HEAPS, COMPLEXES AND CONCEPTS (PART 2)

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In this pair of articles, my focus is on how an individual appropriates notions from the socially-sanctioned body of knowledge which we call mathematics. Specifically, I am concerned with how students internalise mathematical ideas that exist in the social world (on the chalkboard, in textbooks, in the activities of their lecturers and fellow students) and make them their own.

In part 1 (Berger, 2004) I argued for a focus on the learning of an individual student in the context of a traditional undergraduate mathematics course. I posited that the various activities of such a student engaging with a new mathematical object are usefully interpreted as that student's (non-linear) movement through various pre-conceptual stages. These stages derive from an elaboration of Vygotsky's (1986) stages and can be broadly grouped into thinking in heaps or complexes. Complexes can be further divided into different forms each with its own particular character.

For ease of reading, I repeat the diagram (Figure 1) from part 1 which shows these different (non-linear) stages in the appropriation of a mathematical object.

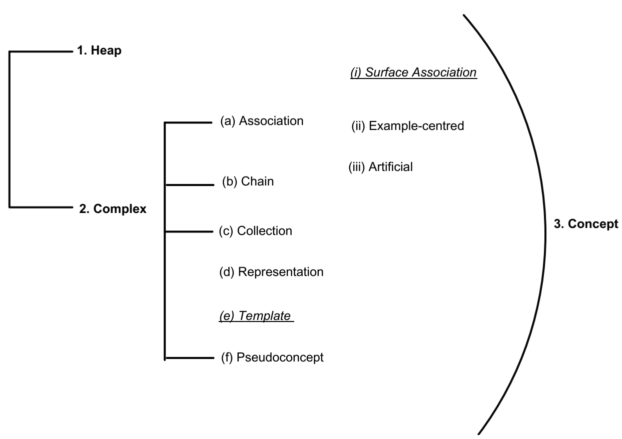

Figure 1: Stages in the appropriation of a mathematical object.

In part 1, I described and exemplified thinking in heaps in the mathematical context. Here, in part 2 of this article, I continue with this elaboration of Vygotsky's theory by describing and exemplifying complexes and concepts in the mathematical domain.

Complex thinking

With complex thinking:

> individual objects are united in the child's mind not only by his subjective impressions but also by *bonds actually existing between these objects.* (Vygotsky, 1986, p.112; original emphasis)

According to Vygotsky (1986, 1994), when someone thinks in complexes that person's thinking is coherent and objective; this is unlike in heap thinking where the connection between different elements of the group is idiosyncratic and incoherent. Complex thinking is also different to conceptual thinking in that the bonds between objects in the groups formed as a result of complex thinking are factual and derive from experience rather than systematic or based on logic (as in conceptual thinking).

Complex thinking is crucial to the formation of concepts in that it allows the learner to think in coherent terms and to communicate via words and symbols about a mental entity. And, as I later argue when discussing the pseudoconcept, it is this communication with more knowledgeable others which enables the development of a personally meaningful concept whose use is congruent to its use by the wider mathematical community. Furthermore, in complex thinking, the learner abstracts or isolates different attributes of the objects or ideas, and the learner organises objects or ideas with particular properties into groups, thus creating the basis for later more sophisticated generalisations.

Vygotsky characterised complex thinking in five different ways: the associative complex, the chain complex, the collection complex, the diffuse complex and the pseudoconcept. But, as previously noted, these types of complexes are not sufficient to characterise the type of sign usage that occurs in the mathematical domain: there is a need to distinguish signifier-orientated from signified-orientated usage and there is also a need to refine certain categories further into subcategories (for example, I have separated the associative complexes into three sub-categories).

In the mathematical domain, complex thinking takes several forms each with its own characteristics and indicators. Although it is important to distinguish these different forms of complex thinking from one another (which I do below), it is essential to note that in all these forms the learner links one mathematical sign with another because of an inherent or factual connection rather than a logical connection.

