The Right to Make Mistakes

ALAIN BOUVIER

Students' mistakes bother us. This assertion has become a sort of leitmotif at many meetings of mathematics teachers. Why is this? Do we see in *students'* mistakes the sign of *our* failure?

The same mistakes appear very frequently, persisting across classes, resisting "correction" week after week, as if there were certain "invariants", as if our teaching were unwittingly teaching mistakes. What is going on here? Why do some kinds of learning, like learning our mother tongue, seem to be successful on a grand scale, while the learning of mathematics is much less successful in spite of many hours of teaching spread over many years?

Let us take a simple example, the calculation of 2^3 Among the possible responses, two are common:

$$2^3 = 6 \text{ and } 2^3 = 5$$

In many cases we have only to ask a student who has written one of these, "Are you sure?" or "Why?", for him to change his response immediately to the expected answer

Some teachers may say that the student made a careless mistake; but, especially since Freud, we know that the smallest "slip of the tongue" has significance. Moreover, although we may meet $2^3 = 6$ or $2^3 = 5$, we never seem to meet $2^3 = 37$ or $2^3 = 100$. *Mistakes are not the result of chance.* They show that the student has used a particular logic, though not the appropriate one

Others may say that the student "confused" 2^3 with $2 \times 3 = 6$. Such explanations explain nothing! Do you know any child who would "confuse" the following two drawings, saying "boat" for the first and "house" for the second?

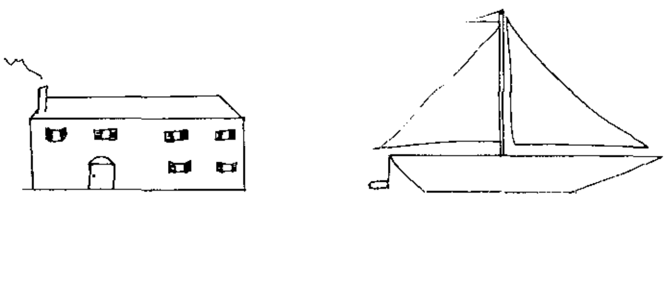

Figure 1

If these swift and summary "explanations" fail to shed light on the interactions between student and subject matter that produce responses that we, as teachers, regard as mistaken, and if we want to go further in order to understand what is happening, to fathom the meaning of the signals that the students sends us, then we must go back to the difficult and central question:

What do we mean by learning?

Starting from the assumption that it is by practising something that one often learns something else, we will first digress with a few reflections on the historical and social development of knowledge Then we will return to learning, to the cognitive development of our students, to our role in the classroom and the *meaning* of the learning situations we present there. Finally, and most importantly as far as education is concerned, we ask what conclusions we can draw.

1. Ruptures

I want to talk essentially about mathematics, but let us begin by considering other disciplines in order to try to pull together certain characteristics of the development of the sciences. We must try to understand how *science opposes obscurantism by questioning the status quo* and how it advances by "scientific revolutions", by abrupt jumps, by "ruptures"

1.1 *Ruptures in conceptions of the world*

— Eastern civilisations before the time of the ancient Greeks had a *cyclic* conception of time: whatever happens today will come again tomorrow This gave way to a *linear* conception of time: water never flows under the bridge twice

— Paleontologists are every day pushing the origin of man and species further back; every day new varieties of organisims are discovered. Before the 18th century scientists believed that certain strange chunks of matter discovered here and there were stones. Their shape and astonishing size were accounted for either as the product of human activity (Mercati, 1541 - 1593) or as the result of natural forces. In ancient Greek times, for example, they were called "thunderstones" During the 17th and the beginning of the 18th centuries, spurred on in particular by explorations in Africa and the South Pacific, where the lifestyles of the people were very different from those of Europeans and where the flora and fauna appeared in unfamiliar varieties, a rupture occurred: these objects came to be seen not as games played by nature, but as fossils, the relics of once living beings

As often happens in the development of the sciences, this "discovery", this new idea, brought fresh problems with it. Among others was the question whether the fossils were the remains of extinct species or of species similar to those then alive. In spite of the evidence, notably the abundance of skeletons quite different from those of contemporary animals and the absence of fossils resembling living animals, Darwin's evolutionist theories (already germinating in the work of Lamarck) only very slowly and with great difficulty established themselves in opposition to the received ideas of the period (as defended by Cuvier, for example) Darwin's theories were at first considered immoral and anti-Christian; they were attacked by distinguished scholars such as Kelvin — and even in the 20th century they have been the object of legal wrangles in certain American states!

