Integrating the Close Historial Development of Mathematics and Physics in Mathematics Education: Some Methodological and Epistemological Remarks

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1. Introduction

Mathematics is conventionally presented deductively, either because this is the quickest way of presentation [1] and/or because the logical clarity to which it may lead is often considered to be identical with the possibility of a complete understanding of the subject By 'understanding', we mean the creation by the learner of links between new information and his or her already-existing conceptual framework (cf. Hiebert and Carpenter, 1992, p. 67) In this way, new information acquires a meaning and becomes knowledge that can be used by the learner, in order to answer questions, solve problems and more generally, organize thought.

On the other hand, the historical development of mathematics suggests that a deductive (or even strictly axiomatic) organization of a mathematical discipline is given only after this discipline has reached maturity, so that it becomes necessary to give an *a posteriori* presentation of its logical structure and completeness [2] However, such a deductive approach usually presupposes a global reorganization of the discipline, often hiding questions and problems, which constituted basic motivations for its development Therefore, a deductive approach can be useful to those who already know the subject, or at least have enough acquaintance with it or related subjects. This is a fact which has been recognized for quite a while [3]

Essentially, a solely deductive approach separates elements which are closely interrelated, namely the learner, the learning process and the knowledge Learners appear as oversimplified receptors of new knowledge, not appreciably modifying their conceptions under its influence. Moreover, its deductive presentation tacitly suggests a straightforward learning process, one in which new knowledge is structured, used and transferred directly and instantaneously. Finally, the fact that new knowledge has a meaning is ignored, not only when it is (mathematically) true, but also when it turns out to be *interesting* for the learner, through becoming linked to previous knowledge in the sense described at the beginning of this section (Brousseau, 1983, pp. 169-171) Therefore, the conventional scheme of a linear, continuous, cumulative learning process, tacitly used to legitimize a deductive approach, presents an oversimplified picture which should be modified.

For a long time, many people involved in mathematics education and looking for another approach have argued that history of mathematics may play an important role in improving the teaching and learning of mathematics (Arcavi, 1985 and references therein). In recent years, there has been an increasing interest in this role (Fauvel and van Maanen, 1997). More precisely:

> the meaning of a concept [theorem or method] is not completely determined by its modern definition [or formulation], but it results from its development in the past as well as in the present. (Sierpinska, 1991, p 13; our translation)

Therefore, a careful 'reading' of the historical development of mathematics may open new possibilities of *understanding* in a natural way how mathematics is created: hence, it may also suggest alternative ways to *present* mathematics and varied conditions under which it can be understood. However, the ways in which knowledge of the historical development of mathematics can influence the teaching and understanding of particular mathematical domains are not obvious At this point, we remark that the above comments are not restricted to mathematics alone, but are equally valid for *physics* as well

Therefore, by adopting a *unified* perspective in the present article, we aim to contribute to the clarification of (some aspects of) the role which history can play in the presentation of *both* mathematics and physics, in response to questions 8 and 5 raised by Fauvel and van Maanen [4] In this connection, and for the sake of brevity, we will occasionally use the term 'science' to denote mathematics and/or physics By considering that an essential part of science itself is the process of 'doing science', we conclude in section 2 that the procedures involved in this process should be analyzed.

In this context, we describe the different types of reasoning in scientific research activity and, supported by historically important examples, suggest that all of them are important in both doing and understanding science. Hence, their use must be explicitly encouraged in science education This leads naturally to an outline in section 3 of three different but complementary possible ways to integrate history in the presentation of science. (In the Appendix, by means of an example, we give a possible illustration of this integration.)

An important point in our analysis in sections 2 and 3 is the recognition of the very close relation between mathematics and physics (which is often ignored in teaching these disciplines), as well as discussing the relations between history of science and science education. We believe that making this relation explicit in the presentation of

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mathematics and physics may be beneficial for an understanding and learning of both disciplines, for reasons explained in section 4 with the aid of historical examples. Finally, a few explanatory comments are given in section 5

2. Types of reasoning and the role of history

An analysis of the role(s) of the history of science in science education refers implicitly to the question "what is science?", an answer to which is necessary in order to formulate a proposal of what to teach and how to teach it. Though the study of this question falls outside the scope of this article, we should mention here that its basic thesis is the point of view that science is not exhausted in its results, i.e. in what can be explicitly stated as scientific knowledge (definitions, axioms, theorems, theories). Equally important is the process of 'doing science', which incorporates discovering procedures, mistakes, misconceptions, retrogressions in the development and understanding of a subject.

Hence, the *meaning* of scientific knowledge is determined not only by those circumstances in which this knowledge becomes a scientific theory, structured deductively, or in which the learner has used it as a means of solving problems It is also determined by the discovery procedure which the learner has followed, and which includes prior conceptions, earlier choices, revised formulations, the economy of thought that has been reached In this context, the didactic interest of a question or problem depends essentially on what the learner used, investigated and presumably discovered, on how important it was for him or her to reject previous, erroneous ideas when forced to do so, on how often the risk was run of being captured by the rejected, erroneous ideas and on their relative importance

These comments are fundamental, in order to be able to determine what may be called a true problem or question and what makes new knowledge interesting (Brousseau, 1983, pp 170-172). At this point, by providing many such examples, history of science can play a crucial, constructive role in science teaching (see the next section) However, this role cannot be realized in the context of the linearly organized, cumulative, conventional presentation, but only in another in which heuristic, discovery procedures are explicitly encouraged. Therefore, in order to reveal this role more clearly, in the rest of this section we discuss in some detail the types of reasoning involved in such procedures, by means of indicative historical examples

A careful analysis of research activity in mathematics and physics and the presentation of its results, as well as of the historical evolution of mathematical and physical ideas, concepts or theories, leads to the conclusion that, in general, we can distinguish three different types of reasoning.