Association complex

Vygotsky explained that with the associative complex, one object forms the nucleus. When the child notices a similarity between an attribute of the nucleus and an attribute of another object, this new object gets included in the group. Vygotsky (1994), although referring to the block-sorting task rather than to the construction of a mathematical concept, described this association thus:

> Any actual relationship which the child discovers, any associative connection between the nucleus and the element of the complex, is enough reason for the child

to include this object in a group selected by him. (pp. 220, 221)

In the mathematical arena, I elaborate the associative complex as follows: the learner uses one mathematical sign as a nucleus and then associates other mathematical signs which have attributes in common with the nucleus, to the nucleus. In this way, a complex is constructed: the connections and associations have some objective and factual justification (as opposed to heap thinking) although they are not based in logical thinking (as in conceptual thinking).

For example, a learner may latch onto one or more familiar signs in a mathematical statement and use these as a nucleus around which to build a new complex (that is, the learner may use these isolated signs to make sense of the entire statement) or the learner may use one example of a mathematical object as a nucleus around which to build a new complex.

Since complex thinking of the associative type is ubiquitous in the early stages of concept construction (as I illustrate below), I have divided it into three subcategories each of which has a different character with a corresponding different indicator.

Surface association: the student isolates a particular aspect or part of the mathematical expression (a set of signifiers or words) and associates these signifiers or words (the nucleus) with a new sign. I call this 'surface' because it appears to be the result of a superficial reading of the original set of signifiers (the statement).

Furthermore the student seems to grasp onto particular words or signifiers often ignoring other significant parts of the mathematical expression. Because the learner's focus is on a signifier rather than on its meaning, this is a signifier-orientated form of complex thinking.

A surface association is empirically indicated by the student's undue focus on a set of symbols or words and their consequent non-attendance to other objectively significant signifiers in the mathematical expression.

For example, when dealing with the greatest integer function, $\lceil\lceil x\rceil\rceil$= greatest integer $\leq x$, many students latch onto the word 'greatest' ignoring the condition $\leq x$. They then link the word 'greatest' (the nucleus) to the idea of 'greater than' and accordingly state that, say, $\lceil\lceil 4.3\rceil\rceil = 5$ (whereas of course, the answer should be 4).

The surface type of complex association occurs frequently in mathematics because a mathematical object is not perceptually accessible and the initial access to the object is through its signifier (Sfard, 2000). Accordingly, some learners conflate the signifier with the object and end up isolating perceptual attributes of the signifier rather than attributes of the object.

Using an example as a nucleus round which to construct a concept: A very common strategy used by learners when appropriating a new mathematical object is to use an example of the mathematical object to construct a new concept. In terms of Vygotsky's theory, the example is the nucleus or core around which the new concept is built.

What happens is this: one particular example (perhaps the only available example) constitutes the core of the complex; attributes of other mathematical examples or mathematical objects are then compared to certain attributes [1] of the core; if they correspond in one or more aspects, the learner adjoins specific attributes of the new example or mathematical object to the core. In this way the learner constructs a new concept.

However since the example is an instantiation of the object (rather than the object itself) irrelevant or extraneous attributes may be included in the core as if they were intrinsic to that concept. If that happens, this use of an example may be disabling of concept construction.

An indicator for this form of activity is when the student uses an example of a concept as if every property of the example was a property of the concept.

For example, we often introduce the notion of an everywhere continuous function which is not differentiable at every point on its domain, through the example:

$f(x) = |x|, x \in \mathrm{R}.$

Students often use this example to assume that all absolute value functions are not everywhere differentiable. Of course, this is not correct. For example:

$f(x) = |x^2|$ is differentiable for all $x \in \mathrm{R}$.

Sierpinska (2000), although working in a different theoretical framework, also gives convincing examples of how some students doing a linear algebra course within the CABRI-geometry II technological environment were impeded by "thinking in terms of prototypical examples, rather than definitions" (p. 222).