In the same field, the laws of heredity formulated by Mendel in 1865 were not taken seriously until the beginning of the 20th century

— For a long time the Earth was thought to be flat, motionless, and at the centre of the world. So what was the point in looking westward for a route to the Indies? There is no need to dwell here on the difficulties confronting Copernicus — who, fearing the reprisals of the theologians, did not publish his work until a short time before his death — and Galileo — whose trial has passed into history and continues to inspire many writers ("And still the Earth moves!")

— In chemistry, Mendeleev's classification of the atoms was not noticed for ten years after its publication in 1869 and 1871. But this delay pales beside the reception of Berthollet's notion of chemical equilibrium In spite of his great reputation as a scholar, his ideas were not taken up and developed for another 50 years.

— In medicine, the observed presence of ferments in the process of fermentation was explained by a doctrine of spontaneous generation until Pasteur put forward his explanation: that ferments are living organisms specific to the substances they cause to ferment. It is acknowledged that this talented researcher sought to be as rigorous as possible by basing his theories on facts, on meticulous experiments, not taking any idea for granted, however widely it was held. The price of his integrity was that throughout his life he ran headlong into the violent opposition of the medical and scientific establishments.

— In political economy the writings of Karl Marx continue to exercise a remarkable grip He brutally introduced a materialist and dialectical conception of history, giving a central place to the balance of forces, the class struggle, the notion of added value; he provided a global theoretical framework to 19th century socialist thought in opposition to the "petit bourgeois" theories of predecessors like Proudhon

— In psychology, Freud rejected the mutual assimilation of psychic and conscious activities. Through a study of slips of the tongue and of dreams, then through the techniques of free association and analysis, he researched the "unconscious" and created the method of psychoanalysis. Struggling alone against his contemporaries, he did not hesitate to compare himself with Copernicus and Darwin. Like them he attracted the hatred and contempt of all conformists.

— Finally in this rapid survey we consider a single example from physics: Einstein's special theory of relativity. At the end of the 19th century, many scientists supposed that physics was substantially complete; they were astonished by Roentgen's discovery of X-rays. Through a profound and critical study of the concepts of time and space — particularly those relating to simultaneity — Einstein in 1905 formulated new principles that ran counter to classical theory: the mass of a body is variable, it is an increasing function of its speed.

Some see Einstein more as a philosopher than a physicist, one who started from the knowledge of his era and completely reorganised it. For him, intuitions, principles, concepts, all preceded experimental verification.

In *Encyclopédie de la Pléiade* we read:

> Einstein attaqua aux notions apparement les plus confirmées et, par une prodigieux effort de pensée, il proposa de l'univers physique une conception entièrement nouvelle. Le relativité devait susciter bien des controverses... *elles tenaient uniquement aux difficultés qu'éprouve l'esprit humain à se débarrasser de ses habitudes* et nullement de contradictions nées de l'expérience.

All the thinkers we have mentioned — Berthollet, Einstein, Freud, Galileo, Marx, Mendel, Mendeleev, Pasteur — worked against received ideas, against the accepted theories of their time, and against the scientific establishment. As disputed as their work was, and in some cases still is, *we cannot think the same way after them as we did before*. They provoked a *rupture* in scientific thought and brought about a complete *reorganisation* of their fields of research.

1 2 *Ruptures in mathematical thought*

It is possible to find the same phenomenon in mathematics, although its occurrences are not as well known as in the other disciplines.

— The most frequently mentioned rupture in mathematics concerns the discovery of irrational numbers and the proof, by Pythagoras, of the irrationality of $\sqrt{2}$

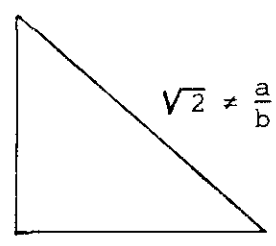

Figure 2

Mathematics suddenly shifted from the idea that "every number is a ratio of two whole numbers" (as we may express it in modern terms) to the conception of two categories of numbers: rational and irrational.

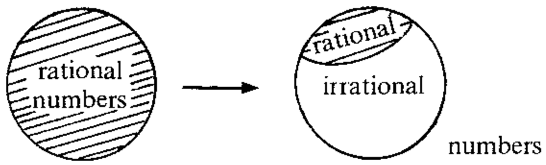

Figure 3

The shock took a long time to digest and traces of it still remain. One of my students told me recently, echoing Kronecker in 1882, that "π doesn't exist!" and I know an able mathematician who develops lectures on the theme that "the real numbers do not exist".

While on the subject of numbers, we might look at those mysterious complex numbers which allow us to perform illegitimate calculations in order to find the legitimate roots of third degree equations. Although we may find $\sqrt{-1}$, $\sqrt{-3}$, etc., written today in books on the theory of numbers [1], in French schools these symbols are forbidden — as if there might remain a touch of sorcery (one never knows ...!)