(i) Deductive reasoning:
on the basis of which complete mathematical proofs can be given and the foundations of a theory can be laid. Therefore, in this way, the validity of scientific results and their subsequent acceptance by the scientific community is ensured. Moreover, a logically clear presentation of a scientific domain can be given, which thus appears as a firm edifice.

For the sake of brevity, formalized procedures (e.g. arithmetic, algebraic or analytic computations) are included in this category, though they differ some from more synthetic, deductive reasoning (e.g. like that often appearing in Euclidean geometry). However, this point will not be discussed here.

(ii) Inductive reasoning:
on the basis of which

(a) statistical inferences are based and conclusions from empirical facts are drawn;

(b) conjectures concerning the generality of certain theoretical results are formulated on the basis of the examination of many, *similar* cases (e.g. the four-colour problem, that any map on the plane or on the sphere can be coloured using four colours only so that any two adjacent regions have different colours);

(c) *confidence* is acquired in the validity of a conjecture, working hypothesis, etc., which has been verified in many cases or which has been used for a long time without leading to contradictions (see e.g. Polya, 1954, p. 22 and p. 30, example 21)

One instance of (c) is the still unproved case of Goldbach's conjecture in number theory formulated in 1742 (any even integer is the sum of two primes), which however has been checked for numbers up to the order of 10^{14} The same holds for the idea of the molecular structure of matter, which had already appeared as a theoretically reliable hypothesis at the beginning of the nineteenth century (Brush, 1983, sections 1 6-1.13), and played a central role in the development of chemistry and kinetic theory, though it was definitely confirmed only in 1909 by Perrin's experiments based on Einstein's work (section 2 7) Incidentally, we notice here that through kinetic theory and statistical mechanics this hypothesis played a key role for important developments in probability theory, exerting a still-continuing fruitful feedback on physics in the twentieth century (e.g. the mathematical theory of Brownian motion and more generally stochastic processes and ergodic theory).

(iii) Analogy arguments:
on the basis of which new concepts or theories can be formulated or already existing ones can be further explored. Here, analogy is meant in the sense of a loose (or, occasionally, even strict) similarity (isomorphism) of structures with which *a priori* different sets of objects are equipped Analogy is a basic mechanism for formulating conjectures [5] Analogies are essential for generalizing concepts, theories, or methods so that:

(a) They become applicable to classes of objects which have a different nature from those to which these concepts, theories or methods originally apply Often this is the way in which one manipulates concepts referring to new objects, but the properties which one knows of from other familiar cases.

For instance, the concept of the integral of functions of one real variable is extended to functions of several real variables or a complex variable, by requiring that the generalized concept have the basic properties of the Riemann integral of functions of one real variable (linearity, positivity for positive functions, geometric interpretation as area, etc.). This is a basic mechanism in algebra as well: for instance, the abstract vector space concept is defined by keeping the basic properties already known in particular, conceptually different cases (e.g. vectors in geometry, solutions of systems of linear algebraic or differential equations).

(b) New research domains appear by naturally establishing an appropriate conceptual and methodological framework. This is often the case in physics: for example, the mathematical analogy between geometrical optics and classical mechanics, known to Hamilton, led to Schrödinger's wave mechanics as the necessary generalization of classical mechanics, *by analogy* with the way wave optics is the generalization of geometrical optics.

More specifically, based on this analogy, Hamilton developed a unified mathematical approach to the description of these two theories, one in which they appear as different but isomorphic structures. Schrödinger's crucial argument was based on the remark that since we know that geometrical optics is only an approximation to exact wave optics, we may look for a new (wave) mechanics, such that classical mechanics is an approximation to it, in such a way that the above mentioned isomorphism is preserved. This was sufficient for arriving mathematically at the formulation of the partial differential equation now bearing Schrödinger's name and which is the cornerstone of wave mechanics (for a reconstruction of Schrödinger's approach, see Tzanakis, 1998).

Didactically, Hamilton's approach may be used at the undergraduate level, either as a method for solving first-order partial differential equations (the so-called Jacobi method) or as a starting point for developing analytical mechanics (more precisely, the Hamilton-Jacobi theory of solving mechanical problems), depending on whether one wants to stress the mathematical or physical aspects of the subject, respectively (Tzanakis, in press, section 3.3).

From what has already been said, the less firm *discovery procedures* are mainly (though not exclusively [6]) related to *inductive reasoning* and *analogies*, whereas the subsequent *logical arrangement* and *ensuring of the validity* of its consequences are achieved mainly by *deduction* (this is consistent with Polya's (1954, p. vi) distinction between 'demonstrative' and 'plausible' reasoning). Therefore, given that besides information and practical activities a basic constituent of learning is understanding (in the sense of section 1), it becomes clear that explaining the motivation, questions, problems and the discovery process that led to new consequences, as well as the more general cultural atmosphere that may have influenced their appearance, should be made explicit in any presentation. It is at this point that history offers many interesting possibilities to which we now turn.

3. The role of the history of mathematics and physics in their presentation

One may argue that an obvious possibility to use history in the presentation of a mathematical and/or physical subject is to retrace its historical evolution. However, the formulation of the problems which led to its birth, and are presented today as part of modern science, would be too advanced for the learners or may look completely foreign to them (Barbin and Douady, 1996, p. 161). Usually, its *strictly* historical presentation is not didactically appropriate, even at university level, because the historical evolution of a scientific domain, contrary to what is sometimes naively assumed (especially in elementary and high school), is almost never straightforward and cumulative. On the contrary, it is rather complicated, involving periods of stagnation and confusion, full of prejudices, misconceptions, etc., and is greatly influenced by the more general cultural *milieu* in which this evolution takes place.

