THREE INTERPRETATIONS OF THE MATRIX EQUATION Ax=b

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Over the past years we have been involved in a series of studies researching the teaching and learning of linear algebra at the undergraduate level in the United States. Through work with our research colleagues and our own teaching of linear algebra we have come to reflect on the nature of the cognitive demands that an introductory undergraduate linear algebra course places on students. Many of the struggles students face are related to a central set of ideas. In this article, we present a framework that offers insight into student thinking related to these central ideas. This framework can serve as a tool for researchers, curriculum designers and instructors to deconstruct their expert knowledge in order to anticipate challenges that students may face. The framework also functions as a diagnostic tool that can be helpful in making sense of the unexpected and seemingly idiosyncratic ways in which students often blend ideas, particularly as they begin learning linear algebra.

The fact that students struggle to develop conceptual understanding of key ideas in linear algebra is well-documented (for example, Carlson, 1993; Dorier, Robert, Robinet & Rogalski, 2000; Harel, 1989b; Stewart & Thomas, 2009). A central theme in this body of research is students' struggles with the formal nature of the course, which tends to be new for students. In relation to students' struggles with formality, some have identified particular modes of representation and reasoning in linear algebra, and documented struggles in moving among these modes of representation and reasoning (Harel, 1989a; Hillel, 2000; Sierpinska, 2000). Some have documented student struggles with particular concepts such as basis, linear dependence and independence, and span (Carlson, 1993; Hillel, 2000; Stewart & Thomas, 2010). Rather than address the difficulty students experience with formality, in this article, we aim to lay out a framework that examines the introductory level in more detail. In particular, we identify a variety of important interpretations of the matrix equation A**x**=**b** underlying many of the central ideas in introductory linear algebra. Throughout the article, A is used to represent an $m \times n$ matrix, **x** is a vector in $\mathbf{R}^n$, and **b** is a vector in $\mathbf{R}^m$. Our framework enables examination of student thinking at a smaller grain size than the existing literature. The framework offers a lens for making sense of the variety of correct and incorrect ways students blend and coordinate ideas.

In our work in introductory linear algebra, we have adopted the recommendations of the Linear Algebra Curriculum Study Group (LACSG; Carlson, Johnson, Lay & Porter, 1993), treating a first linear algebra course as a matrix-based course focused on helping students develop a robust, geometrically motivated understanding of $\mathbf{R}^n$ before delving into an abstract treatment of vector spaces. We note that many of the central ideas recommended for a first course in linear algebra by the LACSG (for example: span; linear dependence and independence; existence and uniqueness of solutions to systems of equations; properties of linear transformations) can be interpreted through the lens of A**x**=**b**.

The examples used in this article come from the first two of a series of four classroom teaching experiments (Cobb, 2000) in linear algebra in which whole class instruction and small group work were video-taped, copies of student work were retained, and individual problem-solving interviews were conducted with students. Each classroom teaching experiment took place in a class of 21-37 students and, in each semester, between one third and one fourth of students were interviewed. Students were selected on a voluntary basis to obtain a representative sample of the class as identified by student performance on classwork prior to the interviews. The students were primarily mathematics, science, and engineering majors at large public universities, and all had completed two semesters of calculus prior to their work in this course. The first three examples come from mid-semester interviews, and the final example comes from an exit interview.

In this article, we synthesize themes from our research on student thinking from a variety of interview items involving the product of a matrix and a vector (for example, Larson, 2010; Larson, Zandieh, Rasmussen & Henderson, 2009). These earlier analyses yielded an interesting set of stories about student reasoning across a set of questions, but these stories lacked an overarching cohesion—an organizing framework that would help make sense of student thinking across a variety of mathematical contexts. The framework we present in this article is the result of an iterative cycle of refinement in which we examined our data, developed a framework to try to capture important trends, applied the framework to our data, then adjusted the framework to more fully account for the trends in the data, until we arrived at a framework that helped us make sense of the data across a variety of settings. The examples shared here were selected to illustrate a variety of ways the framework can be used to make sense of student thinking.

Conceptual framework: three interpretations of the matrix equation Ax=b

In this section, we lay out our framework, which consists of three interpretations of the matrix equation A**x**=**b**: a linear combination interpretation, a systems interpretation, and a transformation interpretation. Each interpretation includes graphic and symbolic representations, and we highlight the very different ways that the vector **x** is conceived in each of these interpretations.