— Another famous rupture concerns the foundations of geometry. After trying to prove Euclid's fifth postulate, notably by developing the consequences of various forms of its negation in the hope that a contradiction would appear, and after hiding these results from the scientific community, it eventually became necessary for mathematicians to cross the Rubicon and allow new iconoclastic geometries to stand beside Euclidean geometry. [2] The old framework suddenly exploded and within the new framework of a plurality of geometries new problems arose, particularly the problem of classification.

— Without it ever having been made explicit, at the beginning of the 19th century it was understood that a continuous function was everywhere differentiable, save possibly at a few exceptional points. To imagine, as Weierstrass did, that functions could be found which are everywhere continuous but nowhere differentiable, was an enormous jump! Hermite refused to take any interest in these mathematical objects, which he termed *monsters*. What would he say today about those who follow Mandelbrot in exploring the properties of even more pathological curves, the fractals, whose dimensions are not whole numbers but real numbers in the open interval (0,1)?

— Those who learned their university mathematics before 1970 must surely have felt the shock waves produced by the creation of set theory, the fantastic, overwhelming, results that Cantor produced [3] — for example, that **N** and **R** are not equipotent though **R** and $\mathbf{R}^2$ are. We remember the terrible cry Cantor uttered in his letters: "I see it but I cannot believe it!" He was not the only one! The hostility of his contemporaries was violent and lasting and drove him mad.

This short list of ruptures is not exhaustive. Beside these spectacular cases we can observe other, local, ones, and today we can see here and there the starting points for new upheavals — non-standard analysis, for instance. Each of us feels confused: on the one hand, the concepts of number, of measure, of curve, have never before been known with such clarity, yet we also know that at any instant there is a chance that they will be put in question all over again and that new ways of seeing things may force us to rethink what for the moment seem the very essences of the things themselves.

2. Ruptures in learning

Our first detour has led us to reflect on the historical and social development of knowledge. We have emphasized the ruptures, the questioning of widely accepted ideas about the nature of things, and the intellectual reorganisations that are characteristic of scientific progress. Our second detour takes us into reflections about the learning that happens in classrooms and which pose the problem of the teacher's role.

2.1 *Small-step pedagogy*

Breaking — that rupture again! — with the psychology of his time, Watson in 1913 proposed the system, called behaviourism, which returned, or tried to return, psychology to an objective study of actions and observable behaviours. It focused in particular on stimulus-response dyads where the stimulus is physically measurable and the response objectively observable. Very soon he applied his research to learning and to animal and human conditioning. When in 1924 he presented his theories that reduced learning to conditioning, they immediately met with an enthusiastic and widespread success in the U.S.A. Following on from this work, the neo-behaviourists such as Hull tried to understand how motivation affects learning. By studying trial-and-error methods they formulated the principle of reinforcement: stimulus-response connections are strengthened by success and weakened by failure or boredom.

In 1954 Skinner made a systematic application of these principles to learning. He considered the learner as a "black box" — it was no more necessary for the teacher to know what went on inside than for the photographer to know what happens inside the camera.

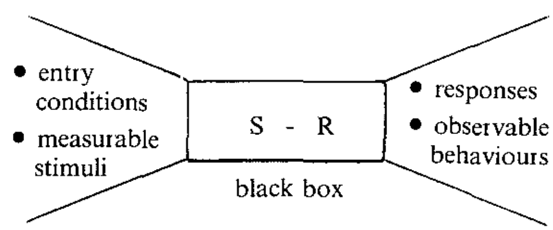

Figure 4

Skinner attempted to single out the "fundamental units of behaviour" and to guide the behaviour of his subjects by rewarding certain behaviours. Here we find the basis of "programmed teaching", leading to the method of small steps, to a fine and very precise control of "individual learnings" and to their organisation into "programmes" The three key words — the charter of programmed teaching — are:

PRECISION — INDIVIDUALISATION — AUTOMATION

each indicating important directions for research.

For us as teachers programmed teaching may be summarised briefly by three characteristics arising from the following hypotheses:

(1) Knowledge can be cut up into clearly defined pieces
(2) One learns (i e. students' learn, we learn, .) by stockpiling, accumulating, items of knowledge (theorem 17, theorem 18, theorem 19,. .)
(3) Errors leave indelible marks prejudicial to future learning.

