L'Hôpital's Weight Problem

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1. An experiment

In this article I shall describe an experiment that I have done in one of my classes. Experiment, in this sentence, has a double meaning: the way in which I have used the history of mathematics in my teaching is experimental (at least, to me it is) and within this experiment I have used a physical experiment. But first let me give some information about the class—a final class—, about my school—a gymnasium—, and about its place in the Dutch educational system.

Dutch children enter the school system at the age of four. They stay in the primary school for eight years, so when they are about twelve they move on to a secondary school. The choice, restricted by the results of the child at the primary school, is made from a broad variety of schools, ranging from direct practical training for a job (lower technical schools, for example) to a general education, which may lead to a job, but which for most pupils prepares for further education. The upper level in these schools for general education is the level which qualifies for admission to the university. Schools at this level have a six years curriculum. There are two types, which differ only at the point of classical languages. At a gymnasium, and that is where I am working, Latin and Greek are in the curriculum, in the other type (the atheneum) they are not. So in the final class of a gymnasium, which is the subject of this article, we meet pupils who are about eighteen years old and who have Latin or Greek in their programme (or both, which is optional, not compulsory) and five or six other options. Mathematics is covered in two of these options, called Mathematics A and B. Mathematics A contains many applied or applicable subjects (differential calculus, matrix theory and graph theory, probability theory and statistics, linear programming) and in the first place aims at pupils that are preparing for a study at university in psychological, social and economical sciences. Mathematics B consists of the more formal differential and integral calculus, including elementary differential equations and solid geometry. In principle Mathematics B leads to science studies (mathematics, physics, technical disciplines). The two options do not exclude each other, and in fact at my school some pupils do both options. Some figures to illustrate the situation: the school has 468 pupils, of which 63 are in the final year. Of those 63 there are 5 who do no mathematics at all, 35 who do Mathematics A, 15 who do only Mathematics B, and 8 who have chosen both options. The class under discussion here is a Mathematics A class of 16 pupils, so it follows the applied programme, but the eight pupils who do both Mathematics A and B and therefore (should) also master the more technical and abstract subjects are all of them in this class.

The leaving certificate exam in the Netherlands consists of two parts. One part is a written test, organised by the central government, held at the same moment for all schools of the same level. For the other part the school itself is responsible. In my school this part is split in three tests, the last of which is a 30 minutes oral exam. And since candidates are not really acquainted with oral tests in mathematics I always include some try-outs so there we are: a class of 16 pupils just before their exam, following the applied maths programme, but half of the class also doing the formal, technical maths. Pupils of about eighteen, many of them hoping to enter the University.

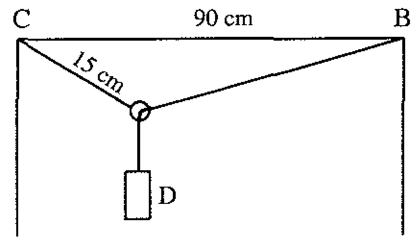

Figure 1
The instrument

A Thursday in February. The day of the announced try-out for the oral Maths A test. When the pupils enter the classroom, they see on my desk the equipment depicted in Figure 1. It consists of a horizontal bar BC. It bears a measuring scale and it has two ropes attached to it at a distance of 90 cm. At the end of the rope attached to B there is a weight D. The other rope is attached to C. It is 15 cm long and has a little ring at the end of F. The ropes and the ring are supposed to have no mass (instead of a ring it would be better to use a pulley, but it was rather difficult to find a pulley that was light enough). The problem now is: if I make the rope which carries the weight pass through the ring and then let it hang freely, can you predict the positions of the ring and the weight when they are at rest?

In this manner I set the problem to my class, and I asked them to try to solve it (or at least to think about it) for a quarter of an hour. This preparing a problem before the questioning begins is in agreement with the structure of the real oral exam, where candidates work on a proposed problem for half an hour after which they are questioned during another half hour, in which discussion the problem that they have prepared has a central position. So at this point the class started to figure out a solution, and we shall return to it in due time. But there is another question to discuss first.

