works. As mathematics educators, we consider it important that students not only know how to apply a procedure, but also why a procedure works. Specifically, we

(1) Describe a way that introductory calculus students could understand not only the 'how' but also the 'why' of implicit differentiation.

(2) Outline the difficulties that students might have in coming to this understanding.

In doing (1) and (2), we present concerns that could guide calculus teachers. In particular, (1) can help them with their initial presentation of the subject, and (2) can sensitize them to the difficulties students might encounter.

Mathematics education literature (*e.g.,* Engelke, 2004; Hare & Phillippy, 2004) tends to treat differentiating equations as an unproblematic application of previously learned rules. We could find only two articles, Thurston (1972) and Staden (1989), that address the legitimacy of differentiating both sides of an equation. While both Thurston and Staden acknowledge that the justification for differentiating equations requires explanation, neither author considers the conceptualizations that might be involved for a student to come to understand when and why differentiating an equation is legitimate.

We therefore provide an approach that entails a justification for implicit differentiation that coheres with the rest of introductory calculus. We hope that this discussion of implicit differentiation will sensitize the reader to the mathematics involved in relation to their own understanding as well as guide them in presenting the topic to their students.

Our approach to addressing the above begins with a *conceptual analysis* (Thompson, 2008) of what it means for an introductory calculus student to understand (the legitimacy of) implicit differentiation. The conceptual analysis addresses point (1) by answering the question 'what does it mean to understand the legitimacy of implicit differentiation in a way that is consistent with introductory calculus?'. Answering this question provides a lens for point (2). Specifically, it helps us begin to answer 'what struggles might a student encounter in constructing such an understanding?' We address this second question by consulting the relevant literature and presenting novel student data. Educators can use this conceptual analysis, together with the discussion of potential student struggles, to help students understand the 'why' of implicit differentiation.

A conceptual analysis

To guide our conceptual analysis, we consider a classic implicit differentiation problem, which we call *Ladder Problem 1:*

> A 3-meter ladder is sliding down a vertical wall. Find the rate of change of the height of the ladder's top with respect to the distance of the ladder's bottom from the wall.

A typical written solution to Ladder Problem 1 involves designating y as the height of the ladder's top and x the distance of the ladder's bottom from the wall and performing this computation:

Ⓐ

$$\begin{aligned} x^2+y^2 &= 9 \\ 2x+2y\left(\frac{dy}{dx}\right) &= 0 \\ 2x &= -2y\left(\frac{dy}{dx}\right) \\ \frac{dy}{dx} &= \frac{-x}{y} \end{aligned}$$

While the solution procedure is correct, there is no explanation for why concluding the second line from the first line is a valid inference; that is, each side is differentiated, but there is no justification for *why* this is OK. We therefore solve Ladder Problem 1 in a way that not only elucidates the 'why' of implicit differentiation, but also does so in a way that is coherent with the rest of introductory calculus. To clarify, we are not claiming that the line of reasoning described in the conceptual analysis below reflects what a student could come up with on their own. Rather, it forms a trajectory of what we believe might be possible given appropriate instructional support.

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Working through Ladder Problem 1

We solve Ladder Problem 1 in a way that reflects a trajectory that an introductory calculus student such as John could be guided through. As before, we have

$$① \quad x^2+y^2=9,\ y>0$$

The first crucial insight John might have, is that any x-value between 0 and 3 has a corresponding y-value that makes ① true. He might then relate this insight to the notion of function and observe that any value of x in this interval determines a *unique* value of y. So, it might make sense to switch to a notation that indicates that type of relationship. Accordingly, we re-write equation ① as