Consequently the organisation of learning must

(1) go from the simple to the complex
(2) construct short sequences of small items (the "small step" pedagogy)
(3) break up difficulties in order to avoid the possibility of error,

all of which can be summed up (as I have shown before) in the following diagram:

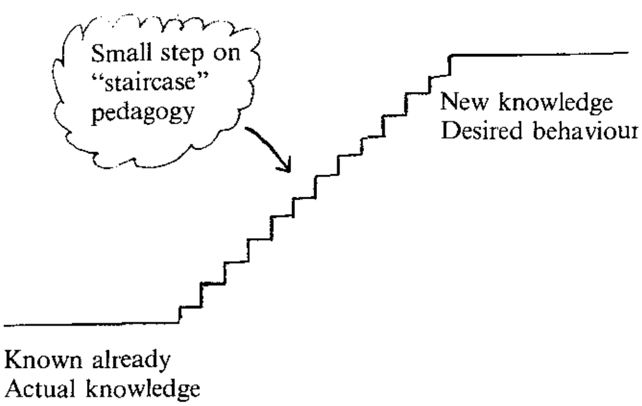

Figure 5

The initial success of programmed teaching was due as much to historical conditions — the need to train a large number of technicians in a short time — as to the effectiveness with which it realised its aims: to allow everyone to learn what he wanted, when he wanted, and with a guarantee of success Soon, however, questions began to be raised about the meaning of this learning, of this observable behaviour. And Crowder, for example, pointed out that in programmed sequences correct answers could be the result of incorrect reasoning — which would thus (according to the theory) be reinforced in this situation!

We all know those materials — books, software videos — which are designed as more or less elaborate sequences of programmed teaching (some being very sophisticated). With the development of microcomputers for home and for school come an almost daily increase in the available software But the principles of programmed teaching have also penetrated the domain of teacher training; microteaching, for example, is a technique of training teachers which consists in getting them to elaborate a short teaching sequence (10 minutes, perhaps), decide on their gestures and behaviours during this sequence, film themselves carrying out the sequence, watch themselves critically on film, and repeat until satisfied.

There are other signs of neo-behaviourism in teaching: in the "teaching to objectives" trend. [4] Indeed, in everything to do with objectives in teaching and with new forms of evaluation we can see teachers recovering the principles of the behaviourist stream in cognitive psychology.

However, researchers and teachers alike are aware that programmed teaching and its derivatives, while aiding the achievement of limited performance of some skills, do not deal with the acquisition of general competence. And all the work on heuristic strategies in problem solving, especially at the metacognitive level, remains to be done. We must still ask, on behalf of the student in the classroom.

How is knowledge really built?

2.2 Non-sense pedagogy

We start with two examples

(1) A student says that the domain of the function $f(x) = 5x - 3$ is $\mathbf{R} - \{1\}$. Why? Because, he replies, $f(x) = (5x - 3)/1$ and the denominator must not be cancelled

(2) Another student, who has written

$$\frac{a+b}{a+c} = \frac{b}{c}$$

is told to replace a by 4, b by 6, and c by 1. He gets, when he has finished the calculation,

$$2 = 6$$

and shows no astonishment. Questioned about his thoughts on the equality, and reminded that a child cannot be 2 and 6 years old at the same time, he explains: "Well. in mathematics things are not the same "

These examples are taken from a book by Stella Baruk, *L'âge du capitaine* (Seuil, 1985), the title of which refers to a study undertaken by the Grenoble IREM a few years ago Recently, thinking that perhaps some evolution had taken place, I undertook an experiment similar to that carried out by the Grenoble IREM, administering the following test to five classes (four elementary school classes, CE1, CE2, CM1 and CM2, and a small 6th level class).

(1)
$$\begin{array}{r} 27 \\ +32 \\ \hline \end{array}$$

(2) On a boat there are 11 cows and 22 goats What is the age of the captain?

In the following table, the first column shows the class with the number of its students, the second column gives the percentage of incorrect answers to the first question, and the third column the percentage of numerical answers to the second question (33 was the most common).

Class and number of students	Addition errors	Numerical answer (33 or other value)
CE(1) 15	67%	100%
CE(2) 18	72%	89%
CM(1) 24	50%	38%
CM(2) 22	5%	95%
6ème 16	0%	81%

Perhaps we should set aside the class CM(1) whose teacher had made the arrangements for me to carry out the experiment in her school. In any case, we are not pretending to have taken the proper methodological precautions, nor that the figures are significant in a technical sense but at first glance two points stand out and are worth examining more deeply.

The school seems to be completely successful in teaching the addition algorithm without carrying for numbers under 100. By the end of the 6ème class (the experiment took place in June) everyone succeeded, the failure rate diminishing steadily from CE(1) to the 6ème. [5]

On the other hand, in the case of an exercise like the second one, *the school seems to have no effect.* More than 80% of the students respond like robots, as Stella Baruk would say, whether they are 7 or 14 years of age.