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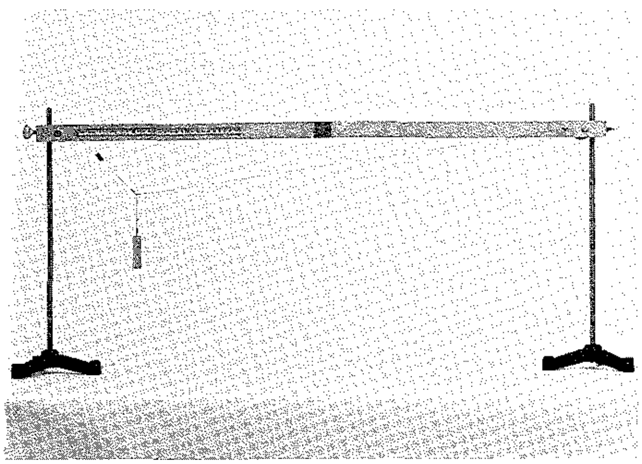

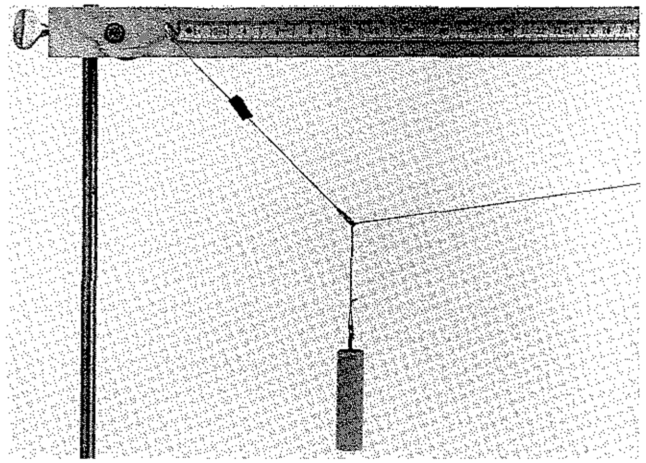

2. What has this to do with history?

The short answer to the question of what this problem has to do with history is that the weight problem is taken from the first textbook in differential calculus, L'Hôpital's *Analyse des infiniment petits* of 1696, a most interesting and very rich book, which should be better known than it presently is. Let me quote the problem in an English translation of L'Hôpital's own words and show you his own diagram (Figure 2), which served as a blue-print for the instrument depicted in Figure 1.

> Let F be a pulley, hanging freely at the end of a rope CF which is fastened at C, and let D be a weight. D is hanging at the end of the rope DFB, which passes behind the pulley F and is suspended at B such that the points C and B are on the same horizontal line. One supposes that the pulley and the ropes do not have mass; & one asks at what place the weight D or the pulley F will be [1]

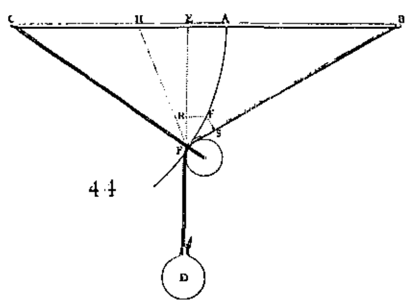

Figure 2:
The diagram in L'Hôpital's *Analyse*

What we see is that history may help us to find (or steal, if you like) interesting problems for an examination, or for other purposes. That is nice and helpful, and everyone will agree that if we are not creative enough to design such instruments ourselves we may very well use history in this way. But is that all? And in this case, with this class, was that the only role that history played? No, it was not, as I shall try to point out now. To do so I have to say a little bit more about the background to the problem. L'Hôpital had a definite purpose in introducing it in his calculus textbook: he used it in his attempts to convince his fellow mathematicians and physicists that differential calculus, which was brand new at the time and not at all fully accepted, is a sound and powerful method. He did so by comparing the new analytical solution of the problem with a classical solution The classical solution used the addition of forces by the parallelogram law and the fact that a body is at rest when the resulting force that acts on the body is zero These physical arguments led to the conclusion that in the geometry of the problem, when some artificial line was drawn, some angles had to be equal, after which the solution was found by a combination of similar triangles and algebra. A really tricky solution The analytic solution used the principle of minimization of potential energy, which means no more than that the weight will search for the lowest possible position, and therefore the only thing that L'Hôpital had to do was to introduce a coordinate system, to express the ordinate of the weight in terms of its abscissa, and to maximize it using differentiation. In this manner the solution is straightforward (as we shall see in the next section); it does not depend on tricks but only on a short algorithm. To summarize: what makes this weight problem interesting is not only the mathematics, but also the motivation behind it. To us it is one problem from a long list of applications of the calculus, to L'Hôpital it is a means to fight for the acceptance of the calculus. I shall come back to this at the end of this article it is now time to return to the classroom.