The notion of the mathematical example as the nucleus around which a concept is built resonates with the notion of a mathematical prototype developed by Schwarz and Hershkowitz (1999). They focus on the different ways in which some students in different learning environments use or do not use certain functions (such as linear or quadratic functions) as prototypes for all functions. They also discuss how the use of these prototypes may be enabling or disabling of mathematical concept learning.

Unfamiliar objects are artificially associated with a familiar object: When confronting a new mathematical object or expression, part of the object or expression (perhaps a sign or symbol) may remind learners of another mathematical sign with which they are more familiar and which is epistemologically more accessible. This more familiar sign may then become the nucleus of the new concept. But the connection between the two signs may be artificial or not relevant.

For example, students often associate the properties of $\int f(x)dx$ with the properties of $f(x)$, presumably because they are more familiar with the properties of $f(x)$ than with the properties of $\int f(x)dx$. So, f(x)becomes the nucleus and its properties are artificially connected to the properties of $\int f(x)dx$.

As another example, students often associate the properties of matrices with properties of determinants. I recently asked the following question in a revision lecture:

If det $A = k$ and A is an $n \times n$ matrix, what is det $(2A)$?

Many students responded that the answer was $2k$. Few responded correctly that the answer was $2^n k$. I suggest that this is because students are associating properties of determinants with the properties of matrices to which they were introduced just before being introduced to the determinant. (In our institution, the determinant is introduced as a scalar associated with a matrix).

An indicator of an artificial association is when students associate a new and unfamiliar object to an object with which they are familiar and which reminds them of the new object in some way. (I exclude those resemblances which occur solely due to the sameness of template; such resemblances are classified as indicative of a template-orientation (see p. 14).

Chain complex

In the chain complex, an object is included in a group because it shares an attribute with another object already in the group. The new object enters the group with all its attributes and the learner then uses any of these attributes to include yet another object with a shared attribute in the group.

Vygotsky gave the example of a child who, while trying to pick out objects which are in the same category as a yellow triangle, picks out triangular blocks of various colours. At some point the child stops focussing on the shape (the triangle) and rather focuses on the colour of an object she has picked out, for example, blue. Accordingly she starts picking out blue shapes that are not necessarily triangles. Vygotsky (1986) argued that the original block has no special significance for the child:

> Each link, once included in a chain complex, is as important as the first and may become the magnet for a series of other objects. (p. 116)

Unlike in the associative complex where the nucleus is stable, the nucleus in the chain complex is continually in flux.

In the mathematical domain, the student associates one mathematical sign with another because of some similarity (for example, a shared word or a shared property). This new sign (or an aspect or attribute of it) is then linked to yet another sign by virtue of a different attribute thus forming a chain of signs [2].

The indicator for a chain complex is the association of one sign with another through *several* links. If there is only one link, the complex is categorised as some sort of associative complex (see p. 11).

Here I give an example of a chain complex taken from a set of interviews which I conducted with several first-year mathematics students (Berger, 2000). A student, David, had just been given the following written definition of an improper integral Type I:

If f is continuous on the interval $[a, \infty)$, then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$.

If $\lim_{b\to\infty}\int_a^b f(x)dx$ exists, we say that the improper integral converges. Otherwise the improper integral diverges.

The student was asked to explain what this definition represented graphically.

David argued that the improper integral gave the area between a "graph with a repeating pattern" (Berger, 2002) which "repeats to infinity" and the x-axis. He further explained that the graph oscillated around the x-axis with the gaps (between graph and (x-axis) getting "smaller and smaller" (*ibid*) when the improper integral was convergent or "bigger and bigger and bigger" (*ibid*) when it was divergent.

Clearly this is an incorrect interpretation of an improper integral – the integrand in an improper integral needs neither be oscillating, nor repeating.

