Two Examples of "a Split Situation" in a Mathematics Class

CLAUDE COMITI, DENISE GRENIER

Nous présentons ici quelques-uns des résultats d'une recherche ayant pour objectif général de décrire et de modéliser le fonctionnement de situations de classes ordinaires Notre travail met en particulier en évidence comment certains évènements, pouvant apparaître comme contingents dans le déroulement de la classe, se révèlent , à l'analyse, s'expliquer par la distance existant entre la situation didactique que le professeur a voulu créer et celle dans laquelle travaillent certains élèves. Après avoir décrit notre cadre théorique, les questions de notre recherche et présenté et justifié notre méthodologie, nous montrerons, à partir de l'analyse de deux exemples particuliers concernant l'introduction de la racine carrée en classe de Troisième (élèves de 14 à 15 ans), l'intérêt d'une modélisation en terme de "dédoublement de situation".

Background to the study

I. Introduction

Our study places us alongside other researchers who, rather than using the theoretical/practical link in the unilateral sense of applying theory to practice, use theory in order to describe and interpret practices [COMITI, 1987; GRENIER, 1993; TOCHON, 1992]. From this point of view, the classroom is not simply a site of application but also becomes a site of development for the researcher But what forms and directions might this development take?

In this article we will present theoretical devices developed to conduct this analysis, show that certain methodological precautions should be taken when analysing classroom practice scientifically, and present the initial results obtained

II. Theoretical framework

In this paper two fields of research overlap:

— the study of the way the teacher functions, whether it be what he/she "thinks" and his/her representations [CLARK-PETERSON, 1987; ERNEST, 1989; ROBERT, 1989; THOMPSON, 1992], or his/her practices and decision-making [ARSAC *et al.*, 1992; CLARK-YINGER, 1987; ROMBERG, 1988; SCHOENFELD, 1988; TOCHON, 1989];

— Brousseau's "théorie des situations" which allows situations to be constructed and analysed, and within which the evolution of the pupil's knowledge can be certified. This theory allows us to read and interpret a local situation and give meaning to what the partners (teacher and pupils) do in the situation

The theorisation put forward by Guy Brousseau [1986; 1989] describes these situations as being scenes of an interaction, a game, between pupils and a "milieu" In systemic terms, if the didactic system is structured around the "teacher-pupil-knowledge taught" triangle, the "milieu" is, within this system, the system of the pupil's opponent.

As a model of the environment and its responses which are relevant to the learning at hand [Margolinas, 1994], the "milieu", therefore, plays a central role:

— in teaching, as a reference and as an epistemological object The teacher builds his/her teaching around the hypothesis that the students function within the "milieu" appropriate to the situation he/she proposes.

— in learning, as an impulse towards adaptation. The student placed in a problem-situation reacts by activating the "milieu" which enables him/her to "read" the situation, and the personal devices he/she can bring to bear on its objects in order to understand the questions asked and to answer them.

Through these actions in the milieu, and through an interpretation of the milieu's counteractions, which provide him with elements to validate his solution, the pupil develops ways of adapting to the situation with which he is confronted.

This concept of milieu serves as a model enabling us to describe, explain, and predict the student's behaviour, the feedback from the situation, and its impact on the student's behavior. When student production seems to have no connection with the questions asked, it may indicate that the situation supposed by the teacher is not the one in which the student actually operates.

III. The research problem

The research presented here is the second part of several years' work, the first part of which concentrated on mathematics teachers' beliefs about their discipline, their job, their students, and the way in which the latter learn. This second part focuses on the study of "practices" in the classroom situation

In their lesson preparations teachers plan their forthcoming practice, taking into account the representations they have of their discipline, the mathematics to be taught, the phenomena of the transfer of knowledge, and their students' means of learning. In practice, work undertaken in the mathematics class is often centered on knowledge other than the knowledge the teacher is trying to transmit. It throws light on notions which the teacher considers to be part of the student's acquired knowledge but which, in fact have not yet been fully acquired and therefore pose a problem. So, paradoxically, this results in classroom interaction taking an unforeseen turn: unpredicted events of different types are triggered which impinge on the implementation of the original plan, so the teacher is unable to pursue it It is vital to analyse these triggered events for two reasons: firstly, for what they reveal about the classroom interaction, and secondly for what they tell us about the students' learning strategies

To clarify out study, it is necessary to distinguish three levels of analysis:

— that of "natural" events, which pertain to reality without the interference of an observer,

— that of "observed" events, which are isolated and selected by an observer as data for analysis,

— that of the "didactic phenomenon", which is the researcher's interpretation of the analysed data.