$$② \quad x^2+(f(x))^2=9$$

where $f(x)$ is that unique value of y determined by x in ①. Next, John might be guided toward the insight that f is not the only function involved. Namely, x^2, 9, and $x^2 + (f(x))^2$ can all be thought of as dependent on x. Hence, ② is a statement about function equality. Accordingly, we consolidate some of the functions involved in equation ② by calling the function defined on the left side m and the function on the right side r. So, for $0<x<3$, $m(x) = x^2 + (f(x))^2$ and $r(x) = 9$. Equation ② tells us that $m(x)$ and $r(x)$ are equal on this interval. The final insight involves inferring that because $m(x)$ and $r(x)$ are equal on this interval, they therefore share a derivative on this interval. The inference is central to why differentiating both sides works. Returning to the problem, a student such as John then might feel like he understands why he can differentiate both sides. Differentiating both sides of equation ② yields $f(x) = -x / f(x)$. We summarize this line of reasoning in Computation B.

$$
\text{Ⓑ} \quad ① \quad
\begin{aligned}
x^2+y^2 &= 9 \qquad\qquad\qquad\qquad y>0\\
f(x) &= \text{the value of } y \text{ that makes ① true}\\
m(x) &= x^2+(f(x))^2\\
r(x) &= 9\\
m(x) &= r(x)\\
m'(x) &= r'(x)\\
f'(x) &= \frac{-x}{f(x)}
\end{aligned}
$$

The Conceptual Steps involved in making the inference of taking the derivative of both sides are as follows:

1. Defining f by using ①
2. Viewing both sides of the equation as functions.
3. Recognizing that the functions defined on the left side and the right side are equal.
4. Inferring that, since the functions are equal, their respective derivatives are equal.

Note that we are *not* claiming that first-year calculus students could generate the above line of reasoning on their own. However, we believe that, like Thompson's conceptualization of integration as an accumulation (Thompson & Silverman, 2008), it can form the basis of instruction aimed at student understanding. In particular, the Conceptual Steps cohere with introductory calculus insofar as derivatives are *of* functions and the equation being differentiated is an equation about functions. Both Thurston (1972) and Staden (1989) suggested that the legitimacy of differentiating equations is rooted in function equality. However, as we discussed, much of the current education research on related rates and implicit differentiation problems overlooks the legitimacy of the procedure.

We wish to emphasize an aspect of our conceptual analysis. The conceptual analysis was grounded in the fact that the standard introductory calculus curriculum that proceeds implicit differentiation treats derivatives as being of functions. Therefore, how an introductory calculus student understands implicit differentiation should involve derivatives as being of functions (rather than of, say, expressions) and differentiation rules being applied to functions. This is consistent with how differentiation rules are introduced in many textbooks; for example, the sum rule states that the derivative of the *function* $f + g$ is the derivative of the *function* f plus the derivative of the *function* g. Thus, for example, when students apply the sum rule, power rule, and chain rule to $x^2 + y^2$, they must think of these rules as applying *to functions*. This approach is supported by research that suggests that some students need to see equations explicitly written with standard function notation before differentiating (Engelke, 2008). This involves viewing each side of ① as representing a function, and therefore viewing ① as expressing function identity.

When does the equation serve as a function definition?

Typically, in calculus textbooks, implicit differentiation and related rates problems are introduced with little distinction between the two. Accordingly, we illustrate these differences by briefly reformulating the problem at hand as a related rates problem. We keep the previous ladder situation, but this time we specify that the top of the ladder is sliding down the wall at 0.1 m/s and ask the student to find the speed of the bottom of the ladder. We call this new problem *Ladder Problem 2*, a related rates problem. Procedurally, solving Ladder Problem 2 is very similar to solving Ladder Problem 1; it involves differentiating the same equation ($x^2 + y^2 = 9$) but with respect to t (time) instead of x.

$$
\text{Ⓒ} \quad
\begin{aligned}
x^2+y^2 &= 9\\
2x\left(\frac{dx}{dt}\right)+2y\left(\frac{dy}{dt}\right) &= 0\\
2x\left(\frac{dx}{dt}\right)+2y(-0.1) &= 0\\
2x\left(\frac{dx}{dt}\right) &= 2y(0.1)\\
2x\left(\frac{dx}{dt}\right) &= 0.2y\\
\frac{dx}{dt} &= \frac{y}{10x}
\end{aligned}
$$