Given that, at the time of the interview, David's class was studying alternating sequences that had been extensively illustrated on the chalkboard, I interpret David's explanation as a chaining of the notion of an improper integral to the notion of an oscillating graph through a sequence of links, as follows:

> Properties of a *convergent* or *divergent* improper integral are linked to properties of a *convergent or divergent sequence*. These properties are, in turn, linked to a *graphical representation of converging or diverging sequences* which in turn is linked to a *graphical representation of an alternating sequence*. The *graph of an alternating sequence* is then linked to an *oscillating graph*. As a result the improper integral is interpreted as the area between an oscillating graph and the x-axis.

Collection complex

In this form of thinking, the child groups together objects on the basis of one trait in which they differ, but in relation to which they are functionally complementary (Vygotsky, 1986, p.115). For example, objects like a knife, fork, spoon and plate are grouped together because they have a functional complementarity in everyday life:

> Objects become unified on the basis of a mutual complementing of one another according to some feature, and they form a single whole that consists of different, mutually complementary parts. (Vygotsky, 1994, p. 221)

Although this category does not translate directly into the mathematical arena, I have adapted it so that it refers to the grouping together of mathematical processes or objects which are complementary to each other in the sense that the one undoes what the other does. That is, I categorise mathematical behaviour in which the student performs an inverse process to what is expected, as a manifestation of the collection complex.

An example of thinking with a collection complex, is when the student differentiates rather than integrates. Another example is when a child multiplies rather than divides in a problem such as:

> An insect travels 200 miles in 10 hours. What is its speed?

I suggest that this type of usage of signs comes about as a result of the teacher, lecturer or textbook linking the process and its inverse explicitly. For example, integration is often introduced as anti-differentiation. Division by x ($x \neq 0$), is often referred to as multiplication by $1/x$.

Representation complex

Complex thinking involving the use of one or more different representations (visual or numeric) is a type of thinking that occurs specifically in mathematics. With this form of complex, the visual (geometric, graphic or diagrammatic) or numerical representation of the mathematical object becomes identified by the learner as the mathematical object itself. Thus any properties that can be abstracted from the visual or numerical representation of the mathematical object are deemed by the student to be properties of the mathematical object itself.

A use of different representations may well enable and facilitate the evolution of the mathematical concept for the learner. Indeed, most of the proponents of the reform of calculus strongly advocate the use of multiple representations (*i.e.*, symbolic, graphical and numeric) in the learning of new mathematical concepts. However, because a visual or numeric representation is a concrete representation of an abstract object rather than the object itself, the use of this sort of complex thinking may lead to false and mathematically incorrect conclusions.

For example, the graphical representation of a particular function on a graphic calculator or computer may obscure the existence of discontinuities if the resolution of the screen is too low or the number of pixels too small. Likewise it is impossible to represent graphically or numerically certain "monster" functions (Lakoff and Núñez, 1997, p. 73) such as:

$$f(x)=\begin{cases}\sin\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$$ due to the oscillating nature of the sine function near $x = 0$.

In summary, the use of visual or numeric representations may be helpful although, in certain instances, it may obfuscate the nature of the mathematical object or the representation may be ontologically impossible. But even if the representation is helpful or enabling it is still a concrete representation of an abstract object, not the abstract object itself. Thus, the type of reasoning a student employs when using a visual or numerical representation of an abstract object is necessarily factual or concrete rather than logical or abstract. I thus classify such reasoning as indicative of complex thinking. Nonetheless, I suggest that form of complex thinking, free as it is from obvious contradictions and inconsistencies, stands at the cusp of conceptual thinking.

The indicator for representation-type complex thinking is the student's use of a graphical, geometric, diagrammatic or numerical representation of a mathematical object as the basis for an argument or investigation.