3. The questioning

The class knew that I would ask a volunteer to come to the blackboard in order to be questioned, while the rest of the class attended the questioning as an audience. May I then now introduce to you Redmer, who volunteered to be questioned? Redmer is one of the eight pupils who do both Maths A and B, and his programme also includes physics. So he seemed an appropriate candidate His first remark was that he had not yet solved the problem, but that he had taken some steps that might lead to a solution. So I asked him to present what he had thought out. He then made a drawing on the blackboard and took a non-calculus approach, which to a certain extent agreed with the classical pre-calculus solution sketched by L'Hôpital. Redmer considered a number of forces, starting with the weight D, and decomposed them into their vertical and horizontal components. This took about ten minutes, and then he got stuck. On an earlier occasion I had already told the class not to worry if this happened. With a sensible reaction to a hint they would still make a good exam. So Redmer was stuck, and I was at a crossroad, since there were two possible hints to give him. One was to guide him through the classical geometrical solution with its network of similar triangles and subsequent algebra. The other was to see whether he would be able to take a completely different stand and solve the problem with the help of calculus. I decided on the latter, for two main reasons.

First, the classical geometrical approaches remain intricate. Similar triangles are involved, or trigonometry, and when the geometry of the problem has been exploited there is still some algebra to be done. The geometrical solution given by L'Hôpital in fact is tricky. However, it can be modified so that it no longer requires an auxiliary line but only the fact that both the vertical and horizontal components of the forces acting upon F add up to zero. This then leads to a little more algebra than in L'Hôpital's own solution, but in the end it leads to success. Redmer had more or less taken this direction, but he was only beginning to use the geometrical data.

The second reason for leading Redmer to the calculus solution was that calculus with applications is one of the major topics in the Mathematics A curriculum, whereas classical plane geometry is taught much earlier than calculus (say in grades 7 to 9). Of course candidates should still have mastered plane geometry by the end of grade 12, but in practice their skills at this point are weak. So I suggested to Redmer to take the calculus approach, and with this advice he solved the problem—apart from some errors in calculations—right away.

Let us take a brief look at the solution, which is Redmer's (and L'Hôpital's) solution slightly modified.

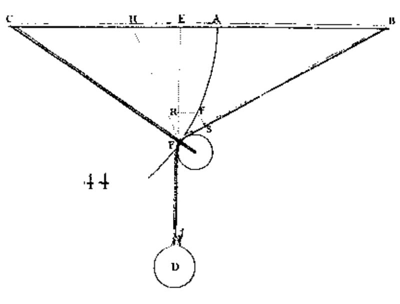

Figure 3:
The diagram again

Given (Figure 3): CB = 90 cm; CF = 15 cm. Take 15 cm as unit of length.

Introduce coordinates such that

$$C(0,0) \quad E(x,0) \quad B(6,0) \quad D(x,y)$$

Furthermore: $CF = 1$.
Call l the length of the rope which carries the weight (so: $BF + FD = l$).
The problem is to maximize y.

Now $y = EF + FD$. We express EF and FD in terms of x.

$$EF^2 = CF^2 - CE^2 = 1 - x^2 \quad \text{whence} \quad EF = \sqrt{1-x^2}$$

FD can be found as $l - BF$:

$$BF^2 = EF^2 + EB^2 = (1-x^2) + (6-x)^2$$

$$\text{so } BF = \sqrt{37 - 12x}$$

and therefore

$$FD = l - \sqrt{37-12x}$$

We now know y:

$$y = \sqrt{1-x^2} - \sqrt{37-12x} + l$$

This y must be maximized, so determine dy/dx:

$$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} + \frac{6}{\sqrt{37-12x}}$$

Putting $dy/dx = 0$ leads, after some calculation, to the equation:

$$12x^3 - 73x^2 + 36 = 0$$

A third degree equation. Redmer remarked here he had a problem unless he could guess a root. And indeed, he managed to find that $x = 6$ satisfies the equation, so

$$12x^3 - 73x^2 + 36 = (x-6)(12x^2 - x - 6) = 0 \quad \Leftrightarrow$$

$$x = 6 \text{ (not acceptable) or } x = -\frac{2}{3} \text{ (not acceptable)}$$

or $x = \frac{3}{4}$ (there it is!!)

