Communications

A mathematics teacher's apology [1]

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*A comment after reading 'Talking about subject-specific pedagogy', **25**(3)*: This discussion immediately caught my attention after having recently participated in an interdisciplinary course on learning strategies in Norway.

While the course was to be independent of subjects taught, the reception among the teachers was not. Mathematics teachers were on the whole less enthusiastic and reported more often than others that they had not implemented the promoted techniques in their classes since the previous session. The language teachers and social science teachers, of course, assumed that the mathematics teachers (as usual) "don't care". This is unfair. Many of us are deeply concerned about strategies for learning mathematics, but the recommended methods somehow do not fit our needs. On the other hand, being mathematics teachers, we did not argue our case very well, and that is why I write this letter. I want to share a picture I tend to draw for myself.

On the course we discussed material from the US, Carol Santa's CRISS Project, which considers the object to be learned as "a certain area". This wording is to emphasize the existence of "horizontal" relations to neighbouring knowledge as a means to convey meaning. *Surveying* is then a natural and tentative starting point, and, even at a premature stage, attempts to organise the material will both motivate the students and promote their learning. It is my impression that a lot of popular learning strategies are based on this reasonable way of seeing things. But does this picture apply to mathematics? It sometimes does, and sometimes does not.

Mathematical objects are given by their definitions. That is very far from saying that such an object comes with *meaning* as well. Meaning is not a property that belongs to mathematical objects, meaning has to do with our relationship with mathematical objects. Learning about a mathematical object is precisely to gain meaning for mathematical objects.

My thesis is that to gain meaning for a mathematical object, there is no substitute for getting to know *how it works*. Until that is done with some degree of success, the object will be "invisible to the mind's eye", and cannot be organised, nor located in any landscape of knowledge [2] – however crystal clear this position might be in the teacher's mind. Such work on an initially meaningless object can be quite a demanding thing to do, and not always easy to schedule. I therefore picture the process of learning mathematics more as *breakthroughs* [3] on particular *points* than gradually *covering an area*. Organising such breakthroughs into a body of knowledge is then a *finalising* thing to do.

I will therefore elaborate on Dave Hewitt's and Kath Cross's view that a difference of degree is to be found. In my view, the timetable can be said to be *anti-symmetric* in the sense that in science or social sciences, I expect to benefit from starting with the broader picture and *then* go for close-up or in-depth studies, whereas in mathematics the sensible thing for me to do is pretty much the opposite.

It may very well be that we mathematics teachers take the finalising part of the job too lightly. But I think there can be little doubt about the reason why many of us show a lukewarm attitude towards central parts of learning strategies. It is because we consider the *breakthrough* part of the job to be the more important and difficult one [4].

So what might be the characteristics of a *pedagogy for breakthroughs*? One thing that quickly comes to mind is more like an attitude than a particular skill. If I pick one single thing that I would wish my mathematics students to obtain from my teaching, it would be to increase their ability to bear or tolerate *not seeing the solution*. Or rather to realise that *not seeing the solution* to the (mathematics) problem you are facing is a normal stage in the process of doing mathematics.

Is it possible to teach students such a thing? By instruction, surely not. Perhaps by modelling. I read something last summer that I found interesting, from Brown – *What should be the output of mathematical education?* [5] Let me quote without further comment:

> I learned a tremendous amount from my supervisor Michael Barratt. I remember thinking after a long session with Michael: 'Well, if Michael Barratt can try one damn fool thing after another, why can't I?' I have followed this method ever since! [6]

I also have some heretical thoughts about the celebrated meta-perspective in learning. These thoughts are based on particular experiences that I have had with adults trying to learn fairly elementary mathematics, and are consequently perhaps of limited validity. I shall therefore here restrict myself bluntly to stating that bringing students with troublesome learning histories to view themselves as *students learning mathematics* is often not a useful way to start.

The emotions that follow pictures of oneself as a scared child behind a desk in a huge classroom struggling with quite incomprehensible mathematical signs may help you at the psychologist's, but are not part of productive learning environments. Emotions like that are obstacles that prevent the student from getting in touch with mathematics. If we could bring such students – if only for a while – to forget completely about the meta-perspective that is haunting them, they might get a glimpse of 'the promised land' and subsequently experience motivation of a kind hitherto unknown to them. Whether this is specific to mathematics I do not know.

Notes

[1] With apologies to Hardy who wrote *A mathematician's apology*.
[2] In Kantian terms, knowing a mathematical concept is to be able to "construct it in the intuition". It is this construction, an act (not a thing), which, hopefully, thereafter can find a place in some cognitive structure.
[3] My notion *breakthrough*, as an undefined term, is to be understood naturally.