Template-orientation complex

Like a usage of signs based on a surface association, a template-orientated usage of signs is signifier-orientated. What is a template-orientation? When a student is presented with a new mathematical sign, the student may associate the new sign with another more familiar sign due to a similarity of the templates of the respective signifiers. As a result of this association, properties of the more familiar sign are applied to the new sign. In this way, students are able to use the new signifier even before they have constructed the concept referred to by the new signifier.

This characterisation of template-orientation derives from Sfard (2000) who argues that the first use of a new mathematical symbol is necessarily template-driven, even if it is of very short duration. She argues that the learner, when confronted with a new mathematical signifier, slots this signifier into an "old discursive slot" (*ibid*, p. 58) and uses it in a way suggested by the familiar template. Gradually, "due to some basic misfit" (*ibid*, p. 58) and discursive interactions, the new signifier slips out of its initial slot and starts to acquire a meaning and life of its own.

In my theoretical framework, template-orientated use of a mathematical sign is not necessarily the first step in mathematical object appropriation. Indeed students may also use surface or other associations as their first step in mathematical object appropriation.

A template-orientated usage of signs may be initially enabling or disabling of mathematical object appropriation. For example, many university students link the matrix multiplication statement '$AB = C$' (where A, B, C are matrices) to the statement '$ab = c$' (where $a, b, c \in \mathrm{R}$) because of a similarity of template. Using the familiar fact that the multiplication of the reals is commutative (*i.e.*, that $ab = ba = c$) and that '$AB = C$' has the same template as '$ab = c$', they may write: '$AB = BA = C$'. This is, in fact, incorrect: multiplication of matrices is not commutative.

On the other hand, the fact that the statement '$A + B = C$' has the same template as the familiar statement '$a + b = c$', may enable the student to write and use '$A + B = B + A = C$', *i.e.*, to extend the commutative property of the addition of reals to the addition of matrices (which is correct).

In the empirical setting, template-association is indicated when the student transfers properties of a sign with which they are familiar to a new or unfamiliar sign which has the same template as the original sign.

Pseudoconcept

The sixth type of complex, the pseudoconcept, is a very special type of complex. Vygotsky argued that it is the very existence and use of pseudoconcepts that makes a learner's transition from complexes to concepts possible. In the mathematical domain I maintain that the use of pseudoconcepts is ubiquitous. The distinctive feature of the pseudoconcept is its duality: from the outside (*i.e.*, to the observer) it has the appearance of a concept, but on the inside (in terms of its genesis, the conditions under which it develops and the causal associations of these conditions) it is actually a complex (Vygotsky, 1994, p. 226).

In mathematical activity, learners are using a pseudoconcept when they are able to use and communicate about a mathematical notion *as if* they fully understand that notion, even though their knowledge of that notion may be riddled with contradictions and connections not based in logic.

For instance, consider the student who correctly argues that the function $f(x) = x^3$ is increasing for $x \in \mathrm{R}$, because $f'(x) = 3x^2 \geq 0$ for all $x \in \mathrm{R}$. The teacher may justifiably believe that this student has a true conceptual understanding of the concepts of the derivative and of increasing functions and of the relationship between them (viz. if $f'(x) \geq 0$, $x \in \mathrm{R}$, then f is increasing for $x \in \mathrm{R}$). But students may have little understanding of *why* a non-negative derivative guarantees an increasing function; they may not even know what a derivative is or what an increasing function is. In this case I categorise the student's knowledge as pseudoconceptual.

Before continuing with this discussion of the pseudoconcept as it manifests in the context of a learner constructing a mathematical concept, I would like to discuss Vygotsky's argument about the role of the pseudoconcept in the development of the meaning of a word (or concept) for a child. Basically, Vygotsky (1994) argued that:

> Childhood complexes which correspond to the meaning of words, do not develop freely and spontaneously along the lines marked out by the child himself, but in *certain definite directions which have been predetermined* for the developmental process of the complex by previously established meanings which have been assigned to the words in adult speech [...] By engaging the child in verbal communication, an adult can influence the further progress of this generalization process, as well as the end outcome of that journey which will be the result of the child's generalizations. (pp. 227, 228; italics added).