So CE=3/4 15=11.25 (cm) and from this the maximal length of ED can be calculated [2]. This result was reached by Redmer right before teh end of the lesson. The final question was whether the predicted value of CE agreed with the value which could be measured at the instrument. And indeed, the agreement appeared to be close.

Here we stop this description of the try-out exam and the mathematics involved. Let's go back to the question: what is the role that history plays in this way of teaching mathematics?

4. How to use history in mathematics education?

Up to this point history played only one role: it served as a source of inspiration for the teacher. No prominent role, but no bad role either. However, only the first act of the play was finished, and in the next lesson, one day later, the curtain opened for the second act. I used this lesson to discuss some aspects of the try-out. The main reactions were:

- the problem was difficult, but fun
- more specifically the pupils found it difficult to find the good beginning; but they agreed that after the hint it was fairly easy to proceed
- many of them were in favour of this kind of problem, since here they could apply maths to solve a realistic problem, not such a one-after-another problem which their textbook is stuffed with.

My contributions to this discussion of the try-out were a brief review of the mathematics of the problem, which went very quickly when Redmer had found his way, and furthermore I revealed to them the origin of the problem. I told them that it already featured in the first textbook on differential calculus, I showed the problem to them on xeroxes from the original 1696 edition of the *Analyse,* and I stressed the parallel between the try-out and L'Hôpital's reason for including the problem in the *Analyse.* Redmer started to take the classical geometrical approach, working with forces. That did not work, or at least not quickly enough. And then he changed to the calculus approach, which led to success quickly. The parallel with L'Hôpital, who in his book advised his colleagues in this and similar cases to use Leibniz's "New Method", is striking. [3]

To summarize I sketch what history has done in this case, and what it may perhaps do in other cases:

- history is a source of interesting mathematics. Even if—as a teacher—you do not use old documents in your lessons directly, it is fun to read them.
- comparison of our present-day mathematics with older techniques enables us to determine the value of our modern mathematics, and to point out this value to our students.
- history shows that mathematics is more than a collection of truths and facts. In the example discussed here mathematics was part of a propaganda machine: it was a weapon used by a youngster against the prevailing but awkward ancient methods. Do not misunderstand the word "awkward" here. L'Hôpital thought the old methods to be awkward since he had a better method. When these first cartesian methods to determine extreme values had been thought out, they were the summit of modern mathematics.
- thanks to L'Hôpital's propaganda, calculus replaced the old methods, and solving extreme value problems could be done almost without thinking. Sometimes this makes calculus a boring subject (to teachers *and* to their pupils). In fact the weight problem is just one of those boring exercises in applying the calculus. The fascination of the problem can only be highlighted in the Dark Past.
- history provokes discussion *about* mathematics, about its methods, about its value for humanity. It helps pupils to make up their minds about what they are doing.

In my opinion, in this try-out examination something essential has happened. History has drawn some lines on the plain face of mathematics. It is an old face, and an old face must have lines. So let's draw them!

Notes

[1] [G F A de L'Hôpital,] *Analyse des infiniment petits. Pour l intelligence des lignes courbes,* Paris (de l Imprimerie Royale) 1696. Text: p 51-52 figure: Plate 4, figure 44. The first edition of the *Analyse* was published anonymously, but later editions appeared under L'Hôpital's name

[2] The difference from the original text by L'Hôpital is that L'Hôpital did not specify the lengths of *CF* and *CB* but took $CF=a$ and $CB=c$. With $CE=x$ as above he found that x has to satisfy the equation

$$2cx^2-a^2x-a^2c=0$$

Since equations with too many parameters often confuse pupils and the problem was already complex enough, a and c were chosen such that the resulting quadratic equation has a "nice" rational solution, an operation which led to another branch of 17th century mathematics: Pell's equation

[3] L'Hôpital had a second, important reason to include the weight problem, which was that the arising extreme value problem involves the square root function, for which earlier methods to calculate maxima and minima — L'Hôpital explicitly attacks Descartes' normal method and Hudde's rule — could not very well be used

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What is required now is that educators of all kinds make themselves vulnerable to the awareness of awareness and to mathematization, rather than to the historical content of mathematics.

Caleb Gattegno