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[4] I suppose Fermi made the same judgement when he replied to the student who asked for an account of elementary particles, "If I could remember the names of all these particles, I'd be a botanist." The kicks are from the breakthroughs, not the bookkeeping.
[5] Sierpinska, A. and Kilpatrick, J. (1998) *Mathematics education as a research domain: a search for identity*, Dordrecht, The Netherlands, Kluwer Academic Publishers, p. 468.
[6] It is hard to image Barratt (and Brown) spending "long sessions" doing "one damn fool thing after another" on *different tasks* – they're obviously struggling with one single problem. The situation then fits nicely to the notion of breakthrough.

Researching linguistic discrimination

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*A response to 'Impact of language on conceptualization of the vector', Hayfa, **26**(2)*: Hayfa (2006) discusses how semiotic representations of vectors in mathematics textbooks can be understood through different registers such as verbal, geometric, algebraic and analytic. Hayfa's key point is that particular semiotic representations lead students to specific forms of reasoning. The form of reasoning, based upon the semiotic representations, impact the ultimate resolution of a problem for a student.

The word register has been widely defined, both in mathematics and linguistics. Halliday (1978) for example, says that

> [a] register is a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings. (p. 195)

Pimm (1987), consistent with Halliday, proposes that

> [r]egisters have to do with the social usage of particular words and expressions, ways of talking but also ways of meaning. (p. 108)

Halliday (1978) talks in more specific terms about the mathematics register as

> meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

I question the usefulness and intent of Hayfa's (2006) defining of semiotic representations as distinct registers. Are semiotic representations simply alternative representations of the same ideas within the mathematical register?

Hayfa (2006) says,

> [i]n general, the register or language of the statement of the problem leads to the register of the reasoning. (p. 39)

I am not convinced that this is necessarily the case, despite Hayfa's claims and correlations. I have found from my own teaching of secondary mathematics that students' reading and interpretation of a problem is highly subjective and informed by mathematical baggage carried by the student. Sfard (2000a, 2000b) supports these perceptions of students' conceptualization; that is, students' sense making is a complex process that is informed by a multitude of factors, including prior learning, social, degree of shared focus and cultural.

Thus, the register of a mathematical problem and the register of reasoning can potentially be unrelated in instances where conceptual misunderstandings prevail. On occasions where understanding *is* achieved by students, the register of the problem and the register of reasoning may still differ. Furthermore, students may be unaware that the register of the mathematical problem embedded within the problem is a clue to a desired register of reasoning.

I question also the importance and necessity of consistency between the register of the problem and the register of reasoning if the reasoning is justified. *Are alternative representations acceptable as resolutions to mathematical problems?* Is a graphical representation less acceptable than, say, an algebraic representation? On some mathematical occasions, yes, it might be the case that a highly specific resolution is of the utmost importance. However, as the National Council of Mathematics Teachers (2000) suggests, students must be encouraged to show their thinking in a variety of ways. This creates added dilemmas for educators who might find comfort in a particular form of resolution (*e.g.*, graphical versus algebraic) based upon their own content knowledge and therefore their own register of reasoning.

Hayfa's (2006) article leaves me thinking that that she is suggesting that greater clarity and purpose in mathematical problems will result in greater incidence of an appropriate register of reasoning and thus a greater incidence of anticipated, shared or desired resolution. Shared resolution between teacher or textbook and students need not imply that the choice of representation, or register, is the same (*i.e.*, as in identical) provided that the ideas are congruent. Approaching mathematics from this perspective requires a considerable amount of content and pedagogical knowledge by the teacher (Shulman, 1986).

The dissection of the mathematical register that Hayfa (2006) makes is interesting, but is it necessary? More pedagogically pressing is understanding how linguistic discrimination (*i.e.*, making sense of the use and purpose of words in mathematical contexts) within the mathematics register occurs for students (Pimm, 1987). Hayfa's research *can* perhaps contribute in this way by providing a framework for such analysis.

Understanding linguistic discrimination may yield some insight into the sorts of conceptual challenges and interferences experienced by students making sense of language used in mathematics. Theorizing about how language informs students' conceptualization from student work alone may not provide sufficient insight into how linguistic discrimination occurs. More direct research with children is needed to make sense of linguistic discrimination and how it informs conceptual understanding. Ultimately, we need to ask children about their (mis)understandings and what factors influence the particular representations they make. As researchers and educators, we need to be asking children about their own register of reasoning.

References

Halliday, M. A. K. (1978) *Language as social semiotic: the social interpretation of language and meaning*, London, UK, Edward Arnold (Publishers) Ltd.