Our hypothesis is that numerous unforeseen events in the classroom situation might be explained by the gap between the "milieu" supposed by the teacher and that in which some students actually function and, therefore, by the distance between the teacher's situation and that of the student. In this context, our research sets out to develop theoretical and methodological devices in order to create models in terms of didactic phenomena that give significance to unforeseen events in the classroom situation which we have identified as important for students' learning.

IV. Methodology

We are confronted with the problem of how to observe ordinary classes, a problem which differs from that of observing situations set up for research purposes. How should the problem of classroom observation be raised in terms compatible with scientific practice? It is a problem which has concerned us for a long time [COMITI, 1988]. Let us briefly recount our point of view on the subject.

The class being observed is a "system" Observing this system involves taking information concerning its "state" The taking of information from the classroom viewed as a system depends on what we mean by observation, and also, obviously, on the theory available for identifying what counts as "relevant" information.

The data collected in observing is not neutral, it is in fact construed. If we do not construe it ourselves, we only collect what the institution itself presents to us But there is no reason why this institutional data should be relevant to our research problem: this is why one cannot reduce observation "of" the classroom to observation "in" the classroom

In order to overcome these problems as far as possible we set up an experimental protocol, described below. It involves on the one hand the contract agreed with the teachers concerned, and on the other hand the setting up of a means of gathering data which allows the taking of "mixed" information.

The contract with the teachers:

1. They are willing to participate in the research, which means that they must agree with the question, that they participate in setting up the protocol defined by the transcription of recordings and the notes of the observers, and they contribute towards the analysis of the data gathered on their lessons

2. The choice of the mathematical content on which their practices are observed is made in common, but devising and managing the teaching sessions is entirely their responsibility.

The collecting of data. The data collected is of several types:

1. The scenario for the sequence of different learning stages, written before the beginning of the lesson by each of the teachers concerned.

2. Interviews with each teacher:
   - An interview prior to the sequence, about the scenario for a planned lesson, on the notion to be taught, the activities planned, the teaching methods envisaged, the objectives;
   - An interview after the teaching sequence aimed at collecting what the teacher has to say on what happened during the lessons, as well as his/her analysis of any differences with what had been planned

3. The recording of discussions at the very end of the sequence.

4. Observations and sound recording of sessions in the classroom

The study

1. The learning context

The study presented here is on a sequence of about ten sessions of 55 minutes in two classes of 14-15 year olds from two different schools. The sequence is on the teaching of the square root. Our study reveals different types of events, unplanned by the teacher but which take place in the classroom situation, which we can classify as follows: the factors which trigger them, the role they play in the students' learning, and the different possible means available to the teacher to control this or that type of event. We will leave aside events which are not relevant to the teaching/learning project, events which could occur in any teaching (language teaching or history teaching, for example), not because they are without interest (they would be to the psychologist or sociologist), but because they are not relevant to our particular research question.

In what follows we have chosen to analyse a type of didactic phenomenon which seems essential as it corresponds to the kind of event in our typology that is very often observed in classrooms: the appearance of student production that seems to have no connection with the question asked Below we present analyses of the first session, dealing with the introduction to the square root, in each of the classes. Teacher A used an arithmetic introduction, starting from a reminder about the properties of squares, Teacher B used a geometric introduction In each case, we infer the teacher's project from the confrontation of the planned scenario with the different statements which preceded the sequence and the observations obtained while this sequence was being carried out.

II. First example (Teacher A)

AN ARITHMETIC INTRODUCTION OF THE SQUARE ROOT
a milieu of signs and writing rules … versus
a milieu of numbers equipped with multiplication

II.1 *First analysis*

Let us analyse the beginning of the sequence.

T(eacher):	Take out your rough paper I am going to ask you three questions, which I will put on the board. Don't copy the questions, just try to answer them individually and then we'll discuss them 1. "Are there any numbers whose square is -1?" 2 "Can two different numbers have the same square?" 3 "Are any of the following numbers square integers?"
She writes on the board: 40, 9, -16, 0, 25/4, 1, 400, 10^5, 121, 0.04, 9^{10}.	