In addition, the conceptual framework and the mathematical terminology and notation used varies from one period to another: hence, if the presentation strictly respects the historical order, understanding the subject becomes more difficult. [7] Therefore, what the teacher or author could do motivated by history is to find one or more situation from which the subject to be presented will emerge. In this context, in the presentation of mathematics and physics, history may appear in at least three different but *complementary* ways, in the sense that each one, if taken alone, is insufficient to exhaust the multifarious constructive influences history can exert on mathematics and physics education.

3.1 Providing historical information directly

The usual deductive presentation may be kept, but historical comments, historical introductions or historically presented bibliographical surveys are added. In this connection, the role of history is mainly to give particular factual information, e.g. names, dates, events, the historical order of appearance of ideas, concepts, problems or theories presumably by also giving references to original works. This is an auxiliary way of integrating history which, by itself, does not modify the structure of the presentation of particular scientific content.

Nevertheless, it advocates the point of view of mathematics and physics as human activities in time: hence, it may act towards destabilizing the widely spread misconception of them (which is especially strong for mathematics) as systems of rigid truths given once and for all. However, given that the emphasis in this connection is on history rather than on scientific topics, the learner is not helped to place him- or herself at the heart of creative scientific activities, something which can lead to a deep understanding of science

3.2 Using original and/or secondary sources

As already mentioned, a strictly historical presentation of a subject is usually confusing rather than enlightening Nevertheless, it is often possible to develop aspects of the subject on the basis of original sources, e.g. by structuring (part of) a course based on them or by including small extracts in worksheets given to the learner. (For a detailed discussion of possible practical implementations, see e.g. Arcavi *et al.*, 1982; articles in Fauvel, 1990; Barbin and Douady, 1996, Ch. I.1 and III.1.)

A critical reading of such sources gives the opportunity to both the teacher and the learner:

(i) to appreciate the general conceptual framework and associated questions and problems which have led to the development of (aspects of) particular mathematical domains (e g. Hamilton's ideas about the generalization of complex numbers, so that this generalization admits a geometric interpretation, analogous to that of complex numbers - see Tzanakis, 1995)

(ii) to see that science is evolving not only in its content but also in its *form*: notation, terminology, favoured computational methods, modes of expression and representations Therefore, history may help students to understand the mathematical (verbal or symbolic) language of a given period and to compare it with its modern form: for instance (a) the convenience in making calculations with the aid of the currently used Leibnizian notation for derivatives compared with that of Newton; (b) the advantages of the modern notation and terminology of vector analysis compared with its premature form that appears e.g in Maxwell's (1873/1954) work on electrodynamics.

(iii) to have at least a partial understanding of the difficulties faced by mathematicians and physicists, in connection with specific questions and problems: for instance (a) the complicated character of the purely geometrical proof Galileo used to derive the laws of uniformly accelerated motion (Galilei, 1638/1954), a result which can be obtained today in high school by elementary calculus or even by a simple algebraic analysis of appropriate geometrical diagrams; (b) the cumbersome form of metric Euclidean geometry and trigonometry in the context of ancient Greek mathematics, due to the absence of a sufficiently convenient algebraic (symbolic) language. See e.g Archimedes' solution of cubic equations, Hero's formula for the area of a triangle or Ptolemy's proof of elementary trigonometric identities (Thomas, 1939, chapters XVII(c), XXII(b), XX(b) respectively).

This is the case in which it is the learner who, under supervision, comes into direct contact with the historical evolution of the subject. (i)-(iii) above provide the opportunity to think critically about science in the past and to become more aware of social and cultural aspects of scientific activities Therefore, learners are helped to see mathematics and physics as an ever-changing human activity, to understand its relative nature (with respect to time), while recognizing those factors, whether inherent to science or external to it, which have played a decisive role in the way a certain scientific domain evolved

However, a word of caution is needed here concerning the use of original sources in teaching *today's* mathematics and physics: usually, the difference between the content of original works and its modern formulation is so large that in following such a procedure care must be taken so that enough time is left to the learner, in order to have sufficient practice in the use of presented concepts, methods or theories and through that to obtain an (at least partial) understanding of them (cf the above examples) Original sources may be used complementarily to the approach described in the next sub-section

3.3 Following an approach inspired by history

This is an approach which is neither strictly deductive nor historical, but its fundamental thesis is that a subject is studied only after one has been *motivated* enough to do so This means that questions and problems, which the presentation of the subject may answer, have been sufficiently elucidated and appreciated (cf Toeplitz, 1963; Edwards, 1977) Thus, the subject (e g a new concept or theory) must be invaluable for the solution of problems, so that the properties or methods connected with it appear *necessary* to the learners, so that they become able to solve these problems

This character of necessity of the subject constitutes the central core of the meaning to be attributed to it by learners (Barbin and Douady, 1996, p. 161). In this sense, the emphasis in such an approach is less on *how to use* theories, methods and concepts, and more on *why* these theories, methods and concepts provide an answer to specific mathematical problems and questions, without however disregarding the 'technical' role of scientific knowledge (cf. Sierpinska, 1991, section II)

From such a point of view, history offers interesting possibilities for a deep, global understanding of the subject, according to the following general scheme (Tzanakis, 1996).

(a) The teacher or author, though not necessarily a historian, has a *basic* knowledge of the historical evolution of the subject.

(b) On the basis of (a), the *crucial steps* of this historical evolution are appreciated, by identifying key ideas, questions and problems which provided new research perspectives

(c) These crucial steps are *reconstructed* in a modern context, often using modern terminology, notation and conceptual frameworks so that they become didactically appropriate

(d) Many *details* in (c) are presented as sequences of exercises, often historically motivated, of an increasing level of difficulty such that each one presupposes (some) of its predecessors.