As with a word, a mathematical object is a discursively constituted and socially sanctioned object.

Thus, paraphrasing and adapting Vygotsky's argument to the usage of mathematical signs rather than the meaning of words, I claim that:

> Learner's complexes which correspond to the usage of specific mathematical signs in the mathematical community, do not develop freely and spontaneously along the lines marked out by the learner herself or himself, but in *certain definite directions which have been predetermined* for the developmental process of the complex by previously established usages which have been assigned to the signs in mathematical discourse. By engaging the learner in verbal or written communication, a teacher or text can influence the further progress of this generalization process, as well as the end outcome of that journey which will be the result of the learner's generalizations.

In other words, I am arguing that a use of the pseudoconcept enables learners to communicate effectively about and engage fruitfully in activities with the mathematical sign even before they fully know what the relevant mathematical object is. This communication and these activities, which take place with people and texts whose usage of the relevant sign is concurrent with that of the mathematical community, guide and influence the development of learners' generalisation processes.

Although all complex thinking provides access to the new and unknown mathematical object and allows for some communication about this object, complex thinking is by its very nature, associative rather than logical. Thus, other than in the case of pseudoconceptual thinking, the complex thinker's use of signs often appears muddled and idiosyncratic. Accordingly, such complex thinking lacks the potency and developmental potential of the pseudoconcept. Indeed it is primarily the use of the pseudoconcept in a social context (constituted by texts or members of the mathematical community) that guides the learner in the development of a concept which is both personally meaningful and whose use is commensurate with its use by the mathematical community.

In line with Vygotsky, I thus regard the pseudoconcept as a bridge between the mathematical complex (which gives the initial access to the mathematical object) and the mathematical concept (which has internal consistency and logical connections within itself and to other mathematical concepts).

Pseudoconcepts are often difficult to detect. This is because they appear as if they are concepts (discussed below). However, a way of empirically distinguishing between a concept and a pseudoconcept is by examining the integrity of the student's prior or post activity. For example, if a student is able to detect whether a particular series is convergent or divergent but, soon after or immediately preceding the task, reveals an incoherent or inconsistent idea of what convergence or divergence is, that student's usage of signs is pseudoconceptual.

Diffuse complexes and potential concepts

For completeness, I need to explain why I have excluded diffuse complexes and potential concepts from Appropriation Theory.

In a diffuse complex, the child uses vague or remote similarities (for example a similarity or sameness of signifiers) to make connections between different objects or ideas. For example, in Vygotsky's block-sorting experiment some children placed trapezoids and triangles in the same category, presumably latching onto a vague similarity (a trapezoid is like a triangle with the top cut off).

In the mathematical domain, one could argue that a diffuse complex is manifested when learners use vague similarities between signs to connect one sign with another. But this is precisely what happens in the artificial association complex (described above, p. 12). Given that the latter category is more descriptive of the mathematical domain, I have subsumed the category of diffuse thinking into that category of complex thinking.

With regard to potential concepts, Vygotsky (1986, p. 135) argued that complex thinking creates the basis for later generalisations in that the learner classifies different objects into groups (or complexes) on the basis of particular characteristics. However classification would not be possible without abstraction of these particular characteristics. Thus the learner engages in abstractions concurrently with complex thinking. Vygotsky called the formation that results from the grouping together of objects on the basis of a single attribute or a set of attributes, a 'potential concept'.

Certainly, abstractions are inherent in the construction of any mathematical concept and so potential concepts in the Vygotskian sense abound in mathematical thinking. But the abstraction of attributes is so profoundly intertwined with the formation of complexes in advanced mathematical thinking that it is impossible to distinguish potential concepts from mathematical complexes. For this reason I suggest that the potential concept is not a particularly useful or appropriate category of analysis, particularly in the advanced mathematical domain.

