TEACHER BELIEFS AND THE DIDACTIC CONTRACT ON VISUALISATION

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In the last twenty years or so the debate about the potential contribution of visual representations to mathematical proof has intensified (*e.g.,* Mancosu *et al.,* 2005). Central to this debate is whether visual representation can be used not only as evidence and means of insight for a mathematical statement but also as part of its justification (Hanna & Sidoli, 2007). For example, Giaquinto (2007) argues that visual means are much more than a mere aid to understanding and can be resources for discovery and justification, even proof. Whether visual representations need to be treated as adjuncts to proofs, as an integral part of proof or as proofs themselves remains a point of contention.

Visualisation has gained analogous visibility within mathematics education. Its richness, the many different roles it can play in the learning and teaching of mathematics – as well as its limitations – are increasingly being written about (*e.g.,* Arcavi, 2003). The foci of these works are as diverse as: curriculum development with an emphasis on visualisation (and often on related IT); mathematicians' perceptions and use of visualisation; students' seeming reluctance to engage (and difficulty) with visualisation; gender differences; links with embodied cognition; *etc.* – see Presmeg (2006) for a substantial review. Overall we still seem to be rather far from a consensus on the many roles visualisation can play in mathematical learning and teaching. So, while many works clearly recognise these roles, several also recommend caution with regard to 'the 'panacea' view that mental imagery only benefits the learning process' (Aspinwall, Shaw, & Presmeg, 1997).

Research into the relationship between teachers' beliefs (epistemological and pedagogical) and pedagogical practice has attracted increasing attention by mathematics education researchers (*e.g.,* Cooney, Shealy, & Arvold, 1998; Leatham, 2006; Leder, Pehknoen, & Torner, 2002; Thompson, 1992). With regard to teacher beliefs about mathematical reasoning and proof, as Harel and Sowder (2007) observe, 'the emphasis that teachers place on justification and proof no doubt plays an important role in shaping students' "proof schemes"' (p. 827). The not very extensive research in this area shows that this emphasis is insufficient both in terms of extent and in terms of quality. Internationally, in most educational settings – even those with an official curricular emphasis on proof – little instructional time is dedicated to proof construction and appreciation. Furthermore, teachers' own proof schemes are often predominantly empirical and teachers do not always seem to understand important roles of proof other than verification. For example, in Knuth's (2002) study of practising secondary mathematics teachers, while all teachers acknowledged the verification role of proof, they rarely talked about its explanatory role. With regard to their proof schemes many of the interviewed teachers: felt compelled to check a statement on several examples even though they had just completed a formal proof; considered several of given non-proofs as proofs; accepted the proof of the converse of a statement as proof of the statement; and, found arguments based on examples or visual representations to be most convincing.

One of the aims of the study we draw on here is to explore the relationship between teachers' beliefs about mathematical reasoning and pedagogical practice. Our particular focus is on teacher beliefs about the role of visualisation as evident in the reasoning and feedback they present to students. Even more specifically we aim: to explore teachers' beliefs about the sufficiency of a visual argument, their views on the persuasiveness of a visual argument, their beliefs about the role of visual thinking in their students' mathematical learning and their personal mathematical images; and, to discern the influence of these beliefs, views and images on the didactical contract (Brousseau, 1997) they offer their students with regard to the role of visualisation. One such potential influence originates in the 'inherent sensibility' of the teachers' belief systems (Leatham, 2006).

The study and the Tangent Task

The study we draw on is currently in progress in Greece and in the UK. In this project we engage teachers with tasks that feature classroom scenarios that are hypothetical, but likely to occur in actual practice, and grounded on learning and teaching issues that previous research and experience have highlighted as seminal. The structure of the tasks is as follows: reflecting upon the learning objectives within a mathematical problem (and solving it); interpreting flawed (fictional) student solution(s); and, describing, in writing, feedback to the student(s). [1] In brief, in the terms of Adler and Davis's (2006) classification, our tasks are of the $MU^{+}tU^{+}$ type: 'a clear mathematical object that is primary … and a teaching object that is secondary… . In both cases explicit reasoning of various solutions and pedagogical steps is required' (p. 286). Overall our tasks aim to help teachers develop their capacity to transform theoretical knowledge into theoretically-informed practice – see (Watson & Mason, 2007) – a transformation that has been described by concepts such as Hill and Ball's (2004) mathematical knowledge for teaching and Shulman's (1987) pedagogical content knowledge.