The aim of the problem situation is to be sure that the students know how to find the squares of positive and negative integers, of decimals and rationals, and to institutionalize the properties of squares, in order to introduce afterwards the square root of a number (condition of existence, definition, and properties).

Here is our first analysis What we describe below corresponds to the teacher's perception of the situation.

> The M-milieu on which students are supposed to be able to act is formed of sets of numbers **N, Z, D, Q,** equipped with multiplication. The devices presumed to be available are:
> —the definition of the square of a number,
> —the properties of multiplication (sign rule) in **Z**.
> These devices functioning in the M-milieu should allow the students to produce couples (a, b) where b is positive and equal to a^2.

However what happens is that unexpected events disturb the lesson which are inexplicable in the terms of the first analysis.

T:	OK, go! Begin to correct what you have done. Can you find any number whose square is "minus one"? …
Michael:	(Yes) If you take the negative square …
T:	Michael, what do you want to say? The negative square, the square of a negative number? Let's listen to Michael.
Michael:	You take one, you put minus.
T:	I take one and how do I write it? I put minus one … [she writes $(-1)^2$ and that makes?
Michael:	No!!!
T:	Come and write it for us
Michael goes to the board and writes underneath: $-(1)^2 = -1$.	

Michael's affirmation is correct but entirely unexpected by the teacher. According to the teacher's situation (M-milieu), the students know the sign rule, and for this reason she expected the answer: "No, because a square number is always positive". The teacher does not understand Michael's answer because what he wrote on the board is inexplicable in terms of the analysis conducted above In the M-milieu, $-(1)^2$ cannot be the writing of a square number!

However, Michael's answer turns out to be quite acceptable to a considerable number of pupils in the class While the teacher had planned that the answer to the three questions asked at the beginning of the session, and the reminder about the properties of squares, would only last about ten minutes, she ends up spending more than half an hour trying to help the pupils understand that, of course, Michael's equality is correct, but that it has "nothing to do" with the question: Are there numbers whose square is -1?

II.2 *Second analysis*

Michael's answer and the other students' replies which follow it are inexplicable in terms of the analysis conducted in the M-milieu, but we can explain them if we suppose they take place in another situation S′ centered on another milieu M′.

> The M′-milieu in which these students work is defined by the set **N** and signs such as: parentheses ( ); the minus sign -; the fraction sign /; the comma between two numbers; the square sign (^{2}). The devices which function naturally in this "milieu" are the usual rules for writing these signs (knowledge of conventions), of the type: a; - a; a/b; a,b; a^2; - a^2; $(- a)^2$; - $(a)^2$. The possible productions are thus the writings that can be obtained by combining numbers with the signs available.
> When Michael replies: "Yes, because minus (one squared) equals minus one", this answer leads to a discussion which confirms that other students function like Michael. Briefly, in the sequence that follows, the reactions of several students show that their explanations are interpretable in the second situation

To come back to her initial objective, the teacher asks "good" students, those who work with the good M-milieu

T:	Lise, the answer to question 1 is what?
Lise:	Well, it's no!
T:	It's no! Why?
L:	The square of a negative number is always positive!
T:	The square of a negative number is always positive. And the square of a positive number, then?
L:	Positive
T:	It's always positive. How come the square of a negative number is always positive, Adhil!
Adhil:	It is the property of multiplication of negative numbers.
T:	Yes, go ahead, explain.
A:	If we multiply two negative numbers the result will be positive
We hear a student saying "why"?	
T:	If we multiply two negative numbers the result is always a positive number Thus the square of a negative number is always positive and the square of every number is always positive You'll write it in your notebooks. Is everybody sure of that? OK.

This does not solve the problem as, on several occasions later in the lesson, we see arguments arising from M' and even, at the end of the session, whilst the teacher is writing on the board the property "The square of both -5 and 5 is 25", several students protest.

T writes on the board:	
	"The square of -5 is 25, the square of 5 is 25"
E:	No! That's wrong!
A lot of students:	
	The square of -5 is -25!
T:	Oh! you don't agree?
E:	If you put -5 without brackets you get -25
T:	I did not put a bracket. You want me to put a bracket *there*? (showing -5)
T adds a bracket as follows:	
	"The square of (-5) is 25, the square of 5 is 25."
T:	Does that satisfy you?
E:	Yes!