We use Computation C to stress the procedural similarity between Ladder Problem 1 (Computation A) and Ladder

Problem 2. This similarity may contribute to the common conflation of implicit differentiation with any differentiation of equations using Leibniz notation. Consider, for example, Hare and Phillippy (2004), who explain "Implicit differentiation must be used whenever the differentiation variable differs from the variable in the algebraic expression" (p. 9). Thus, the authors appear to be conflating implicit differentiation with use of the chain rule. That is, when we 'take *d/dt*' of $x^2 + y^2 = 9$, we have to use the chain rule with x and y. Similarly, when we 'take *d/dx*' of the same equation, we have to use the chain rule with y. So, viewed procedurally (without attending to the legitimacy of the procedure), these problems are almost identical.

However, as we now illustrate, the conceptual operations entailed in Ladder Problem 2 are not identical to those in Ladder Problem 1. In fact, Ladder Problem 2 is not truly an implicit differentiation problem. In order to provide this comparison, we solve Ladder Problem 2 in a way that an introductory calculus student such as John might understand.

We begin as before with the equation $x^2 + y^2 = 9$. Unlike in Ladder Problem 1, where we conceptualized y as a function of x, we conceptualize y and x as functions of t: for all t, $(x(t))^2 + (y(t))^2 = 9$. Similar to our earlier discussion, if we give labels to the functions on the left and right sides of the equation, say $m(t) = (x(t))^2 + (y(t))^2$ and $r(t) = 9$, then the equation $x^2 + y^2 = 9$ simply asserts that $m(t)$ and $r(t)$ are equal for all t. As with Ladder Problem 1, this statement of function equality implies that $m'(t) = r'(t)$. So: $2x(t)x'(t) + 2y(t)y'(t) = 0$, which, since we know $y(t) = -0.1$, yields: $x'(t) = 0.1y(t)/x(t)$. We summarize this reasoning in Computation D:

Ⓓ ①
$$\begin{aligned} x^2+y^2 &= 9, \qquad y>0 \\ m(t) &= (x(t))^2+(y(t))^2 \\ r(t) &= 9 \\ m(t) &= r(t) \\ m'(t) &= r'(t) \end{aligned}$$

Unlike with Ladder Problem 1, solving Ladder Problem 2 does not involve Conceptual Step 1, as there was no function of t implicitly defined by the equation. This difference is what distinguishes related rates from implicit differentiation problems. Instead, the equation $x^2 + y^2 = 9$ describes a relationship between yet unspecified functions of t. We carefully contrasted the two types of problems to emphasize that, while these problems have similar procedural solutions, they differ in Conceptual Step 1 when attending to the legitimacy of differentiating the equation.

Summary of the conceptual analysis

We are addressing an aspect of deep procedural knowledge, specifically, how introductory calculus students can understand the legitimacy of differentiating each side of an equation. In order for such a student to see the legitimacy of this procedure for differentiating an equation, they must understand the equation as asserting a statement of function equality (Conceptual Step 3). Doing so requires viewing each side of the equation as defining a function (Conceptual Step 2). Viewing each side of the equation this way in Ladder Problem 1 (an implicit differentiation problem) requires a significant conceptualization (Conceptual Step 1) that Ladder Problem 2 (a related rates problem) does not. Yet, when viewed as symbol manipulation exercises, Ladder Problems 1 and 2 are nearly indistinguishable, which could explain why some educators appear to conflate related rates problems with implicit differentiation problems. Importantly, the conceptual analyses described above present implicit differentiation (and related rates) problems in a manner that makes the role of functions more transparent and foregrounds the reasons for the legitimacy of the procedure.