In what follows we focus on one of the tasks (fig. 1) we

Year 12 students, specialising in mathematics [2], were given the following exercise:

Examine whether the line with equation $y = 2$ is tangent to the graph of function f, where $f(x) = 3x^3 + 2$.

Two students responded as follows:

Student A:

'I will find the common points between the line and the graph solving the system:

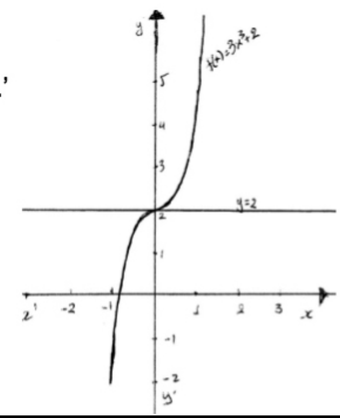
$$\begin{cases} y = 3x^3 + 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 + 2 = 2 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} 3x^3 = 0 \\ y = 2 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 2 \end{cases}$$

The common point is $A(0, 2)$.

The line is tangent of the graph at point A because they have only one common point (which is A).'

Student B:

'The line is not tangent to the graph because, even though they have one common point, the line cuts across the graph, as we can see in the figure.'



Questions:

a. In your view what is the aim of the above exercise?

b. How do you interpret the choices made by each of the students in their responses above?

c. What feedback would you give to each of the students above with regard to their response to the exercise?

Figure 1. The Tangent Task

have used. The Task was one of the questions in a written examination taken by 91 candidates for a Masters in Mathematics Education programme. All were mathematics graduates with teaching experience ranging from a few to many years. Most had attended in-service training of about 80 hours.

The first level of analysis of the scripts consisted of entering in a spreadsheet summary descriptions of the teachers' responses with regard to the following: perceptions of the *aims* of the mathematical exercise in the Task; mathematical *correctness; interpretation/evaluation* of the two student responses included in the Task; *feedback* to the two students. Adjacent to these columns there was a comments column for our observations regarding the approach the teacher used (verbal, algebraic, graphical) to convey their commentary and feedback to the students across the script. On the basis of this first-level analysis we selected 11 of the participating teachers for interviews. Their individual interview schedules were tailored to the analysis of their written responses and, mostly, on questions we had noted in the comments column of the spreadsheet. Interviews lasted approximately 45 minutes and were audio recorded.

The mathematical problem within the Task in figure 1 aims to investigate students' understanding of the tangent line at a point of a function graph and its relationship with the derivative of the function at this point, particularly with regard to two issues that previous research (*e.g.,* Biza, Christou, & Zachariades, 2008; Castela, 1995) has identified as critical:

- students often believe that having one common point is a necessary and sufficient condition for tangency; and,
- students often see a tangent as a line that keeps the entire curve in the same semi-plane.

The studies mentioned above attribute these beliefs partly to students' earlier experience with tangents in the context of the circle, and some conic sections. For example, the tangent at a point of a circle has only one common point with the circle and keeps the entire circle in the same semi-plane.

Since the line in the problem is a tangent of the curve at the inflection point A the problem provides an opportunity to investigate the two beliefs about tangency mentioned above - similarly to the way Tsamir, Rasslan, and Dreyfus (2006) explore teachers' images of derivative through asking them to evaluate the correctness of suggested solutions. Under the influence of the first belief Student A carries out the first step of a correct solution (finding the common point(s) between the line and the curve), accepts the line tangent to the curve and stops. The student thus misses the second, and crucial, step: calculating the derivative at the common point(s) and establishing whether the given line has slope equal to the value of the derivative at this/these point(s). Under the influence of both beliefs, and grounding their claim on the graphical representation of the situation, Student B rejects the line as tangent to the curve.