Concepts

I use the term mathematical concept to refer to the mental idea of the mathematical object. In terms of my usage, and consistent with Vygotsky's theory of concept formation (1986, 1994), a mathematical idea becomes a concept (rather than a complex) when its internal links, *i.e.*, the links between the different properties and attributes of the concept are consistent and logical, and the external links, *i.e.*, the links of that concept to other concepts, are consistent and logical.

Vygotsky's distinction between a concept and a complex further illuminates the nature of a concept. Whereas the bonds between the elements of a complex are associative and empirical, the bonds between a concept are abstract and logical. Furthermore in a complex no attribute is privileged over another whereas in a concept particular attributes are privileged over others

Within the mathematical (or any other academic) domain, the evolution of concepts is systematic and deliberate (Vygotsky, 1986, pp. 148, 149). Moreover scientific concepts [3] (which include mathematical concepts) are interrelated within a system:

> Concepts do not lie in the child's mind like peas in a bag, without any bonds between them. If that were the case, no intellectual operation requiring coordination of thoughts would be possible, nor would any general conception of the world. Not even separate concepts as such could exist; their very nature presupposes a system. (Vygotsky, 1986, p. 197)

With regard to scientific concepts, "the relation to an object is mediated from the start by some other concept" (*ibid*, p.172).

Since I am using the term 'mathematical concept' to refer to the mental idea of a mathematical object whose existence is indicated by a specific use of signs and since Sfard's (2000) object-mediation refers to the discursive use by the learner of an abstract object as if it were a concrete and material object, mathematical conceptual thinking has many of the same features as Sfard's (2000) object-mediated stage.

Sfard (2000) characterises object-mediation as the transformation of the "signifier-as-an-object-in-itself to signifier-as-a-representation-of-another-object" (p. 79). I suggest that in conceptual thinking the signifier is similarly transformed. That is, for the preconceptual learner, the signifier or a set of signifiers, rather than the signified is a primary focus of attention. In contrast, for the conceptual learner the signifier is transparent so that the signified (*i.e.*, the idea of the mathematical object) shines through. Sfard (2000) argues as follows:

> The transition from signifier-as-an-object-in-itself to signifier-as-a-representation-of-another-object is a quantum leap in a subject's consciousness [...]
>
> [O]nce the symbol takes the role of representation, the whole discourse undergoes a modification. The old ways of expressing mathematical truths give way to new formulations. [...] [T]he ongoing mathematical conversation, perhaps more than any other type of discourse, is similar to a living organism that incessantly grows and mutates without losing its identity. (p. 79)

Finally and again consistent with Sfard's (2000) idea of object-mediated sign usage, the learner in the conceptual stage is able to attend to the mathematical object in its entirety, not just to isolated or fragmented aspects of the object (although this may sometimes also be required) or just to its signifier. Indeed, "the new sign gets life of its own and becomes an integrated signifier-signified unit" (*ibid*, p. 59).

For example, consider the mathematical object, the natural logarithm function. In many undergraduate mathematics courses this object is introduced via its definition:

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0\,,\ x \in \mathrm{R}.$$

From this definition, the student is expected to deduce various properties of the natural logarithm. For example, the domain is $(0, \infty)$ and the range is $(-\infty, \infty)$; the function is continuous, increasing and one-to-one; the graph is concave down.

Indeed all these properties can be logically deduced from the definition of the ln function and their deduction is, of necessity, mediated by prior knowledge of various other mathematical concepts, such as domain, range, continuity, increasing functions, one-to-one and concavity.

Similarly other properties of the natural log function (*e.g.*, $\ln ab = \ln a + \ln b$, $\ln a^b = b \ln a$) can be proved using properties of differentiable functions, the Second Fundamental Theorem of Calculus, and so on. These properties are also consistent with the properties established at school level (where the log function is treated as the inverse of the exponential function which is defined only for integer powers).