Potential student struggles: insight from previous studies

Viewing an equation as implicitly defining a function (Conceptual Step 1) might be problematic for students. Notice that defining f takes the form of '$f(x)$ is the y such that the proposition $P(x,y)$ is true.' Conceiving of a function definition that involves outputs according to whether or not a proposition is true requires a process conception of function, which many students have not yet developed (Breidenbach, Dubinsky, Hawks, & Nichols, 1992).

As discussed above, a key aspect of understanding the legitimacy of differentiating an equation is conceptualizing that equation as expressing function equality (Conceptual Steps 2 and 3). Doing so requires *not* thinking of the equation as merely expressing numerical equality. For example, thinking of $x^2 + y^2 = 9$ as referring to fixed specific values of x and y is antithetical to thinking of it as a statement of function equality. Knowing when an equation does or does not express only numerical equality seems to be difficult for students (White & Mitchelmore, 1996; Engelke, 2004). Engelke (2004) argues that a major student impediment in solving related rates problems involves difficulty in viewing equations and problem situations *covariationally* (in the sense of Confrey & Smith, 1995). Engelke found that students tended to label their diagrams with constants when they should have been using variables. These observations suggest that students struggle with Conceptual Steps 2 and 3. If students are viewing x and y as constants, then they are not viewing $x^2 + y^2 = 9$ as representing a statement about function equality. However, while viewing equations covariationally might be necessary for understanding the legitimacy of differentiating equations, it is not sufficient. As we will illustrate with John's clinical interview, a student might think of an equation as expressing a relationship between varying quantities while not considering the role of functions. Students cannot think of function equality if they are not even thinking of functions, impeding Conceptual Step 3.

Even *if* a student views functions as being involved (Conceptual Step 2), they still might struggle with the notion of function equality (Conceptual Step 3). Mirin (2018) suggests that a strong understanding of function equality may be absent in a number of calculus students. Specifically, students struggle with inferring that sameness of graph (pointwise equality) implies sameness of function. Thankfully, students need not have a fully developed sense of function equality in order to achieve Conceptual Step 4; they need only reason that because the function on the left side and the function on the right side agree on all inputs, their respective derivatives agree on all inputs (that $m(x) = r(x)$ for all x implies $m'(x) = r'(x)$ for all x). However, Mirin (2018) reports that this inference might be especially problematic for students. When presented with a piecewise-defined version of the function defined by $y = x^3$, only 32% of the first-semester calculus students who assessed it as sharing a graph with the function $y = x^3$ believed that their derivatives agreed at a particular point. So, not only did some students struggle to infer function equality, some did not use equal graphs on an interval to infer equal derivatives at a point on that interval. Hence, even if students were to consider equations that they differentiate as statements of function equality, it is not clear that they would infer that the derivatives of those functions are also equal. Consequently, they would not understand why taking the derivative of both sides of an equation is ever a valid procedure (Conceptual Step 4).

The literature discussed above provided insights into how students might understand the Conceptual Steps. Specifically, we used our conceptual analysis as a framework for investigating the literature to delineate where students might struggle with the necessary conceptualizations for understanding the legitimacy of implicit differentiation. Although the existing literature provides guidance regarding where students might struggle, it does not directly address student understanding of the legitimacy of differentiating equations. Our conceptual analysis is itself novel and provides a lens for incorporating information from previous studies, and as we will see in the next section, it will also provide a lens for making sense of how students understand (the legitimacy of) implicit differentiation.

A clinical interview

Equipped with our conceptual analysis, we return to John's interview from the opening vignette. At the time of the interview, John was enrolled in second-semester introductory calculus at Anonymous State University (ASU). He had, the semester prior, taken first-semester introductory calculus. John was a successful calculus student in that he earned a 'B' in his first-semester calculus course. John had learned about derivatives *as* functions and being *of* functions (we reviewed videos of his lectures). In the first-semester calculus course, John had learned to take the derivative of both sides of an equation in solving 'implicit differentiation problems' and 'related rates problems' in a similar procedural way as illustrated in Computations A and C, without an explanation for why this procedure works.