Evidence of teacher beliefs about the role of visualisation in the scripts

One of the themes that emerged from the comments recorded in the spreadsheet with regard to the means the teachers used (verbal, algebraic, graphical) to convey their feedback to the students concerned the *beliefs (epistemological and pedagogical) of the teachers about the role of visualisation.* In the course of this part of our analysis we noticed several influences on the teachers' responses: for example, almost all teachers distinguished between (and often juxtaposed) Student A's *algebraic* approach and Student B's *graphical* approach. In many cases they referred explicitly to their beliefs about, for example, the sufficiency/acceptability of the graphical approach; or about the role visual thinking may play in their students' learning. The teachers' responses also appeared significantly influenced by the mathematical context of the problem within the Task; namely, by their own perceptions of tangents and their own views as to whether the line in the Task must be accepted as a tangent or not.

At this point of our analysis we were somewhat surprised by the fact that 43 of the 91 participating teachers supported Student B's claim that the line in the Task is not a tangent line – explicitly (25/91) or implicitly (18/91). From the analysis of the data from the 25 teachers who explicitly supported Student B's claim three interrelated themes regarding teachers' perceptions of tangents and their beliefs about visualisation emerged: *Mathematical views* on whether the line is a tangent or not, *beliefs about the sufficiency/acceptability of the visual argument* used by Student B and *beliefs about the role of visual thinking in their students' mathematical learning.* Below we take a closer look at the evidence from these 25 teachers. [3]

Mathematical views

Of the twenty-five teachers who explicitly accepted Student B's claim, ten rejected the line as a tangent without stating an argument (phrasing their responses as if this was obvious). The other fifteen stated that the line intersects with the curve without being its tangent either because point *A* is an intersection point but not a tangency point; or because it 'cuts across' the graph as student B argued. Three of these fifteen based the rejection on the fact that the line does not keep the entire curve in the same semi-plane. For example, one teacher claimed that 'it is not sufficient that the tangent line has only one common point, but it must keep the graph on the same side' and offered the graph in figure 2. Taking a *local* perspective on Student B's *'cutting across'* argument the teacher also offered figure 3 and said: 'the tangent could cut across the curve … the line is a tangent at x_0 [in figure 3] although it cuts across the curve [at another point, *our addition*]'.

Sufficiency of the visual argument used by Student B

Ten of these twenty-five teachers did not dispute the sufficiency of the visual argument used by Student B. Of these ten, eight made no reference to an algebraic argument. One teacher made some reference to both the 'algebraic and graphical methods' implying that she accepted the validity of both. Another set out with some reference to an algebraic

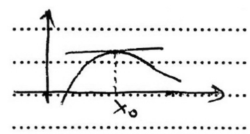


Figure 2.

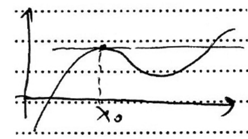


Figure 3.

argument that involved accepting the line. As she proceeded to a consideration of Student B's solution she deleted the algebraic argument and concluded her response with agreeing with Student B.

The other fifteen teachers, while basing their inference on the graph and supporting Student B's claim, stated the need for supporting and verifying the claim algebraically (11 explicitly and 4 implicitly). These teachers, although they hinted at the algebraic argument for the justification of the answer, did not employ it in the argument they offered the students. As a result they did not confront the inconsistency between rejecting the line and the algebraic argument. The teachers proposed the algebraic argument either in order to support (*e.g.,* 'more feasible', offering a 'more rounded view of the problems', *etc.*) or to validate mathematically the graphical one (*e.g.,* 'the graph not necessarily constituting a valid complete proof'). So, the teachers, while appreciating their students' employment of visualisation to reach a conclusion, were keen to stress that ultimately students are expected to demonstrate their capacity to complete the Task algebraically. It is therefore possible that these teachers' embrace of the visual approach evident in Student B's solution is driven more by their belief in the gradual enculturation (Sierpinska, 1994) of the students into formal mathematical practice and their belief in the assistance that visualisation can provide towards reaching a conclusion (rather than a belief in the completeness of a graph-based argument).