Concerning this scheme, the following remarks are helpful:

(i) In contrast to sub-section 3.2, it is mainly the teacher or author who (presumably) comes into contact with original sources

(ii) Mainly in (b) (and partly in (c)), the teacher or author makes an effort to grasp the difficulties inherent in the subject, to unearth possible obstacles in its understanding. Then, motivated by history, he or she can understand and subsequently choose those questions and problems which could activate the curiosity of the learner, by creating and/or explaining the necessary motivations for studying new theories, methods and concepts

In this way, one could have an answer to the important question put forward by Brousseau (1983):

> A pupil doesn't do mathematics [or physics], if he is not given problems [to solve] and does not solve problems. Everybody accepts this fact. Difficulties arise once it is required to know which problems must be given to him, who poses them and in what way (p 167, our translation)

At this level, inductive reasoning and analogies dominate as creative and discovery patterns, emphasizing the activity itself, rather than the well-organized arrangement of its results (see section 2)

(iii) In the reconstructions in (c), history may enter either explicitly or implicitly. This duality has been stressed by several authors, from Toeplitz's work to recent research - see e.g. the distinction between "direct and indirect genetic approach" (Toeplitz, 1927, 1963), "forward and backward heuristics" (Vasco, 1995, pp. 61-62), "explicit and implicit use of history" (Schubring, 1978, 1988; Menghini, 1998, section 2)

In a reconstruction in which history is *explicitly* used, scientific discoveries are presented in all their aspects. Different teaching sequences can be arranged according to the main historical events, in an effort to show the evolution and the stages in the progress of mathematics and physics by describing a certain historical period (Menghini, 1998, p. 3; cf. Hairer and Wanner, 1996, preface; Friedelmeyer, 1990, especially p. 3)

In a reconstruction in which history is *implicitly* used, history *suggests* a teaching sequence, one in which use may be made of concepts, methods and notations which appeared later than the subject under consideration, always bearing in mind the general didactic aim, namely to *understand* mathematics and physics in its *modern* form. In such an approach, the teaching sequence does not necessarily respect the order in which the historical events appeared: rather, one looks at the historical development from the *current* stage of concept formation and logical structuring of the subject (for examples, see Kronfellner, 1996; Flashman, 1996; Tzanakis, 1995, 1999).

At this point, it is important to stress that the above two possible types of reconstructions of the historical development are *not mutually exclusive*: rather, they have a dual character with respect to each other and both may be used in the presentation of a subject in *complementary* ways (cf. Ofir, 1991, p. 23). In an explicit use of the history, emphasis is on a rough but nevertheless more or less accurate mapping of the path network that appeared historically and led to the modern form of the subject; in an implicit use, the emphasis is on the redesigning, shortcutting and signalling of this path network (Vasco, 1995, p. 62)

(iv) A reasonable criticism of such an approach is that it takes time and/or leads to voluminous textbooks. However, the presentation may be 'compact' in the sense of (d) above, at the same time giving the opportunity to the learner to arrive at presumably non-trivial results, starting from easy corollaries of the main subject and often following the main steps of the historical path (see Tzanakis and Coutsomitros, 1988; Tzanakis, 1995, 1999).

In this way, the solution of exercises becomes an essential ingredient of the presentation, leading to the acquisition of the necessary 'technical' knowledge, on the basis of interesting problems and not on the basis of exercises, artificially constructed and often devoid of interest. However, it should be stressed here that one must be careful not to abuse this point, by presenting fundamental aspects of the subject (e.g. basic concepts or difficult theorems) in the form of exercises

In summary, by integrating history in the presentation of mathematics and physics as described in this section, it is possible:

1. To *suggest* several possible *ways of teaching* a subject, by providing a vast reservoir of relevant questions and problems that could be taken into account, according to the specific needs of the classroom and the curriculum; e.g. emphasize the historical aspects, present specific ideas, presumably on the basis of extracts from original sources, illustrate interrelations between different domains.

2 To look for and *recognize* obstacles that appeared historically and may reappear in the teaching process In addition, the teacher may become aware of the fact that although a subject may have a simple form today, it is the product of a gradual evolution of its original conception based on concrete questions and examples, which are not evident if the subject is presented in its modern form right from the beginning. But these questions and examples may presuppose a mathematical maturity on the part of the student that may not yet exist. In this sense, history may help the teacher to become more aware of (and possibly to decide) whether and to what extent a subject can be presented at a particular level of education [8]

3. By reconstructions of examples (point (c) and remark (iii) above), to provide the learner with the opportunity to *understand the motivation* behind the introduction of a new concept, theory, method, or proof and grasp their content more profoundly Implicitly, learners may be thus encouraged to formulate their own conjectures and to realize why these conjectures (or similar ones that have been put forward in the past) do or do not supply satisfactory answers to already existing problems

4. By point (b) above, to *reveal interrelations* between domains which, at first glance, appear completely different, thus making it possible to appreciate the fact that fruitful research activity in a scientific domain never stands in isolation from similar activities in other domains On the contrary, it is often motivated by questions and problems coming from apparently unrelated disciplines, with which a close interaction is thus established In particular, this holds for the relation between mathematics and physics (see the next section).

5 *To make* the solution of *problems and exercises* an essential ingredient of the presentation for understanding a subject (point 4 and remark (iv) above). Actually, interest in them may be naturally induced by historically important and scientifically fruitful questions, without however neglecting their role as a means to improve one's knowledge of 'mathematical techniques'.

6. The teacher has the opportunity to *compare modern mathematics and physics* with their *form in the past* (i.e the notation, terminology, methods of proof and of computation, etc). Presentation of aspects of this comparison, presumably on the basis of original texts, may be beneficial for the students.