The analysis undertaken above in terms of splitting into two situations:
—enables us to interpret the interaction between the student and the situation and to give meaning to the student's behavior, throwing light on the mathematical notions with which he is actually working,
—reveals a dysfunction in the didactic contract (there is a double misunderstanding: the student does not understand what the teacher expects of him, and the teacher does not understand the answer the student gives),
—shows how unforeseen events may be explained by the distance between the situation supposed by the teacher and that in which some students actually function,
—shows that in such an emergency situation, the immediate reaction of the teacher is to try to obtain the answer she is expecting from the "good" students, that is those who function in the same milieu and therefore in the same situation as she does

We characterise the didactic phenomenon analysed as A SPLIT SITUATION.

III. Second example (Teacher B)

A GEOMETRIC INTRODUCTION OF THE SQUARE ROOT
a milieu of "drawing/measurement" ... versus
a milieu of "construction/geometric object"

At the beginning of the sequence, two exercises were considered, on which the teacher had planned to spend only a little time (which is what he did). Their goal was to bring the pupils to investigate the following properties relating the sidelength and the area of a square:
—when the sidelength of a square is doubled, its area is not twice as much, but four times as much;
—the area of a square of sidelength a is equal to a^2

Then teacher B gave the following problem to his students:

Construct a square of sidelength 2 cm. Let ABCD denote this square. What is its area? Using this square, construct a square of area equal to two times the area of ABCD. Let VOLE be this new square.

The mathematical problem consists in the geometric construction of a segment of length $\sqrt{8}$ (the side of VOLE).

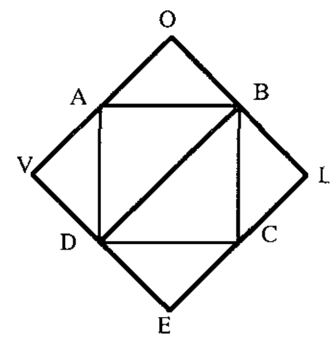
This problem is proposed by the teacher with two objectives:
—to show that $\sqrt{8}$ exists as a number since one can construct a segment of this size, even though one cannot give the value of this number in the sets of numbers **N, D+, Q+**;
—to show that one cannot write its exact value without introducing the symbol √

III.1 *First analysis*

In order to solve this problem, one must first construct the square ABCD of sidelength 2 cm, then compute its area (4 cm^2) and deduce from this that the desired square VOLE has an area of 8 cm^2. All this is nothing but a preliminary tour around the crucial question: construct VOLE from the already existing square ABCD.

Students have to produce pairs (a, a^2) with $a^2 = 8$ and recognize that no such number a exists in **N, D+, Q+**. For this they can use the √ key of their calculator to obtain $\sqrt{8}$, but then must reject this solution because it does not accord with the request to construct the "exact value"

It is therefore necessary to change the viewpoint and to construct the required square by means of geometric data extracted from the given square ABCD. The solution, as expected by T, consists in the construction of the square VOLE by using the diagonal of the square ABCD. One must therefore imagine this diagonal (which is not given *a priori*), compute its length by Pythagoreas' rule, $\sqrt{8}$ cm, and recognize that this length is equal to the sidelength of the desired square



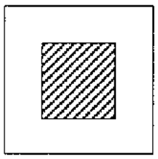
Let us build a model of the theoretical situation (fitting the teacher's expectation) allowing us to analyze how the rejection of the $\sqrt{8}$ value obtained by the √ key of a calculator can be achieved, and how Pythagoras' rule can be brought into play

> The M milieu on which the students are supposed to be able to act is the following:
> —the sets of numbers **N, D+, Q+**, equipped with multiplication,
> —the instruments of construction and measurement,
> —the square geometric object, its graphic representation (whatever its position on the sheet of paper), its sides, its diagonals, its area ... , and the relationship between diagonal and side lengths

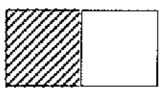
III.2 *What do students do?*

Let us analyse the work produced by the students under observation. Two kinds of productions are found:

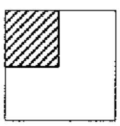
1) A large number of students really need to know the sidelength of the square to make the construction. They start their investigation with computations, using the fact that the area of a square of sidelength a is a^2 These reasonings do not rest on the previous construction of the square ABCD, although some students believe they match the requirements when they try to construct a square of sidelength 2.82 cm (rounded from the value given by the calculator), placing it in a "good position" with respect to ABCD.