Views on the role of visual thinking in student learning

Nine of these twenty-five teachers declared overtly their view of the graphical approach employed by Student B as evidence of 'conceptual' understanding (*e.g.,* 'understands to a good degree what mathematical thinking is about', 'imaginative', 'quick and ready', 'clever', 'not wasting any time', 'natural', 'real'; as opposed to the algebraic method being 'traditional', 'mechanical', *etc.*). In this vein, several teachers, while expressing their appreciation for the graphical approach, stressed that students are not always at ease with it and are often reluctant to use visualisation – they therefore saw its use by Student B as meriting appreciation because of its rarity.

With regard to how these twenty-five teachers responded to Student A – particularly given that student's exclusive use of the algebraic method – many attempted to balance their feedback ('the teacher must encourage students to work in both ways'). They therefore encourage Student A to also work graphically ('this would help him') and Student B to also 'justify their answer algebraically'.

Probing further – particularly through the interviews – into these teachers' commitment to accepting visual arguments, such as the one used by Student B, revealed a more ambiguous landscape than some of their statements in the written responses had suggested. We exemplify the complexity of this landscape through the case of one of the teachers, Nikos (see also Biza, Nardi, & Zachariades, 2009).

Variation and relativity of teachers' beliefs about the role of visualisation – the case of Nikos

Nikos initially claimed that, generally, he would accept a proof based on a graphical argument. Then he added that 'it depends on the case' and he offered a case in which the line is a tangent without splitting the curve (*e.g.*, a tangent at a point of the graph of $y = x^2 + 1$). 'For this case', he said, 'I would like the algebraic solution' because 'when you need to prove something … [the graph] is not enough … the graph could be inaccurate'. When asked 'why he accepted Student B's graphical response', he said that he believed that this line was not a tangent and 'if you make a graph in which the line is not a tangent, I will accept it … that means, from the graph I will accept it' and he added that 'if I believed that the line is a tangent … I would not accept only the graphical [solution]'.

It appears that Nikos is ready to accept the visual argument without any algebraic verification as the information in the image is, to him, a clear and convincing proof that the line is not tangent. Furthermore, he made clear that to prove that a line *is* a tangent an algebraic argument is necessary. What is not clear from the written response and the interview is whether Nikos intended to make this distinction clear to his students. Keeping this distinction covert is likely to have serious ramifications for the didactical contract a teacher offers their students with regard to mathematical reasoning and proof. Below we draw on the evidence from two teachers, Spyros and Anna, in order to discuss briefly some of these ramifications. We note that Spyros is one of the 38 (out of 91) teachers who rejected Student B's claim that the line is not a tangent. Anna is one of the 25 teachers who agreed with Student B's claim specifically. [3]

Covert variation of teacher beliefs and the didactical contract – the cases of Spyros and Anna

While discussing Student B's response we asked Spyros to elaborate on whether he would accept an argument based on a graph. His answer was firm: 'No, first of all it is not an adequate answer in exams'. (We note that in the Year 12 examination, which is also a university admission exam, there is a requirement for formal proof). We asked him to let aside the examination requirements for a moment and consider whether an argument based on a graph would be adequate mathematically. He replied: 'Mathematically, in the classroom, I would welcome it at lesson-level and I would analyse it and praise it, but not in a test'. Asked to elaborate he said: 'Through [the graph-based argument] I would try to lead the discussion towards a normal proof…with the definition, the slope, the derivative, *etc.*'. Asked to justify he said: 'This is what we, mathematicians, have learnt so far. To ask for precision. … we have this axiomatic principle in our minds. … And this is what is required in the exams. And we are supposed to prepare the students for the exams.'

In the above, Spyros's statement is clear: while he cannot accept a graph-based argument as proof, he recognises graph-based argumentation as part of the learning trajectory towards the construction of proof. He seems to approach visual argumentation from three different and interconnected perspectives: the *restrictions of the current educational setting,* in this case the Year 12 examination; the *epistemological constraints* with regard to what makes an argument a proof within the mathematical community; and, finally, the *pedagogical role* of visual argumentation as a means towards the construction of formal mathematical knowledge.

These three perspectives reflect three roles that a mathematics teacher needs to balance: *educator* (responsible for facilitating students' mathematical learning), *mathematician* (accountable for introducing the normal practices of the mathematical community) and *professional* (responsible for preparing candidates for one of the most important examinations of their student career). Spyros' awareness of these roles, and their delicate interplay, is evidence of the multi-layered didactical contract he appears to be able to offer to his students.