4. On the historical relation between mathematics and physics

At any level of education, more often than not there is a strict separation between mathematics and physics However, this is not compatible with the fact that throughout their history the two sciences have always had a close relationship. This is suggested by the examples presented in the previous sections and is also clearly revealed by the work of great mathematicians from antiquity to the present day, whose contributions to physics rival their purely mathematical works [9] Therefore, any treatment of the history of mathematics independent of the history of physics is necessarily incomplete.

Accordingly, in the light of the reasoning developed in this article (see section 1, especially [4]), the *interaction between the two sciences should also be revealed in their presentation* It should be stressed here, however, that this interaction has a multi-dimensional character, which cannot be exhaustively understood simply by adopting the conventional point of view - namely, that physics is only a domain of application of already-existing mathematical tools, hence that physics is simply a huge exterior to mathematics, a reservoir of problems to be solved.

In this connection, a preliminary discussion of two epistemologically and didactically important points that should be taken into account will be given in this section, but the subject needs further elaboration.

1. *Mathematics and physics*, as embodiments of general attitudes towards the description and understanding of empirically and mentally conceived objects, *are so closely interwoven, that any distinction between them is related more to the point of view adopted while studying particular aspects of an object than to the object itself* A historically inspired approach, though not necessary, is well-suited to illustrate this point The following are a few brief comments on some historically important, suggestive examples

   (i) It is known that the Special Theory of Relativity is based on the so-called 'Lorentz coordinate transformations' between inertial co-ordinate systems moving relative to each other with constant velocity In 1905, both Einstein and Poincaré arrived at this transformation, starting from the 'Relativity Principle' that in all inertial co-ordinate systems the laws of physics have the same form (Einstein, 1923/1950, pp. 37-40; Poincaré, 1954, pp 494-495). However, they developed the subject quite differently Einstein translated this principle and the principle of the constancy of the light speed in vacuum into exact *analytic formulas* useful in actual calculations (Einstein, 1950, section III 3), whereas Poincaré used the *mathematical structure* of the researched transformations, namely their group structure (Poincaré, 1954, pp 513-515).

   Actually, for high school or early university students who know some matrix algebra, one may use Poincaré's ideas together with elementary matrix algebra to obtain the Lorentz transformations and their group structure as a mathematically and physically relevant non-trivial example of a group of transformations (Tzanakis, 1999, section 3). [10]

(ii) In 1925, originally Heisenberg and later Heisenberg, Born and Jordan developed matrix mechanics as a new theory of atomic phenomena, based on matrix algebra, a subject unfamiliar to physicists at that time. In 1926, Schrödinger founded wave mechanics based on the familiar theory of partial differential equations in the way outlined in section 2. The main mathematical problem of the two theories was, respectively, the diagonalization of a certain (often infinite-dimensional) matrix, the Hamiltonian matrix, and the solution of Schrödinger's equation (see e.g. Heisenberg, 1930/1949, Appendix).

The strange thing was that these two *conceptually* totally *different* theories gave *identical* results compatible with experiments. Hence, the question of finding their relation naturally arose. It was tackled by both Schrödinger (in 1926) and von Neumann (1927-1932) in different ways.

Schrödinger established the equivalence of these two *particular* formalisms by providing a *formal* proof that, by choosing a basis for the wave functions he was using, solving his partial differential equation becomes a matrix eigenvalue problem identical to that of matrix mechanics, and *vice versa* (Schrödinger, 1982, article 4).

von Neumann's approach was of a rather different character (von Neumann, 1932/1947, Ch. I). He tried to identify the basic properties of the objects with which the two theories were dealing, thus emphasizing the *linear structure* of the *function spaces* underlying the two theories. In this way, he was led to define axiomatically what became known as a separable Hilbert space (von Neumann, 1932/1947, Ch. II). Then he proved that all such spaces are isomorphic, thus giving a definitive answer to the question above: the two conceptually different theories were just different representations of the same abstract mathematical structure that forms the mathematical substratum of the formalism of quantum mechanics (von Neumann, 1932/1947, Ch. II, theorem 9).

Reading these works concerning the same problem and to a large extent having the same motivation stresses clearly the differences between the point of view of mathematicians and physicists. In fact, this example may be used for didactical purposes at the university level to motivate the introduction of the concept of a Hilbert space in its abstract form, by first presenting the formal but suggestive approach of Schrödinger and then following the essential steps of von Neumann's treatment (see Tzanakis, in press, section 3.4).

2. The intertwining referred to in the previous subsection can be better appreciated by following an approach to the presentation of both mathematics and physics inspired by history. The latter offers many examples in which we may not only see the *use of mathematical methods in physics*, but may also *equally well talk about the use of physical concepts, thinking and arguments in mathematics* - cf. Polya's 'physical mathematics' (1954, Ch. IX).

Many examples can be given, ranging from the simple case in which the name and partly the meaning of concepts is motivated by physics [11] to the use of purely physical arguments in formulating or giving a plausible justification of mathematical consequences. [12] The following indicative examples should be taken only as hints to a subject useful in science teaching, but one needing further study and clarification based on a historical-epistemological analysis.

(i) Clearly, the derivative concept is partly the result of a long-term search for a precise, quantitative description of the velocity concept, in its full generality. More precisely, it took almost three centuries before the velocity concept was partially formulated by Galileo, who discussed only uniformly-accelerated motion (Boyer, 1959, pp. 72-73, 82-83, 113-114; Dugas, 1988, pp. 57, 59-61, 66-67). It is known that this fact influenced the emergence of the concept of instantaneous velo-city (Boyer, 1959, pp. 177, 180), which in turn acted as a basic motivation for the formulation of the derivative concept by Newton (Hall, 1983). [13]

This fact can be illustrated by a short extract from Newton's *Principia*. After Newton introduces his conception of the derivative as "an ultimate ratio of evanescent quantities", he tries to refute possible objections to it by writing:

> Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But *by the same argument*, it may be alleged that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to a place, is not the ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the

motion ceases, nor after, but at the very instant it arrives. [.] *And in like manner*, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. (Newton, 1686/1934, p 39; *our emphasis*)

It is clear that Newton tried to legitimize the concept of the derivative on the basis of the physical concept of instantaneous velocity which he considered intuitively more clear (and apparently he never defined it exactly; see Boyer, 1959, pp 193-194), which thus appears as the prototype example of a derivative.