Interview protocol

The interview was an hour-long semi-structured clinical interview aimed at discovering and identifying the student's mental structures (Ginsburg, 1981). Throughout the interview, John was asked to think about ideas regarding implicit differentiation and function equality that he had perhaps not reflected on before. John might have never considered these matters and might therefore have improvised explanations.

Four prompts guided the interview:

Prompt 1. What is your meaning for "implicit differentiation"? How do you interpret the word "implicit" in this situation?

Prompt 2. Find dy/dx for $x^2+y^2=1$ when $y>0$.

Prompt 3. A 10-foot ladder leans against a wall; the ladder's bottom slides away from the wall at a rate of 1.3 ft/sec after a mischievous monkey kicks it. Suppose $h(t)$ = the height (in feet) of the top of the ladder at t seconds, and $g(t)$=the distance (in feet) the bottom of the ladder is from the wall at t seconds. Then $(h(t))^2-100 = -(g(t))^2$. How fast is the ladder sliding down the wall?

Prompt 4. True or false. Suppose $f(x) = g(x)$ for all values of x. Then $f'(x) = g'(x)$.

Only the most pertinent highlights of the interview are reported here.

Prompt 1

In response to Prompt 1, John expressed that he did not remember exactly what the procedure of implicit differentiation was, but that it was something that must be done when there is no function (due to failure of the vertical line test). He did not have an idea of what the *implicit* referred to in *implicit differentiation,* suggesting a difficulty with Conceptual Step 1.

Prompt 2

John did not have an idea of how to approach Prompt 2, so the interviewer reminded him that $x^2 + y^2 = 1$, $y>0$ defines the top half of a circle and that a particular procedure was done in his first-semester calculus class: replacing y with $f(x)$ before differentiating the equation. The interviewer then asked him to elaborate on what $x^2 + (f(x))^2 = 1$ means. He explained that 1 is "the radius", and having $f(x)$ (in place of y) "makes the computation easier". He was then asked explicitly what it means for the right-hand side of $x^2 + (f(x))^2 = 1$ to equal the left hand side, and he responded "It's a circle. I just see a circle". When prompted to explain what the circle has to do with the equation, he graphed two parabolas on the same axes: a sideways parabola (representing y^2) and an upright parabola (representing x^2) and asked "how is that a circle?". In this situation, it seems that John was not thinking of y (or $f(x)$) as a function of x. Instead, he seemed to be thinking of y^2 as denoting the parabola associated with the equation $x = y^2$. This association indicates that he was engaging in what Moore and Thompson (2015) call 'shape thinking' (associating shapes with symbols), rather than understanding the equation as a statement of function equality (Conceptual Step 2).

After reasoning with a graph was unhelpful to John, he began considering specific values of x and y, observing that "as they change together, in this equation here, they have to change together in such a way that it always equals 1". It seems that here, John began thinking covariationally, but it was still not clear how John's approach related to his understanding of the legitimacy of the differentiation procedure.

As discussed in the opening vignette, John justified the procedure of implicit differentiation by drawing an analogy to algebra. He subsequently related the procedure of taking d/dx to inferring equal rates of change: "if you take the rate of change of this [left side], it is the rate of change of this [right side]. They're equal to each other, so the change in one is gonna be the change in the other". Since John believed the inference of equal rate of change came from *something* being equal, to get at what that *something* was, the interviewer asked him what happens if he differentiates each side of $x = 1$. As discussed in the vignette, John noticed that it results in $0 = 1$, which he said did not make sense. It appears that John was struggling with Conceptual Steps 2, 3, and 4; he was not viewing the equation as an equation of functions (Conceptual Step 2 and 3) and, despite being explicitly prompted, did not justify use of the differential operator (Conceptual Step 4).