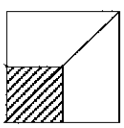
2) A part of the class actually works directly with the given square, trying to use it as a "unit element" to be doubled. These pupils produce figures such as rectangles of area 8 cm^2, squares of sidelength 4 cm, or squares of diagonal equal to twice the diagonal of the square ABCD. These constructions consist in taking the double of a given length a They could also be achieved independently of any measurement. In that sense, they actually rest on a previous construction of the given square.



However, this is not a square



However, the area is not the double.



However, the area is not the double

These fake solutions are easily rejected; the pupils are then led to a numerical evaluation of an approximate sidelength for the required square (2 82 cm) and to its construction by means of a slide-rule.

The problem is then considered as being "well solved" by the students, who therefore do not understand why their solution is rejected by the teacher The production by students of 2 82 as the sidelength from which they construct the square, is unacceptable to the teacher who, on the contrary, wants to justify the existence of $\sqrt{8}$ based on a geometric construction

III.3 *Second analysis*

Another modelisation can give sense to these productions.

The M′ milieu on which these students work is defined by :

—positive numbers as measurements of size in centimetres and millimetres,

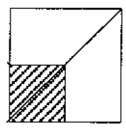
—the slide-rule and the calculator,

—drawings of squares with sides parallel to the side of the sheet of paper

The tools which function naturally in this milieu are the measures with the help of a ruler, the relationship between sidelength and area of a square

A split situation

The situation for the teacher is not the same as the situation for the pupils: it is split into two situations According to the terminology introduced before, we will refer to the class situation of the above example as being a split situation. To manage this event, the teacher will be led (as in Example 1) to rely on the pupil who, in spite of a wrong answer, is recognized by him as running in the "good context" M, because he has introduced a diagonal.



Moreover, in this case, we put forward the hypothesis that no means can allow the two situations to converge towards the same didactic situation.

Concerning the *contract* induced by the situation: The question itself, and the existence of measurement specifications in the problem statement, put pupils in a context of drawing and measurements, and not in a context of geometric constructions of plane figures. The question is thus interpreted by them as "Draw the required object with the instruments as accurately as possible" Here we are faced again with difficulties mentioned by various researchers in geometric didactics: the difficulty students have in differentiating drawing, figure, and geometrical object [Laborde & Capponi, 1993; Parzysz, 1988]; a confusion between "draw accurately" and "construct" [Grenier, 1988]; the consequences of the overwhelming importance given in school situations to geometric knowledge over spatial knowledge [Berthelot & Salin, 1994].

Concerning the *situation context* (the knowledge brought into play): The relations between diagonal and sidelength of a square, as well as between diagonal length and area of a square, are not institutional objects. The institutionalized knowledge in connection with the concept of square (at high school level) is the relation between sidelength and area, as well as right triangular areas, and the relations between the edges. Moreover, Pythagoras' rule is strongly associated to the concept of triangle, not to that of square.

Discussion

These analyses which take into account the teaching project but also the actual students' work in the classroom situation

—enable one to step back from the situations observed, and

—are a relevant and efficient way of interpreting events which perturb the teacher's project in the classroom situation.

This connection appears to be a relevant model-setting device as it allows one not only to interpret but also to predict events that are likely to arise in a didactic situation.

Our analysis and in particular the study of the events developed above shows that the work which really takes place in the classroom (sensitive objects) is often concentrated on different knowledge than that which the teacher wishes to convey It demonstrates that the objects which, according to the teacher, are supposed to be known by the students, and thus considered as having been previously grasped, are in fact issues remaining to be learnt by the latter.

This phenomenon is already known, but our work provides a sharper analysis by showing how the disturbance created by a students' intervention will display the objects on which the latter is actually working, creating an indispensable event—a chance intervention, but a necessary event—for the teacher to seize in order to carry out his/her project. If this event does not occur, there will be a loss of meaning of what is happening on the students' side.

Other types of didactic phenomena have been highlighted using the same situations, but here we do not analyse them. One can find their analysis in Comiti *et al.* [1995].

The relevance of these didactic phenomena remains to be studied in modelling the interactions arising in other didactic situations.

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