(ii) It is well known that vector concepts and methods are related both to geometrical and physical problems. In fact, by the mid-nineteenth century there were important physical investigations containing deep mathematical insights that form part of the foundations of modern vector analysis. Green's (1828) essay 'The applications of mathematical analysis to the theories of electricity and magnetism' containing his and Gauss' theorems, Thompson's early work on analogies between electric phenomena and heat conduction and elasticity (1846-47; Whittaker, 1951) and Stokes' Smith Prize Essay of 1854 in which the theorem bearing now his name is contained (for more details on the history of these theorems, see Crowe, 1985, note 29, pp. 146-147).

That the general significance of these theorems, as well as the importance of vector methods, are revealed in the context of physical problems is explicitly acknowledged by Maxwell in his classical *Treatise on Electricity and Magnetism* published in 1873 (Maxwell, 1873/1954, vol I, sections 16, 21, 24, 95; see also below). His views on the role of physical insight for appreciating the significance of mathematical results appear clearly in his preface, where he wrote:

> I also found that several of the most fertile methods of research discovered by mathematicians could be expressed *much better* in terms of ideas derived from Faraday than in their original form. [.] Hence many mathematical discoveries of Laplace, Poisson, Green and Gauss *find their proper place* in this treatise and their *appropriate expressions in terms of conceptions mainly derived from Faraday* (pp. ix-x; *our emphasis*)

On the other hand, the parallel and partly related development of the algebra of Hamilton's quaternions and of vector analysis, in the second half of the nineteenth century, both influenced and was influenced by problems in mechanics and electromagnetism Its importance was emphasized by Maxwell, who wrote in 1872 that:

> A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities [.] and Vectors [. .] The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared for its importance with the invention of triple co-ordinates by Descartes *The ideas* of this calculus, as *distinguished from its operations and symbols*, are fitted to be of the greatest use in all parts of science. (quoted in Crowe, 1985, p 131; *our emphasis*)

Actually, in his *Treatise*, Maxwell used both formalisms with emphasis on vector methods; these qualitative remarks were transformed into exact mathematics in his *Treatise*, in which on the basis of the calculus of quaternions he stressed the importance of the fundamental operators *grad, div, curl* of modern vector analysis (originally introduced by Tait; Maxwell, 1873/1954, section 25) and revealed the general *significance* of its basic theorems, already known in special cases (theorems of Stokes, Gauss and Green) In fact, according to Maxwell:

> the doctrine of Vectors [ ] is a method of *thinking* and not a method for saving thought (quoted in Crowe, 1985, p 133, *our emphasis*)

Thus, in Crowe's view:

> by means of the vectorial approach, the physicist attains to a direct mathematical representation of physical entities and is thus aided in seeing the physics involved into the mathematics. (p 134)

It is through such deep insights into the mathematical structure of physical theories, together with an outline of vector methods and concepts contained in his *Treatise*, that modern vector analysis emerged and was established in the hands of physicists like Gibbs (1881-1884) and Heaviside (from 1883 onwards) (Crowe, 1985, pp. 138-139).

In conclusion, in this section we do not suggest the use of a non-rigorous mathematical treatment of particular subjects of mathematics or physics! We wanted only to emphasize that both history and research practice indicate that a rigid attachment to the requirement of extreme rigour may impede the development of a researcher's ideas and may often be a rather time-consuming luxury

On the other hand, paying insufficient attention to rigour by following a formal mathematical development may lead to severe problems, e.g to contradictions or to consequences with no mathematical and/or physical meaning In our opinion, both these facts should be taken seriously while teaching mathematics and physics, and a balance between the two extremes must be found This means that students should be encouraged to use non-rigorous, heuristic procedures, while studying a new subject and solve exercises related to it.

At the same time, the necessity of an at-least *a posteriori* control over what the students have obtained and how they obtained it should be emphasized, and self-correcting procedures must be pointed out to them. The way by which this could be accomplished is a subject for further research on the basis of particular examples (e.g. the one given in the Appendix).

6. Discussion

To avoid misunderstandings concerning the role which history can play in the presentation of (the historical relation between) mathematics and physics, as it has been described in sections 3-5, some additional comments are necessary.

It was not our intention to give a method of presentation in the strict sense of a model, one which can be applied more or less as an algorithm. Rather, we outlined the role history can play in mathematics and physics education by describing some of its decisive aspects that can be better understood only through the detailed analysis of specific examples. Implicit in this description is a general attitude towards the presentation of a scientific subject, in which the desire prevails to explain the motivation behind the introduction of new concepts, theories or methods on the basis of the historical evolution of the subject

Adopting a historical approach to the presentation of a subject does not mean that one should also adopt the scheme "ontogenesis recapitulates phylogenesis" (cf. beginning of section 3). This is evident once it is appreciated that what has been said in section 3 in no way implies a uniquely specified presentation of a given subject. It simply suggests some possibilities for presenting the subject on the basis of its historical development, which at the same time renders the teacher or the author more sensitive to the difficulties inherent in the subject and which can lead the learner to a deep understanding of it

History can be integrated in any one of the three ways described in section 3, or presumably in all of them. Moreover, its role is very different at different levels of instruction. For instance, in elementary school, except for historical comments that can be given (cf. sub-section 3.1), an approach inspired by history can be adopted mainly in the sense of points (a) and (b) of sub-section 3.3: whereas, points (c) and (d) concern mainly (but not exclusively) students at the university level or the last years of high school.