Prompt 3

Prompt 3 is a related rates problem, like Ladder Problem 2. In this prompt, function notation is provided explicitly in order to encourage the student to talk about functions, and an animation of the problem situation is provided in order to give the student a context to refer to (Engelke, 2004, suggests that having a dynamic image of the problem situation is helpful for helping students reason about related rates problems). John was reminded that he could take the derivative of both sides of the equation, and he did so. He explained that the ladder's distance from the wall, $g(t)$, and the ladder's distance from the floor, $h(t)$, "change together". When pushed, he did not say why taking the derivative of both sides is a valid procedure. Instead, John continued to express an understanding of the two distances as changing together with time and did not mention each side of the equation as representing a function:

> We take the derivative of both sides because *[pause]* you need to have the two rates change together, in order for this scenario to work. Because if they don't with respect to each other, then uh *[pause]* it just doesn't hold true. So we do it on both sides in order to have the scenario change together and everything stay true to itself *[pause]* maybe.

Even when asked *what* exactly is being differentiated on the left hand side, John talked about only $h(t)$ as a function and did not seem to consider the entire left hand side as representing a function. This suggests that John was struggling with Conceptual Step 2. The only further justification he gave for the legitimacy of differentiating both sides was covered in the opening vignette: "because math teacher says so".

Prompt 4

Since John was not using the language of functions on his own, the interviewer decided to move to Prompt 4 in order to see if he would relate taking the derivative of each side of an equation to an inference from function equality. John almost immediately provided what he viewed as a counterexample to the assertion that if two functions are equal for all inputs, then their derivatives are also equal for all inputs. By misapplying the quotient rule, he argued that $f(x) = x$ and $g(x) = 2x/2$ are equal for all values of x but have different derivatives, which is the antithesis of Conceptual Step 4. John continued his explanation that, if he were to simplify $g(x)$ prior to differentiating it, he would end up with the same derivative as that of $f(x)$. However, he noted that simplification before finding derivatives is not permitted in his calculus class. John's response highlights that he had a fundamental misunderstanding of how the derivatives of equal functions relate, a key aspect in understanding the legitimacy of applying the differentiation operator. This shows us that, for John, Conceptual Step 4 is problematic. Even *if* he had viewed equations he was differentiating as statements of function equality, he still would have the obstacle regarding understanding the differentiation inference. In other words, John not only struggled to understand *why* it was acceptable to differentiate equations of functions, but he also misunderstood *that* it was acceptable to differentiate such equations.

Interview results: discussion

The fact that John reasoned covariationally, yet still struggled with Conceptual Steps 1–4, indicates that understanding the legitimacy of differentiating equations (and hence implicit differentiation) is a significant challenge for John. Specifically, it provides an existence proof that there is more to understanding implicit differentiation than correct mathematization, covariational reasoning, and algorithm implementation. The analysis of John's work also serves to demonstrate the utility of our Conceptual Steps framework for highlighting which components of John's knowledge could benefit from reinforcement.

Discussion

Being aware of the difficulties students might encounter in developing the conceptualizations described in the conceptual analysis (the Conceptual Steps) can be useful to calculus instructors when working with their students. We wish to emphasize that the conceptual analysis was helpful in delineating which conceptualizations impeded John when formulating his explanations. Importantly, our work reformulates the topic of implicit differentiation in a way that coheres with typical calculus curriculum. Developing the topic in this way can serve to enrich the connections students make between implicit differentiation and the differentiation that precedes it.

Two natural questions emerge from this work. First, how might the conceptual analysis presented here inform the creation of implicit differentiation and related rates units? Adequately addressing this question involves both the development of such units and studying their implementation. Second, what can we discover from analogous work in alternative instructional paradigms such as infinitesimal calculus? The conceptual analysis used in this study was predicated on derivatives being of functions, which is the dominant calculus instructional paradigm.