First, one can interpret the equation A**x**=**b** to mean that the vector **b** is a linear combination of the column vectors of the matrix A, with **x** functioning as the set of weights on the column vectors of A (an interpretation related to developing an understanding of ideas such as span and linear independence). We refer to this interpretation as the *linear combination (LC) interpretation.* In the second interpretation, **x** is a set of values that satisfy the system of equations corresponding to A**x**=**b** (an interpretation related to developing an understanding of ideas such as existence and uniqueness of solutions to systems of equations). We refer to this interpretation as the *systems of equations interpretation.* In the third interpretation, the equation A**x**=**b** corresponds to a linear transformation, in which the vector **x** is related to the vector **b** through a transformation via multiplication by the matrix A (an interpretation related to developing an understanding of properties of linear transformations such as injectivity, surjectivity, and invertibility). We refer to this interpretation as the *transformation interpretation.*

Table 1 presents the interpretations of A, **x**, and **b** entailed by each of the three interpretations discussed in this article. Note that Table 1 depicts a prototypical case in $\mathbf{R}^2$ in which the solution is unique. Our framework is not restricted to this case; it is intended to apply for well-defined multiplication by any m × n matrix A. While the imagery shown here is most applicable in 2 and 3 dimensional contexts, we intend the framework to also be applicable in higher dimensional contexts. This will be further discussed in the conclusion section.

What we highlight in this article is how markedly *different* these interpretations are from one another, and we identify geometric and symbolic coordinations needed to operate flexibly within and across settings. In particular, we discuss how differently students must interpret the vector **x** in each of these three interpretations of A**x**=**b**. However, we first draw on the literature to develop some useful language for talking about these varied interpretations of **x**. Student struggles with the use of variables in real-valued contexts are well-documented in the research literature at the secondary and tertiary levels (for example, Schoenfeld & Arcavi, 1988; Trigueros & Jacobs, 2008; Wagner, 1983). Philipp (1992) uses the term *literal symbol* to refer to “the mathematical use of a letter” (p. 558). He discusses the varied use of literal symbols in mathematics as a potential challenge for students in middle grades, identifying mathematical conventions for the use of letters to represent conceptually different entities ranging from universal constants (*e.g.*, π), to unknown values (*e.g., x* in $2x - 1 = 4$), to varying quantities (*e.g., x*, *y* in $y = x + 2$), to parameters (*e.g., m*, *b* in $y = mx + b$). Such distinctions can shed light on the variety of interpretations students need to develop in linear algebra. We focus on the role of **x** both symbolically and geometrically in each of the three interpretations of the matrix equation A**x**=**b** described in our framework (see Table 1).

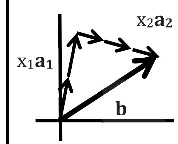
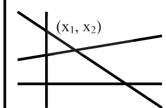
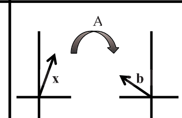
Interpretation of A**x**=**b**	Symbolic Representation	Geometric Representation
Linear combination (LC) interpretation $x_1\mathbf{a_1} + x_2\mathbf{a_2} = \mathbf{b}$	A: set of column vectors $(\mathbf{a_1}, \mathbf{a_2})$ **x**: weights (x_1, x_2) on column vectors of A **b**: resultant vector	
System of equations interpretation $a_{11}x_1 + a_{12}x_2 = b_1$ $a_{21}x_1 + a_{22}x_2 = b_2$	A: entries viewed as coefficients $(a_{11}, a_{12}, a_{21}, a_{22})$ **x**: solution (x_1, x_2) **b**: two real numbers (b_1, b_2)	
Transformation interpretation T:**x**↦**b**, T(**x**)=A**x**	A: matrix that transforms **x**: input vector **b**: output vector	

*Table 1. Three views of A***x**=**b**.

Symbolic role of x

Under the linear combination interpretation, **x** is interpreted as a set of *weights*. This view of **x** as a set of weights most closely matches Philipp’s characterization of literal symbols that function as parameters—values that scale the column vectors of the matrix A. Under the systems interpretation, **x** is interpreted as a *solution*. This view of **x** as a solution most closely matches Philipp’s characterization of literal symbols that function as unknowns—the entries of the vector are values to be found by solving the system of equations. Under the transformations interpretation, **x** can be interpreted as an *input vector*. This view of **x** most closely matches Philipp’s characterization of literal symbols that function as varying quantities—in this case, the vector **x** is viewed as something that is coordinated with the output vector **b**.