As a final comment, two proposals could be made:

(i) The intertwining of mathematics and physics curricula and teaching can be deepened at all levels of education, by integrating historical aspects of their development into their presentation in various ways which take into account the particularities of each level. This can have a beneficial influence on the teaching of both mathematics and physics

(ii) Though the role of history in teaching can be relevant at all levels of education, from what has been said in section 3, it can be made explicit more easily at the university level. Given that often implicit in teaching at this level is the belief that "once students have made the choice to study (either 'pure' or 'applied') mathematics or physics, they have to learn it independently of the way it is presented", we think that history can play a decisive role in the qualitative improvement of university mathematics and physics education.

Appendix: an example

The following outlines a possible implementation of the approach in sub-sections 3.2 and 3.3 to an example for final-year high school students. At the same time, it illustrates a presentation at this level of the close relation between mathematics and physics referred to in section 4.

For students who have already been taught the derivative concept and its use in the study of extrema, it is possible to present the laws of geometrical optics as simple applications of differential calculus.

1. The following historical comments are didactically important (sub-section 3.2 in connection with sub-section 3.3, points (a) and (b)):

   (i) The law of reflection was derived by Hero in antiquity, as the result of the study of an elementary geometrical extremum problem, based on the assumption that nature does nothing worthless, hence that light follows the shortest path (Thomas, 1939, pp. 496-499). This follows easily from figure 1: $AO+OB$ is a minimum if $A'OB$ is a straight line, where A' is the symmetric image of A with respect to the reflecting surface.

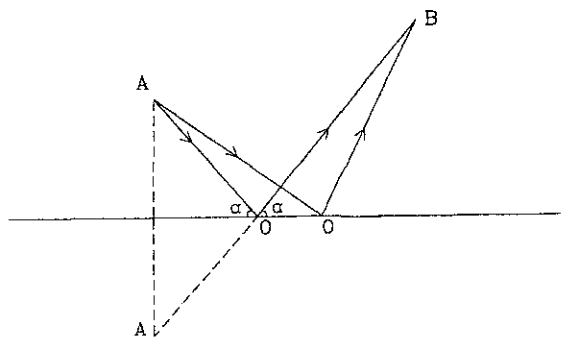

Figure 1

   (ii) The law of refraction is derived by Fermat (1662) on the basis of the principle "that Nature always acts in the shortest ways" (Dugas, 1988, p. 254), which he interprets by saying that light follows the path of shortest time. Hence, the law of refraction follows geometrically from Figure 2

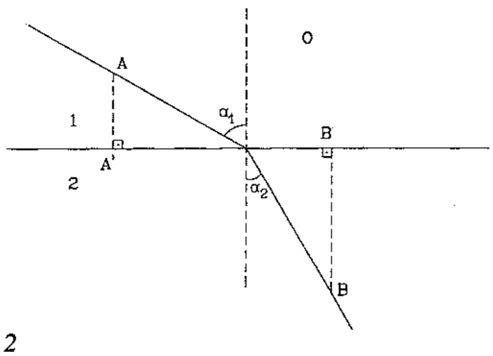

Figure 2

If the path of light from medium 1 to medium 2 is AOB and its speed is respectively v_1, v_2 ($v_1 > v_2$), then O is determined by the relation $A'O/B'O = v_1/v_2$, where we took $AO=BO$ (Dugas, 1988, part III, section V 1)

(iii) In 1696, Johann Bernoulli formulated and solved the 'brachistochrone problem': given two points A, B in a vertical plane, find the trajectory AMB of a point M, which, starting from A and moving only under its weight, arrives at B in the least time (Hairer and Wanner, 1996, pp. 136-137). His solution is based on the *analogy* (cf section 2 (iii)) of what this problem asks for with Fermat's principle of Least Time ((ii) above) and it turns out to be the cycloid.

2. On the basis of the above historically crucial steps, the following approach can be followed (see sub-section 3.3, point (c)).

(i) By Fermat's principle and since we have to find the point O in Figure 2, the time needed to follow the line AOB ($AO \neq BO$ in general) is expressed analytically as a function of one variable, e.g. of OA'. By requiring the vanishing of the derivative of this function, the law of refraction follows in the usual form:

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}$$

(ii) On the basis of Figure 1, the simpler problem of deriving the law of reflection by a similar procedure may be solved as an exercise (possibly in several steps; sub-section 3.3, point (d)).

(iii) With the aid of the proof in (i) above, Bernoulli's solution is reconstructed in modern language, by dividing the path AMB into infinitesimal *rectilinear* pieces, such that the passage from the one to the other is done in least time. Hence, by analogy with (i), a 'refraction' law is obtained, which with the aid of the law of the conservation of energy is expressed analytically as the first order differential equation:

$$y(1 + (y')^2) = 2c = \text{constant}$$

for the unknown curve $y(x)$ (section 3.3, point (c) and Simmons, 1972, section 1.6)

(iv) That the cycloid, given in parametric form as:

$$x = c(\theta - \sin\theta)$$
$$y = c(1 - \cos\theta)$$

satisfies this equation may be given as an exercise (sub-section 3.3 point (d)). Moreover, its geometrical and physical properties may be studied.