I have recounted an analogous experiment elsewhere [6] involving mathematics teachers and teacher trainers which revealed similar behaviour and percentages just as spectacular.

What is the significance of these observations?

I mentioned earlier how, starting from certain theoretical assumptions about learning, Skinner and his successors developed programmed teaching and other by-products of "small-step" pedagogy. It is probable that these ideas have spread progressively among teachers; it is still more probable that the latter have created, independently or not, a practical pedagogy which consists in breaking down difficulties into smaller and smaller constituents. The result is that today's students, like yesterday's (their teachers today), receive instruction based on different variations of the staircase pedagogy Now, the more we cut up a task for a student, the more he loses a view of the whole, of *what it signifies,* and so we train, we progressively condition our students — alas, from an early age — to follow the rules or, worse still, *what appear to them to be the rules* and not to look for meaning.

A small-step pedagogy is therefore a non-sense pedagogy.

2.3 *The class as a scientific community*

At a time when Watson was successfully laying the groundwork of behaviourism, the zoologist Jean Piaget was beginning the work in psychology which he was to pursue for half a century, attempting to understand the mechanisms by which knowledge develops, starting (around 1920) with the intellectual operations and logic of children He was to concentrate on the development of ideas of number, space, time, speed and chance

Little by little the basic ideas of what has come to be called *constructivism* were put in place:

> I know an object, mathematical or not, only by *acting on it*.
>
> My intelligence is built up through use, by changing points of view, by *stages* [7] in my awareness of the new ideas I generate.

The movement from one stage to another is reminiscent of the leap, the break, the rupture, of Bachelard's theory. I symbolise the contrast between this picture of learning and that of behaviorism and small-step pedagogy by the following schema [8]

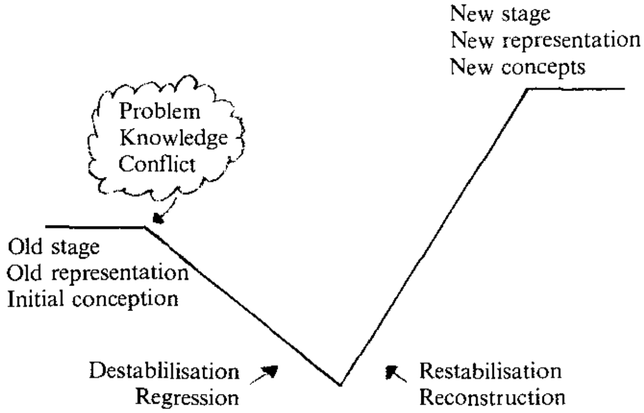

Figure 6

For Piaget and his successors — Wallon, Doise, Mugny, Anne Perret-Clermont — *the child is a scientist* who develops his knowledge and his intelligence through interaction with others in *problem-situations* [9]

Pursuing the practical consequences of the ideas of constructivism for teachers, I take up again the points touched upon in the previous paragraph.

> *Hypotheses*
>
> (1) Knowledge probably cannot be cut up into small distinguishable pieces; one can only work on the *conceptions* (mental representations) [10] of the concepts of a given subject (a student).

(2) We learn from knowledge conflicts [11] in problem-situations, shifting from one conception to another

(3) "Mistakes" are part of the process of learning and their analysis, by the learner himself, can only aid learning.

Practical consequences for the organisation of learning
(1) Go from the complex to the simple by starting with very open problem-situations; for example, we can give students in the second year the task of finding the number of solutions of the equation $200 \sin x - x = 0$ *before* systematically studying the trigonometrical functions. *It is complexity that gives meaning*

(2) Encourage the students to criticize and analyze every partial result and conjecture, whether it seems obviously true or false to us as teachers *Learn to find and analyze the simple in the complex*

The classroom can then be considered as *a small scientific community* with its own rules of procedure (very often implicit). The major role of the teacher is to help this scientific community function by supplying appropriate problems which will initiate socio-cognitive conflicts, by organising scientific discussions, and by suggesting validation procedures for new knowledge. But there remain several difficult and key questions for the teacher who would like to work with this point of view

— How can I know the *conceptions* my students have?
— How do I discover and help my students get around the *obstacles* to learning?
— How do I construct conflict-situations?

Finally and above all, even if I manage to resolve these problems, how do I encourage the analysis, related to the synthesis, of the different conceptions that the student has constructed? How is the "balkanisation" of these conceptions to be avoided? How do I help him relate the "knowledge" he has made, organise them, and give them coherence?