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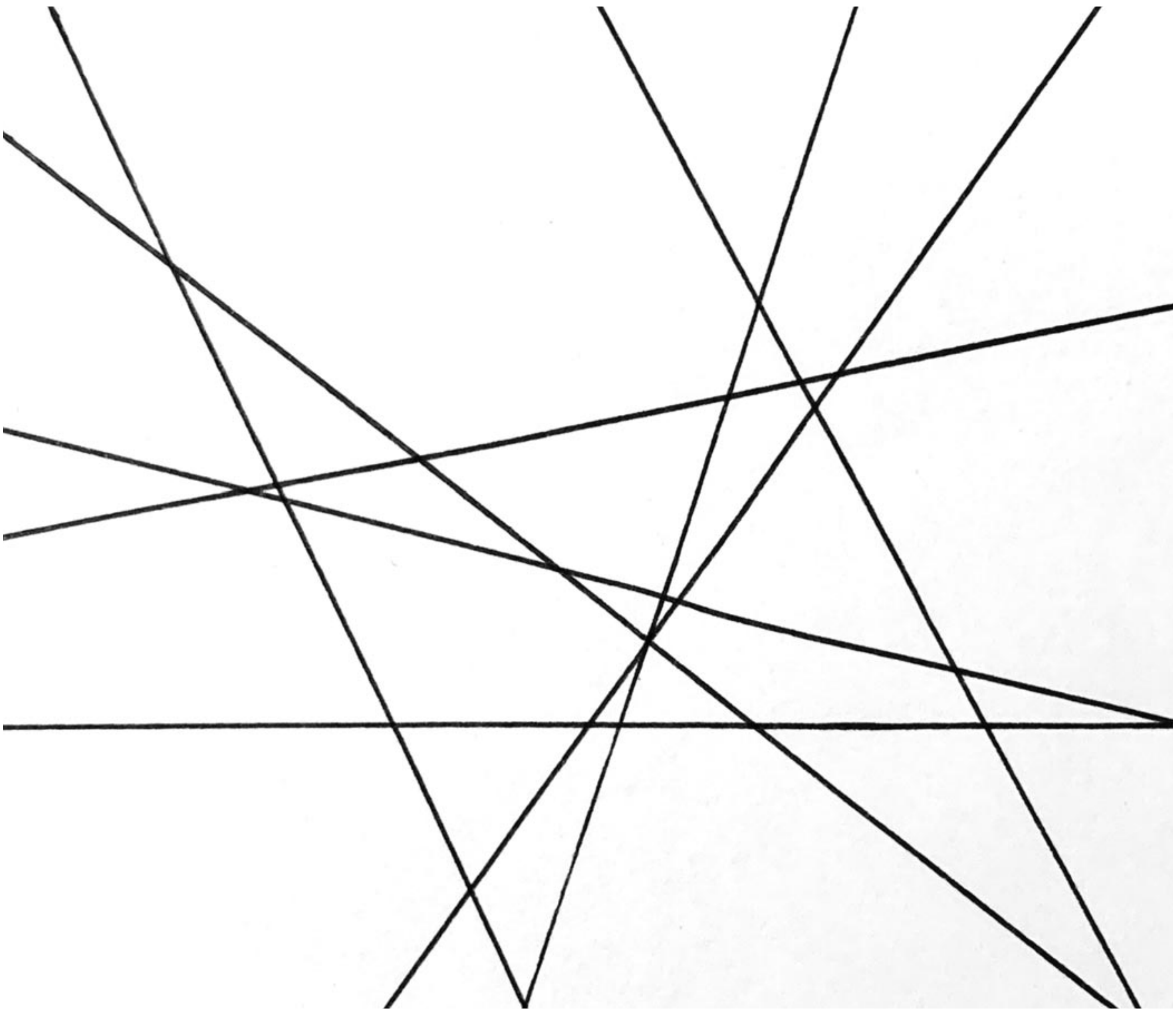

Drawn by Arthur, age 15, on being asked to "Draw something mathematical."