(v) A good though somewhat more advanced example for the latter, is Huygens' use of the cycloid (1673) to construct a clock with a period independent of the amplitude of the oscillation. By using Newton's dynamic law and the differentiation rules (especially the chain rule), one derives the equation of motion of a point particle which is constrained to move along a cycloid under its own weight alone. The result is (g = the acceleration of gravity):

$$\frac{d^2}{dt^2}\left(\sin\frac{\theta}{2}\right) = -(g/4c)\sin\frac{\theta}{2}$$

which is the equation of motion of a simple pendulum ($\sin\frac{\theta}{2}$, denoting the amplitude of its oscillation, is proportional to the arc length of the cycloid). This is an important result, since on the one hand it describes a strictly isochronous oscillation (i.e. its period is independent of its amplitude) and on the other it is only an *approximation* for the *simple* pendulum, but an *exact* result for cycloidal motion. This was the basis of Huygens' construction of the first isochronous clock based on the motion along a cycloid (cycloidal pendulum; Sommerfeld, 1964)

The above steps (i)-(iv) illustrate a possible way to realize points (b)-(d) of sub-section 3.3 (the present description corresponds to an implicit use of history; sub-section 3.3 (iii)). At the same time, by giving a natural explanation for the formulation and solution of a technical problem (the brachistochrone problem) and a meaning to the corresponding computational exercises, it makes a nice introduction into the reasoning and the methodology which at a higher (university) level can be sufficiently generalized to lead to the calculus of variations (sub-section 3.3, point (b)).

Notes

[1] By 'presentation', we mean both teaching in a classroom and writing a textbook or even a monograph in which the author aims at giving the opportunity to the reader to obtain a good understanding of a presumably advanced subject. We fully appreciate the complicated, interactive nature of the teaching process; hence, we realize that this terminology may be somewhat misleading. However, it is used for the sake of brevity only, and is in no way intended to mean that teaching is simply a one-way process, from the teacher ('emitter') to the learner ('receiver')

[2] For instance, though the concept of a finite group had been defined by Cayley in 1854, it was only in the 1890s that it was defined in full generality by Weber. This was done after a certain lapse of time, necessary for the study of many concrete examples, which finally made its more general, abstract definition appear necessary (Dieudonné, 1978, vol II, p. 116)

Conversely, the abstract algebra of De Morgan and Peacock apparently had not much impact when it appeared (around 1830-40), simply because it was premature at that time A basic motivation for the development of abstract algebra was given only after particular structures appeared (the first being Hamilton's quaternions), which justified the search for an abstract formulation of ordinary algebra and its generalizations (Crowe, 1985, p. 26).

[3] See Klein's (1928/1979, p 316) criticism of abstract group theory and Lakatos' (1976, appendix 2) thorough analysis.

[4] We aim to discuss some aspects of these questions, which refer to the "listing of ways of introducing or incorporating a historical dimension" (p 257) (mainly section 3) and to the fact that "historical knowledge [..] could lead to suggestions for new topics to be taught" (p. 256) (mainly sections 2 and 4)

[5] For instance, from the fundamental result of linear algebra that any quadratic form in n dimensions can be transformed to diagonal form by a change of basis (i.e the so-called transformation to principal axes), one may conjecture that the same may be possible for quadratic forms in infinite dimensions, presumably under some additional restrictions In fact, this was a basic motivation of Hilbert in his study of integral equations (Dieudonné, 1981, section V.2) and led to important developments in what later became known as the spectral theory of operators. In fact, this analogy may be used at the university level to motivate both the formulation and basic steps in the proof of the spectral theorem of self-adjoint operators (roughly speaking, that every such operator can be diagonalized)

[6] It often happens, for example, that the effort to prove a mathematical theorem leads to new propositions (in the form of lemmas), which subsequently acquire a mathematical value independent of the theorem for the proof of which they were originally conceived

[7] It would be rather confusing, for instance, to start the study of Diophantine equations using Diophantus mathematical language; or to present classical mechanics following Newton's *Principia*; or to develop the algebraic structure of complex numbers, passing through the more general epistemological thinking of Hamilton, for the nature of algebra as a science of pure time, in analogy with the nature of geometry as a science of space (Crowe, 1985, section 2 III).

[8] For about 25 years in Greece, final-year high-school students were taught about abstract algebraic structures (fields, groups, rings, vector spaces) and it was noticed by many that though the relevant chapter included only the definition of these structures, students were unable to answer questions and exercises that were simply direct and naive applications of these definitions On the other hand, most (if not, all) relevant, non-trivial examples of such structures that might help the student to appreciate their real mathematical significance, and which played a crucial role in their gradual formulation in today's abstract form in which they were presented to students, are out of the reach of high-school students, and have no place in any reasonable high-school curriculum. Therefore, these difficulties could have been forseen on the basis of a rough *a priori* knowledge of the historical development, thereby concluding that such a subject cannot successfully be presented in high school (Tzanakis, 1991).

[9] For examples in this century, think for instance of Hilbert in connection with the foundation of general relativity, Minkowski's (1923/1950) geometrization of special relativity, von Neumann's (1932/1947) *Grundlagen der Quantentheorie* and Kolmogorov's revitalization of classical mechanics and dynamical system theory

[10] It is on the basis of many such concrete examples that the significance of abstract algebraic concepts like that of a group can be appreciated, thus preparing the way for assimilating their general, abstract form

[11] The centre of gravity in elementary geometry, for instance, or the spectrum of a linear operator in functional analysis (Dieudonné, 1981, p 150).

[12] Archimedes' 'Method' provides one instance where, by using arguments from mechanics, mathematical results become plausible (Thomas, 1939, section XVII j) Liapounov's 'direct method' in the stability theory of differential equations offers another, where the motivation is the stable equilibrium of a mechanical system, with minimum energy at equilibrium (e.g. Simmons, 1974, section 43)

[13] Though infinitesimal methods had already been used in antiquity (particularly in the calculation of areas and volumes, and in the drawing of tangents to particular curves), the derivative concept and its relation to the integral did not exist before Newton's era It was Newton who realized the enormous advantages of calculating a quantity not by direct 'summation' of its infinitesimal parts (integration), but by first determining its rate of change

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