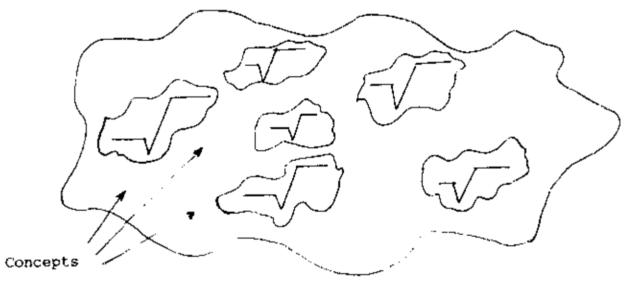

Figure 7

Before we examine two examples of conflict situations in detail, let us note one difficulty which has to be kept in mind: *in spite of the construction of new conceptions, new models all the old ones are still alive and operational.*

Who doesn't have the feeling that the Earth is flat, stationary, and that *the sun rises in the East and sets in the West?* Who hasn't seen the sun sinking below the horizon? (Besides, in California, there is a special society for people who believe the Earth is flat. It supplies *rational* explanations that counter the scientific arguments: plane trips, satellites, etc.)

The ancient models have left their indelible traces in our collective and individual memories. Here are two other examples, not from mathematics

(1) Three identical receptacles contain respectively water, sand, and iron fillings. They are placed in an oven at a temperature of 80° for several days. When they are taken out of the oven, which will have the highest temperature? People will often choose the iron "because it is hotter than sand".

(2) Students are asked the following question: Imagine that you are drinking a litre of water; draw its path in your body. Many drawings show no exchanges in the intestines: it is a pipe and so it is non-porous

These two simple, not to say simplistic, examples [12] illustrate what Bachelard has been bringing out for a long time: there are *obstacles* to the development of knowledge; these obstacles are not lack of knowledge but *badly constructed knowledge*

2 4 Evidence and obstacle

- Consider the following problem often ascribed to Tartaglia.

> Two Roman soldiers have 2 and 3 loaves of bread respectively and decide to share them A third soldier arrives and as he hasn't any bread the others share the 5 loaves with him. On leaving he gives them 5 gold coins as thanks for their generosity. How should they share the coins?

Let us call the three soldiers A, B and C. The reasoning often used by people trying to solve this problem reduces to a consideration of the following situation

	Soldiers	Loaves	Coins
A	2		
B	3		
Total	5	5	

They conclude that A should get 2 coins and B should get 3. Where is the problem? The solution is so easy, so obvious. And yet . . !

In the following drawing, Step I represents the initial situation. In Step II, to help in sharing the loaves between 3 people, each loaf is divided into 3 parts. Finally in Step III the final sharing takes place

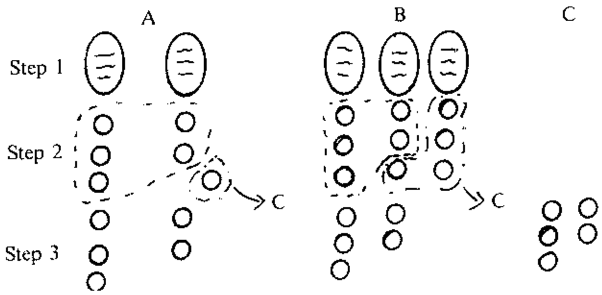

Figure 8

What can we say? B gave C four times as much bread as A did! This could have been calculated directly:

— each person's share: 5/3 of a loaf
— A gives C 6/3 – 5/3 = 1/3 while B gives C 9/3 – 5/3 = 4/3

Summarising, B should have four coins and A one.

- Let us now consider another problem quoted by Nicolas Balacheff.

> A farmer buys a horse for 400 francs and sells it for 500 francs. He then buys *the same horse again* for 600 francs and sells it again for 700 francs Did he make any money? or lose it? and if so, how much?

I have given this problem on a number of occasions. The audience immediately gets involved in impassioned discussion. For some he lost money, some say he gained, and very different amounts are mentioned Two solutions that come up frequently are:

— He made 300 francs, the difference between the last sale price and the first cost price
— He made 100 francs because he successively made 100, lost 100, made 100

Both solutions are of course wrong: he made 200 francs. But I must point out that the debate was often so heated that some people were not convinced by any of the three solutions that I gave them afterwards

(1) Purchases 400 + 600 = 1,000
Sales 500 + 700 = 1,200
Profit 1,200 – 1,000 = 200

(2) (–400) + 500 + (–600) + 700 = 200

(3) Departure Arrival

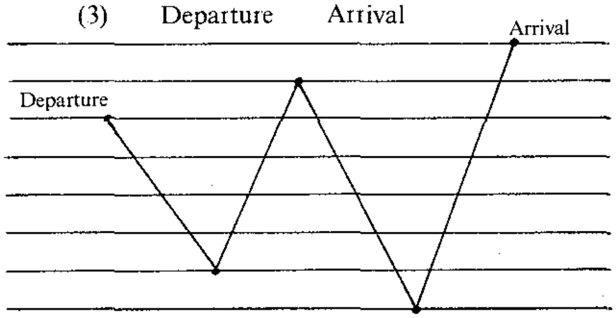

Figure 9

- Let us look at still another problem shown to me a long time ago by Claude Gaulin.