Anna's case is rather different. Her beliefs about the acceptability or not of a visual argument appear less clear. She appears ready to accept a visual argument without any algebraic justification if the information in the image constitutes, for her, convincing support for a claim. In her script she regarded the image in the Task as sufficient evidence for determining that the line is not a tangent – also drawing on her belief that a tangent cannot intersect the graph. However she stated clearly that to prove that the line *is* a tangent an algebraic argument was necessary. Later, she stated that she could accept a *correct* statement based on the graph. When we shook her faith in the graph she declared the algebraic solution necessary. While initially she did not speak of validation of the visual statement through reference to mathematical theory, she asked for such validation when she realised that the image could be misleading.

Many times in her interview she returned to her appreciation of visual representation and argumentation as evidence of a student's in-depth understanding and as an important means towards students' construction of mathematical knowledge. She did not specify whether she meant *formal* mathematical knowledge (for example, proof). Furthermore, her views with regard to the sufficiency and acceptability of a visual argument appeared rather ambivalent and heavily dependent on the specific images involved in the discussion. In this sense the didactical contract she appears to be able to offer to her students seems less informed by the multitude of issues that underlies Spyros' approach.

Towards a multi-layered and explicit didactical contract regarding visualisation

The Task in figure 1 invited the teachers to offer feedback to two students, one of which had used (incompletely) the algebraic method for deciding whether the line is a tangent and the other had used (incorrectly) a graphical representation of the problem. On this occasion the graph contained information that conjures up images that may lead to the rejection of the line as a tangent. About half of the teachers in our study appeared to get 'carried away' by this information – or, in Aspinwall *et al.*'s (1997) term, by these 'uncontrollable' images – and agreed with Student B's incorrect claim that the line is not a tangent. In the evidence we presented above the teachers appeared to get 'carried away' not simply by the images that they hold about tangents, conjured up by the graph in Student B's response, but also by a compelling tendency to support what they described as the more 'conceptual', 'imaginative', *etc.* approach of Student B. To them the mathematical problem in the Task offered an opportunity to convey their appreciation for the employment of visualisation. However lack of awareness of the problems that certain imagery may cause, in this case the graphical representation of tangency at an inflection point, stands in the way of fulfilling the potential within the employment of visualisation.

As the example of Nikos illustrates, this compelling tendency of the teachers to support a visually based approach by their students is less clear-cut than initial impressions may suggest. Even though, to this teacher, the persuasiveness of the image was so strong that he did not feel compelled to seek further verification, it is perhaps unwise to infer from this response the teacher's stable belief about the sufficiency of a visual argument. As he explained in the interview, this tendency would apply in the cases in which the image is persuasive (in this case an image that suggests that a line *is not* a tangent). This variation of perspective on when a visual argument is acceptable and when it is not is far from alien in the world of mathematics. However, not addressing this variation explicitly in teaching is likely to have serious ramifications for the didactical contract offered to students with regard to mathematical reasoning and proof. The juxtaposition of the evidence from Anna and Spyros offered us an opportunity to explore these potential ramifications a little further.

Spyros' clear insistence on the class' collective arrival at a formal proof as closure to the lesson is distinctly different from Anna's fluctuation between cases where she would and would not accept a visual argument. Her willingness to rely, occasionally, on imagery in order to support a claim is 'a practice that may mislead students into thinking that such are acceptable mathematical "proofs" and reinforcing the acceptability of their empirical proof schemes' (Harel & Sowder, 2007, p. 829). Furthermore, her own criteria about what makes a visual argument acceptable appeared very personal and rather fluid. Within the unstable didactical contract that this vagueness might imply, how would her students distinguish between when a visual argument is acceptable and when it is not? Variation and multi-layeredness are welcome reflections of the complexity that characterises mathematical activity. However, in the already compounded didactical contract of school mathematics, such vagueness can be detrimental.