Hayfa, N. (2006) 'Impact of language on conceptualization of the vector', *For the Learning of Mathematics* **26**(2), 36-40.
National Council of Mathematics Teachers (2000) *Principles and standards for school mathematics*, Reston, VA, NCTM.
Pimm, D. (1987) *Speaking mathematically: communication in mathematics classrooms*, New York, NY, Routledge and Kegan Paul.
Sfard, A. (2000a) 'Symbolizing mathematical reality into being – or how mathematical objects and discourse create each other', in Cobb, P., Yackel, E. and McClain, K. (eds), *Symbolizing and communicating: perspectives on mathematical discourse, tools and instructional design*, Hillsdale, NJ, Lawrence Erlbaum Associates, pp. 37-98.
Sfard, A. (2000b) 'Steering (dis)course between metaphor and rigor: using focal analysis to investigate the emergence of mathematical objects', *Journal for Research in Mathematics Education* **31**(3), 296-327.
Shulman, L. (1986) 'Those who understand: knowledge growth in teaching', *Educational Researcher* **15**(2), 4-14.

In search of practical wisdom

ALF COLES

A response to 'In search of practical wisdom: a conversation between researcher and teacher', Goos and Geiger, **26***(2)*: Reading Goos and Geiger's article in the last issue created strong resonances – both similarities and differences – with my own experience. Having met Merrilyn Goos at PME30 in Prague in a Research Forum on *Teachers working with university academics* adds another dimension and I am going to refer to the authors by their first names because of this.

I am a mathematics teacher and, like Vince, have been involved in a long-term collaboration with a university academic, Laurinda Brown. A difference is that I was an inexperienced teacher and Laurinda had worked for fourteen years in a secondary school before moving to a university position. The fact that Laurinda's research was initially about teacher development meant that, despite the disparity in our experience, I did not feel as though I was having anything 'done to me'. In some ways, I felt like Merrilyn when we first started working together in that "I had no world of my own" professionally and was uncertain about my future as a teacher. I remember first going to conferences, both professional and academic, and, through Laurinda's contacts, feeling as though I had access to some sort of 'elite' (whose work I had read *e.g.*, during my teacher training year or later as part of a Master's in Mathematics Education). This was exciting and energising.

In the early years of our collaboration, we would meet up once a week to plan the lessons that Laurinda would be coming in to observe and sometimes co-teach. These meetings always entailed us working together on the mathematics. I think Laurinda enjoyed having an outlet for fourteen years of accumulated experience of teaching. The starting point of our planning would usually be something she had seen or done before. Over time the issue of my own contingency in my classroom became important to me – I wanted to be in a position to respond to what students did without feeling I needed to force the lesson down a particular track.

I remember a phase in our planning when we would plan certain inputs not in time but in relation to what might happen. For example, in a lesson that involved starting with a class of number puzzles for students to explore we planned that I would offer an algebraic way of solving a puzzle at the time when students seemed convinced that some were impossible. We would also sometimes plan for different eventualities in a classroom – if the students do A then I could do B, or if they do C then D and so on.

Later, I remember we used a phrase 'planning to conviction' that, for me, signified I had got to a place where I knew how I would begin the lesson and was sure that there was enough interesting mathematics involved in the starting point for me to be able to follow questions and leads from the students. There was also something in the phrase about having conviction that this was a lesson for which it was worth my while expending energy on keeping some student behaviour boundaries in place. At this stage, I was becoming comfortable working contingently with the responses of students – my focus had perhaps shifted to being on the mathematical thinking that was occurring and facilitating that, rather than on the content and whether we were or were not straying from any intended path 'through' it.

A lesson that in retrospect I think signalled this shift is one I recall when I had wanted to work with a group of 14-15 year old students on solving simultaneous equations. The lesson beginning, which was an idea of Laurinda's, was my setting the challenge: "Find two numbers such that double the first add the second comes to 48". I was immediately asked to repeat this and responded by asking if anyone could tell me how to write what I had said. A student offered how the challenge could be written algebraically (*i.e.*, $2x + y = 48$). After pausing to give students some time in silence to find a solution, I wrote down students' answers on the board. I remember there being surprise from some students that their solution was not unique and we discussed how there would in fact be an infinite number of solutions.

I continued (this was still part of the original lesson planning) by inviting students to make up a second equation that would limit, somehow, the number of solutions we had obtained from the first equation. As we worked on making sense of students' second equations and how we would find solutions I became aware that a significant difficulty many in the class were having was that they were making errors when substituting negative numbers into equations. After some time of being aware of this difficulty and it recurring in discussion I decided to place 'on hold' further work on simultaneous equations in order to have a focus on operations with negative numbers. I invented ten questions for students to do, trying to capture all the different ways negatives can come in to simple calculations (*e.g.*, from 4×-8, to some substitutions) and we spent the rest of the lesson discussing students' answers. A student commented on how there seemed to be two meanings to the negative symbol – a comment which I remember others said helped them. In the next lesson we returned to the simultaneous equations.

This was perhaps the first time I had contingently offered students an activity, not previously planned, based on my awareness of their responses. Students worked on a skill with a sense of why it was needed and had to immediately apply it back in the context from which the need arose.