> On the Earth a man starts walking at a point A. He walks successively 1 km South, 1 km East, and 1 km North, and arrives at a point B. Is it possible for the departure (A) and arrival (B) points to coincide?

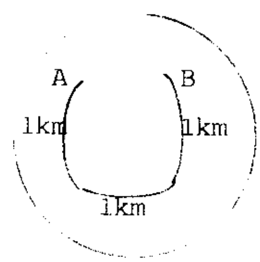

Figure 10

A few seconds reflection will lead us to say, "Yes, if he starts at the North Pole."

> Can the same thing happen if he starts anywhere other than the North Pole? In other words, is the solution unique?

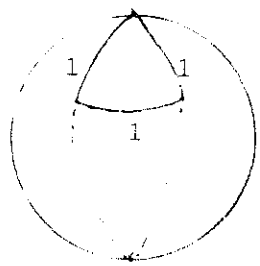

Figure 11

People solving this problem often conclude, for intuitive reasons or after seeing a proof similar to the one that follows, that the North Pole solution is unique

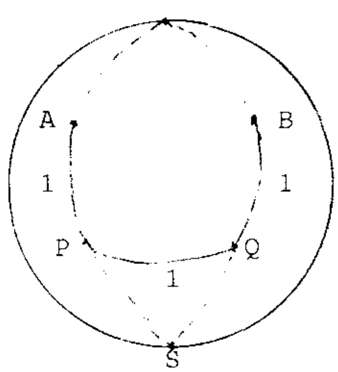

Figure 12

Starting at A, to go 1 km South the walker must travel along a meridian Then to go East he must travel on a parallel of latitude and finally on another meridian going North. For the departure and arrival points to coincide, the two meridians must intersect. Now two meridians can only intersect at the North Pole (which is a possible solution) or the South Pole (which cannot be a solution). There is therefore only one solution: the North Pole.

The proof confirms our intuition But intuition can fool us; the proof is actually false! In arguing this way a particular hypothesis — an insidiously wrong hypothesis — creeps in Read the proof again! Have you found it?

In fact there is an infinite number of solutions Find them!

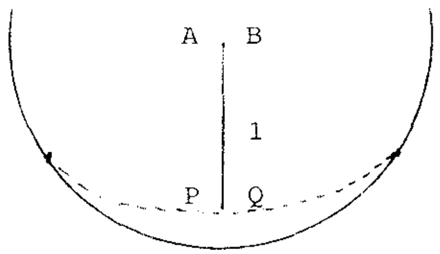

Figure 13

In the "proof" everything is correct except that there is no reason why the point Q must be on a *different* meridian. It could be coincident with the point P To see how, consider a parallel of latitude very close to the South Pole with a circumference of 1 km Do you agree such a parallel exists? Choose any point on this parallel to be P and make A the point 1 km North of P That's one solution so *all* the points on the parallel of latitude 1 km North of the parallel close to the South Pole with a circumference of 1 km are solutions.

But we are still not finished! Other solutions exist, many others, an infinity of other solutions I leave to you the pleasure of finding them!

- For those who would like to reflect on a more mathematical problem, here is one quoted by Alain Robert

Solve in the set of real numbers the equation

$$x^{x^{x^{x}}} = 8$$

Solution. It can be reduced to solving the equation

$$x^8 = 8 \text{ so } x = 8^{1/8}$$

One might also try to solve the equation be means of series, putting

$$x_0 = a, \;\; x_1 = a^{x_0}, \ldots, x_n = a^{x_{n-1}}$$

Study the nature of this series as a function of a Compare with the "solution" above Explain what you notice.

What all the above examples seem to me to have in common can be expressed in the following words:

> *Anything "obvious" constitutes an obstacle* as, for my students, assertions of the sort $x^2 > x$ or $\sqrt{(a+b)} = \sqrt{a} + \sqrt{b}$.

From all this follows the importance for us as teachers of avoiding situations which are too simple. We should train our students in problem solving and in the critical analysis of complex situations which do not readily lend themselves to "automatic" treatments.