A less vague, subtler contract could be as follows: in a classroom discussion where a visually-based (incorrect) claim is proposed, the class employs the algebraic, formal approach to convince the proposer about the incorrectness of their claim. Even when a visually based (correct) claim is unequivocally accepted by the whole class, the class still employs the algebraic approach to establish the validity of the claim formally. In both cases visualisation emerges as a path to insight, and proof as the way to collectively establish the validity of insight. In both cases there is a pedagogical opportunity for linking imagery with algebra and for embedding the algebra in the immediately graspable meaning in the image. The above suggest a role for proof in the mathematics classroom that is not disjoint from the creative parts of visually-based classroom activity and that reflects an essential intellectual need, 'an understanding of proof that is consistent with that shared and practised by the mathematicians of today' (Harel & Sowder, 2007, p. 836).

The identification of subtle issues regarding the teachers' pedagogical and epistemological beliefs (such as the above, about visualisation) – and the raising of teacher awareness of these issues – was made possible, we believe, by the combination 'task engagement-followed by-interview'. In general (Biza *et al.,* 2007), we propose that these tasks can be employed in a teacher education context as follows:

> … as tools for the identification and exploration of mathematically, didactically and pedagogically specific issues regarding teacher knowledge (that purely theoretical questions on pedagogy or mathematics could not have identified); and, as triggers for teacher reflection on these issues. We also note that engagement with these tasks can function as a preliminary, preparatory, smoother transitory phase for pre-service teachers prior to their exposure to real classroom situations. (p. 309)

Notes

[1] For an elaborated description of the theoretical origins of this type of task, see Biza, Nardi, and Zachariades (2007).

[2] For details about the Greek curricular context of the task, see Biza, Nardi, and Zachariades (2008).

[3] For more detail on these views, see Biza, Nardi, and Zachariades (2008).

References

Adler, J. and Davis, Z. (2006) 'Opening another black box: researching mathematics for teaching in mathematics teacher education', *Journal for Research in Mathematics Education* **37**(4), pp. 270–296.

Arcavi, A. (2003) 'The role of visual representations in the learning of mathematics', *Educational Studies in Mathematics* **52**, pp. 215–241.

Aspinwall, L., Shaw, K. L. and Presmeg, N. C. (1997) 'Uncontrollable mental imagery: graphical connections between a function and its derivative', *Educational Studies in Mathematics* **33**, pp. 301–317.

Biza, I., Christou, C. and Zachariades, T. (2008) 'Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis', *Research in Mathematics Education* **10**(1), pp. 53–70.

Biza, I., Nardi, E. and Zachariades, T. (2009) 'Do images disprove but not prove? Teachers' beliefs about visualisation', in Lin, F. L., Hsieh, F. J., Hanna, G. and de Villiers, M. (eds.), *Proceedings of the 19th Study of the International Commission on Mathematical Instruction: Proof and proving in mathematics education,* Taipei, TW, 1, pp. 59–64.

Biza, I., Nardi, E. and Zachariades, T. (in press) 'Teachers' views on the role of visualisation and didactical intentions regarding proof', in *Proceedings of the 6th Conference of European Research in Mathematics Education,* Lyon, FR.

Biza, I., Nardi, E. and Zachariades, T. (2008) 'Persistent images and teacher beliefs about visualisation: the tangent at an inflection point', in Figueras, O. and Sepúlveda, A. (eds.), *Proceedings of the 32nd annual conference of the International Group for the Psychology of Mathematics Education,* Morelia, MX, 2, pp. 177–184.

Biza, I., Nardi, E. and Zachariades, T. (2007) 'Using tasks to explore teacher knowledge in situation-specific contexts', *Journal of Mathematics Teacher Education* **10**, pp. 301–309.

Brousseau, G. (1997) *Theory of didactical situations in mathematics,* Dordrecht, NL, Kluwer.