One or two mathematics staff at my school have occasionally referred to me as a 'creative teacher' (usually in supposed contrast to themselves). I have never seen it that way. My lesson ideas are rarely original in the sense of my creating new activities. I have spent a lot of time and energy in my life thinking about mathematical activities that allow me to work in certain ways with classes. It is not surprising that I can draw on that wealth of experience in planning and in the flow of a lesson but this does not mean I have any special gift.

I like Merrilyn and Vince's phrase "practical wisdom" to denote something of what is needed to bridge university and school domains of discourse. Wiliam problematised the issue of how mathematics education research can deliver knowledge that is both widely shared and used in practice:

> [...] research results that have widely shared meanings appear to be more difficult for teachers to 'make sense of' and to make use of in improving their practice. [...] 'action research' addresses this by not even trying to generalise meanings across readers – what matters is the meaning of the research findings for the teacher in her own classroom [...] Put crudely, in action research, the lack of shared meanings are justified by the consequences, while in other kinds of research, the lack of consequences are justified by their more widely-shared meanings. (1999, p. 327)

This echoes Merrilyn's comment that there are "two mathematical education communities". Four years after Wiliam's paper, Breen (2003) suggested there had been little movement in terms of finding ways for teacher-research to gain wider applicability:

> On the one hand, there is a growing movement for more teachers to become involved in a critical exploration of their practice through such methods as critical reflection, action research, and lesson studies. The contrasting position makes the claim that these activities have done little to add to the body of knowledge on mathematics education. (p. 2)

Vince's experience on entering university life that "some academics didn't think the type of research I did should be taken seriously" seems evidence that Breen's "contrasting position" is indeed around.

The notion of "practical wisdom" suggests to me knowledge that can both be shared and that has consequences for practitioners. Jaworski (2005) has coined the phrase "co-learning partnerships" to describe working relationships such as Merrilyn and Vince's or my collaboration with Laurinda in which both partners take responsibility for learning and development within their respective roles. She believes such partnerships are one way to add to the body of knowledge:

> There is a growing body of research which provides evidence that outsider researchers, researching the practice of other practitioners in co-learning partnerships, contribute to knowledge of and in practice within communities of which they are a part. (p. 2)

The key phrase in this quotation for me is "within communities of which they are a part". It is perhaps part of human nature that 'outsiders' are mistrusted. I know that many teachers I work with will dismiss the findings or suggestions of outsiders if they get a sense that such people, for example, work in a more privileged setting or do not have a similar experience of teaching. This mirrors Vince's experience of being distrusted in a university setting because of his exclusively school-based background. What is perhaps powerful about partnerships between teachers and university academics is that together there is membership of the "two mathematical education communities".

References

Breen, C. (2003) 'Researching teaching: moving from gut feeling to disciplined conversation', *South African Journal of Higher Education* **16**(2), 25-31.

Goos, M. and Geiger, V. (2006) 'In search of practical wisdom: a conversation between researcher and teacher', *For the Learning of Mathematics* **26**(2), 37-39.

Jaworski, B. (2005) 'Learning in practice from a study of practice', available from, http://stwww.weizmann.ac.il/G-math/ICMI/Jaworski_Barbara_ICMI15_paper.doc, accessed 31st August, 2006.

Wiliam, D. (1999) 'Types of research in mathematics education', in Zaslavsky, O. (ed.), *Proceedings of the twenty-third annual conference of the International Group for the Psychology of Mathematics Education*, Haifa, Israel, Technion, Israel Institute of Technology **4**, pp. 321-327.

[These references follow on from page 32 of the article "Lortie's apprenticeship of observation revisited" that starts on page 30 (ed.)]

Romberg, T. and Carpenter, T. (1986) 'Research on teaching and learning mathematics: two disciplines of scientific inquiry' in Wittrock, M. (ed.), *Handbook of research on teaching (third edition)*, New York, NY, Macmillan Publishing Company, pp. 850-873.

Ross, E. W. (1987) 'Teacher perspective development: a study of preservice social studies teachers', *Theory and Research in Social Education* **15**(4), 225-243.

Thompson, A. (1992) 'Teachers' beliefs and conceptions: a synthesis of the research', in Grouws, D. (ed.), *Handbook of research on mathematics teaching and learning*, Reston, VA, NCTM, pp. 127-146.

Wideen, M., Mayer-Smith, J. and Moon, B. (1998) 'A critical analysis of the research on learning to teach: making the case for an ecological perspective on inquiry', *Review of Educational Research* **68**(2), 130-178.

Zeichner, K. and Gore, J. (1990) 'Teacher socialization', in Houston, W. (ed.) *Handbook of research on teacher education*, New York, NY, Macmillan Publishing Company, pp. 329-348.