A fully-fledged concept of the natural logarithm function requires logical links between the different elements of the concept (as made manifest in deductive reasoning), logical links to other concepts (*e.g.*, to the differentiability of a function, the Second Fundamental Theorem of Calculus and so on) and consistency between concepts (*i.e.*, the properties of one concept should not lead to contradictions between the properties of another concept).

Furthermore in order to develop the properties of the natural logarithm function from its definition and to link these properties into one coherent and consistent whole which itself relates in a logical way to other mathematical concepts, students need to be able to attend to all aspects of this object (many of which are mediated by yet other concepts). They cannot just focus on isolated properties or features of the natural logarithm function without regard for other properties. Such a fractured focus would inevitably lead to contradictions and gaps in the concept.

Finally, and most importantly, in conceptual thinking the signifier '$\ln x$' acts as a sort of trigger so that the learner becomes aware of the specific mathematical object, the 'ln function' (and is able to access its properties and links to other mathematical concepts explicitly if required) when she sees those symbols: 'l', 'n', 'x', side-by-side.

Conclusion

In this article I have elaborated Vygotsky's theory of concept formation to the mathematical domain. I have done this so as to provide a "language of description" (Brown and Dowling, 1998, p.2) that may be used to interpret the mathematical activities of an undergraduate mathematics student in a traditional, university, learning environment. In particular, this language describes the stages which learners may zigzag through as they appropriate a mathematical object or some of its properties from its definition or a theorem (possibly guided by pedagogically-designed mathematical activities).

I have illustrated how students use different forms of preconceptual thinking (heaps and complexes) when appropriating a new mathematical object. These preconceptual usages (based on associations, circumstantial conditions, template-matching, specific instances and so on) may explain why many students appear muddled and confused as they work their way through new definitions or theorems, albeit they may eventually (some very quickly, some very slowly) use these new mathematical ideas in a logical and systematic way.

Although it seems that complex thinking gives the student initial access to various mathematical notions even while that student does not fully understand the mathematical concept, it is the use of the pseudoconcept that is particularly interesting. This is because the pseudoconcept functions as a highly effective bridge between complex thinking and conceptual thinking, allowing the learner to communicate coherently with more knowledgeable others and to engage fruitfully with textbooks concerning the use of the new mathematical sign even while that learner's understanding of this sign is limited. Indeed it is this social usage that enables and stimulates the development of a personally meaningful concept that is congruent with that of the wider mathematical community.

Overall I have shown how an elaborated version of Vygotsky's theory of concept formation can be usefully applied to the interpretation of the mathematical activities of undergraduate mathematics students. In particular, I suggest that this theory may give the educator much insight into the sometimes garbled and incoherent explanations of otherwise articulate and coherent students.

Notes

[1] These attributes are those which are epistemologically available to the learner or which the learner regards as important for one of many reasons such as familiarity, explanatory power and so on.

[2] A chain complex in the Vygotskian sense is not the same as the notion of "chains of signification" (Walkerdine, 1988) although some features are similar. In the "chains of signification" approach (*e.g.*Presmeg, 1998; Cobb *et al.*, 1997) the chaining of one signified from one context to a new signifier in a different context is the central tool for explaining how the learner uses the meaning of a particular sign in one context to make meaning of a sign in the mathematical context. In the Vygotskian approach, this chaining (which takes place within the same basic context or between different contexts) is but one possible form of sign usage.

[3] Vygotsky distinguished between scientific concepts and everyday concepts. Unlike scientific concepts, everyday concepts are saturated with experience and constructed unintentionally.

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An extract from an e-message sent by Dietmar Küchemann, Institute of Education, London, UK: Further to the quotation from Mazur, *more is up* (FLM **24**(1), p. 19, original emphasis):

> I've had some lows in my career but this is up there with them.
>
> (Michael Reiziger, quoted in 'Crocked Reiziger out until new year', David Plummer, *The Guardian,* 7th September, 2004