Castela, C. (1995) 'Apprendre avec et contre ses connaissances antérieures:

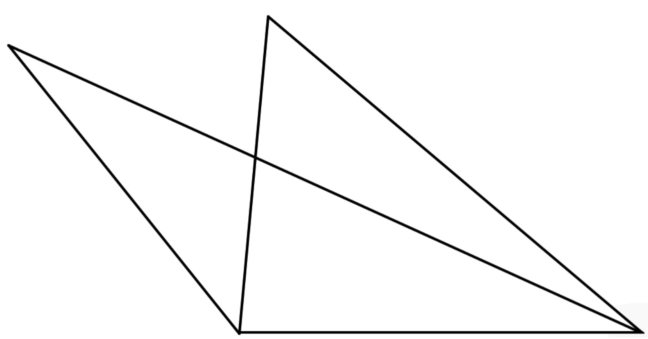
un example concret, celui de la tangente', *Recherches en didactiques des mathématiques* **15**(1), pp. 7–47.
Cooney, J., Shealy, B. E. and Arvold. B. (1998) 'Conceptualizing belief structures of preservice secondary mathematics teachers', *Journal for Research in Mathematics Education* **29**(3), pp. 306–333.
Giaquinto, M. (2007) *Visual thinking in mathematics,* New York, NY, Oxford University Press
Hanna, G. and Sidoli, N. (2007) 'Visualisation and proof: a brief survey of philosophical perspectives', *ZDM Mathematics Education* **39**, pp. 73–78.
Harel, G. and Sowder, L. (2007) 'Towards comprehensive perspectives on the learning and teaching of proof', in Lester, F. K. (ed.), *The second handbook of research on mathematics teaching and learning,* Reston, VA, NCTM, pp. 805–842.
Hill, H. and Ball, D. (2004) 'Learning mathematics for teaching: results from California's Mathematics Professional Development Institutes', *Journal for Research in Mathematics Education* **35**(5), pp. 330–351.
Knuth, E. (2002) 'Secondary school mathematics teachers' conceptions of proof', *Journal for Research in Mathematics Education* **33**(5), pp. 379–405.
Leatham, K. R. (2006) 'Viewing mathematics teachers' beliefs as sensible systems', *Journal of Mathematics Teacher Education* **9**, pp. 91–102.
Leder, G. C., Pehkonen, E. and Torner, G. (eds.) (2002) *Beliefs: a hidden variable in mathematics education?,* Dordrecht, NL, Kluwer.
Mancosu, P., Jorgensen, K. F. and Pedersen, S. A. (eds.) (2005) *Visualization, explanation and reasoning styles in mathematics,* Dordrecht, NL, Springer.
Presmeg, N. C. (2006) 'Research on visualization in learning and teaching mathematics: emergence from psychology', in Gutierrez, A. and Boero, P. (eds.), *Handbook of research on the psychology of mathematics education,* Dordrecht, NL, Sense, pp. 205–235.
Shulman, L. (1987) 'Knowledge and teaching: foundations of the New Reform', *Harvard Educational Review* **57**(1), pp. 1–22.
Sierpinska, A. (1994) *Understanding in mathematics,* London UK, Falmer.
Thompson, A. (1992) 'Teachers' beliefs and conceptions: a synthesis of the research', in Grouws, D. (ed.), *Handbook of research on mathematics teaching and learning,* New York, NY, Macmillan, pp. 122–127.
Tsamir, P., Rasslan, S. and Dreyfus, T. (2006) 'Prospective teachers' reactions to right-or-wrong tasks: the case of derivatives of absolute value functions', *Journal of Mathematical Behavior* **25**, pp. 240–251.
Watson, A. and Mason, J. (2007) 'Taken-as-shared: a review of what is known about mathematical tasks in teacher education', *Journal of Mathematics Teacher Education* **10**, pp. 205–215.

Here are responses given by three different learners in a Grade 8 class to the task, "Draw a triangle with 2 obtuse angles." How would you evaluate these explanations?

1) It is impossible. An obtuse angle is more than 90 degrees, so two obtuse angles gives you more than 180 degrees. The angles of a triangle sum to 180 degrees.

2) It is impossible because you get a quadrilateral, not a triangle.

3) It is impossible, because if you draw an angle close to 90 degrees, say 89 degrees, and you stretch it, the other two angles will shrink so you won't get another obtuse angle.



(adapted from Adler, J. (1999) 'Seeing and seeing through talk: the teaching dilemma of transparency in multilingual mathematics classrooms,' *Journal for Research in Mathematics Education* **30**(1), pp. 47–64)