3. Mistakes and my students' conceptions

Wanting to change my teaching practices, while necessary, is not sufficient. I still need some training which has both individual and collective dimensions.

Since the official introduction of *problem-situations* into the curriculum of the first cycle in France, certain institutional problems seem to have disappeared This makes it easier to identify the real difficulties of introducing such a change in practice.

If I want my students to work regularly on rich, complex intriguing, multipurpose problem-situations, I will find it difficult to think of a large number of problems on my own and I will probably not have enough time to explore their principal possibilities before using them in class.

Such work is easier and richer if undertaken collectively. Teacher training institutions, teachers associations, educational journals, have important roles to play in supporting meetings, exchanges and discussions, capitalising on experience For many teachers their initial training seemed like a rite of passage bearing no relation to the job, and inservice training seems a diversion, a distraction from the daily realities of the classroom

Not only do I not see these attitudes as fatal, but I believe it is possible to provoke a rupture in these attitudes, a change in the relationship between professional practice and teacher training. [13] Teachers might invest the times and places of their training with the possibility of doing some real work on teaching practices, and of capitalising in an organised and critical way on their acquired experiences.

As useful, even indispensable, as this methodical scientific exchange on classroom situations may be, I must at the same time undertake a study of *my* students' conceptions. For these, appropriately guided activities in training situations and/or in research can help me to leap over my own conceptual limitations. I shall not be capable of observing and analysing my students' learnings if I am incapable of analysing my own — in mathematics as much as in any other domain (sport, art, literature, etc.), and not only in the classroom.

As for studying the conceptions my students have, it is up to me to reach them by constructing situations which lend themselves to producing observable facts and to collecting these observations. [13] At each moment I may need to have some idea what a number is for *this* student, what a parallelogram is for him, a function, a derivative.

Would I then still need to make a fuss about my students' mistakes?

Notes

[1] K Ireland and M Rosen, *A classical introduction to modern number theory* New York: Springer-Verlag

[2] This example makes it very clear that the *epistemological obstacle*, in Bachelard's sense, is not a sketchy received idea but the fruit of lengthily elaborated scientific discussions

[3] Alain Bouvier, La théorie des ensembles *Que sais-je* no 1363 PUF

[4] See Alain Bouvier, *La mystification mathématique* Paris: Hermann, 1981

[5] Note in passing, in the highly improbable case that the reader might have missed it. that it took more than five years teaching to achieve this very modest objective!

[6] Que nous apprennent les erreurs de nos élèves. *Bull. APMEP* 335 (1982) p. 657-670

[7] States of reality for the subject

[8] Which I have already used in my previous article ' On strategies for teaching ', *For the Learning of Mathematics* 5 1 (1985) p. 2-11

[9] Since 1986 this term and this idea have been adopted by the official programmes for students of mathematics in French colleges (ages 12 to 16)

[10] On this subject see the excellent book by Jacques Nimier. *Les maths. le français. les langues ... à quoi ça me sert?* Cedic/Nathan, 1985

[11] H. Wallon has stressed the importance of this subject.

[12] See also Laurence Viennot, *Le raisonnement spontané en dynamique élémentaire* Paris: Hermann, 1979

[13] I have recently been given the responsibility of overseeing all the inservice teacher training in the Lyon region.

[14] See *Didactique des mathématiques le dire et le faire* Cedic, 1986 In this book I have collected together the observations of 40 educators from several countries

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The whole part/whole distinctions that proved so important in Western thought and knowledge systems play a minor role in Navajo knowledge. Navajos tend to speak of the world in terms of process, event, and fluxes, rather than parts and wholes or clearly distinguishable static entities. The emphasis is on continuous changes rather than on atomistic structure This becomes more understandable when one envisions the world as constituted of dynamic "things", forces, changes, actions, and interactions: it is then not quite so obvious (as it is in a static world, a world of objects and constant forms) to try to segment perceptual and actional aspects, to distinguish parts in the continuous flux as a first and self-evident characterization. The difference is crucial, I feel, and it must be made totally explicit: "objects" cannot be defined in the same way, "form" cannot be understood in quite the same way as in the Western outlook, since all aspects of reality in Navajo knowledge are process-like and not thing-like. (.) even in Thom's "dynamic topology" (. . .) the dynamic phenomena are rendered understandable and manageable by petrifying them, that is, by defining them as changes between initial and final states In other words, a dynamic phenomenon can be grasped in the Western outlook primarily by defining it as a transformation of a static phenomenon (state, situation) since it can then be segmented. (. .) This approach is generally alien, inappropriate, and unnatural to the Navajo eye . .

Rik Pinxten