Geometric role of x

Under the linear combination interpretation, **x** is a set of weights that stretch, shrink, and/or reverse the direction of the column vectors of the matrix A. Under the systems interpretation, **x** is a solution to the system of equations corresponding to A**x**=**b**. Geometrically, this is conceptualized as zero, one, or infinitely many points of intersection. Under the transformations interpretation, **x** and **b** might be conceptualized as vectors with magnitude and direction that are related to each other in some way.

The transformation interpretation is particularly broad, and we will not go into its full detail in this article. For instance, Sinclair & Gol Tabaghi (2010) documented professors’ ways of using gestures, motion, and time to informally describe transformations in the context of eigenvectors. Zandieh *et al.* (2012) have documented five distinct metaphors students use to make sense of the way **x** and **b** are related to each other in situations consistent with our transformations interpretation: (a) *input-output*, where one puts **x** into the equation, and gets **b** out, (b) *morphing*, where **x** morphs into **b**, (c) *traveling*, in which **x** moves to the location of **b**, (d) *mapping*, in which each **x** corresponds to a specific **b**, and (e) *machine*, in which A acts on **x** to produce **b**. Our framework encompasses such understandings within the transformation interpretation, and we identify students' use of these metaphors in our analysis when possible. However, our focus in this paper is on the interplay between the systems, linear combination and transformation interpretations.

Examples of student thinking

The examples shared in this section were selected to show how our framework can be used to examine differences across students (Example 1), and how it can be used to analyze the thinking of individual students. The latter includes instances when students are struggling (Examples 2 and 3) and when students are appropriately and effectively coordinating ideas (Example 4).

Example 1: Different students, different interpretations

This set of examples serves to show that students, when making sense of a situation, may gravitate toward any of the three interpretations in our framework. Students were asked the question:

> Consider a 2×2 matrix A and a vector $\begin{bmatrix} x \\ y \end{bmatrix}$.
>
> How do you think about $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ geometrically?

This item was originally developed to offer some insight into how students might coordinate the product of a matrix and a vector with the product of a scalar and a vector, specifically in the context of eigenvector equations, prior to receiving formal instruction on them.

One student, Charlene, responded to this question by creating the inscription shown in Figure 1 and said, "Where they meet, it would, the *x* and *y* would be the same. So it would be like where the two vectors meet, would be the solution."

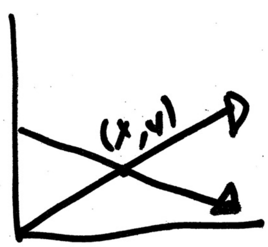


Figure 1. Charlene's interpretation of $A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$

The view that the solution to an equation can be thought of as a point of intersection is consistent with the geometric aspect of the systems of equations interpretation. However, Charlene was unsure about *which* two vectors are meeting:

Interviewer: So when you say the two vectors, because I see you've drawn these two arrows here, what two vectors are you talking about?

Charlene: I have no clue. I just know that they should be equal, because there's an equal sign [...] I just knew that there has to be a solution, and that's how you find the solution.

Interviewer: Oh, so the solution is where the two intersect?

Charlene: Yeah, because there's an unknown x and y, and so there has to be a solution, and usually the solution is where they meet. And so I just drew a picture of the two things meeting, because that's how I understood it.

Charlene's response suggests that the presence of literal symbols *x* and *y* and an equals sign were important in evoking her geometric systems interpretation. Charlene states that it is two *vectors* that are meeting, but she does not know which two vectors are meeting. Charlene's reference to a solution occurring when two things meet uses a systems interpretation. At the same time, she is attending to and attempting to coordinate this interpretation with the presence of vectors, which appear in the linear combination and transformation interpretations, but not in the systems interpretation.

Nico's response to this question suggests a strong transformation view in which he considers A as a matrix that "does something" to the vector $\begin{bmatrix} x \\ y \end{bmatrix}$
He explains:

> If two times the vector x, y is just going to make the vector twice as long, I would hope that A times x, y is doing the same thing if they equal each other. So basically, there is the vector that we're gonna get out of multiplying A by x, y. It's gonna be the exact same vector that we're gonna get by multiplying 2 by x, y.

This explanation conceives of **x** and **b** (in this context $2\begin{bmatrix} x \\ y \end{bmatrix}$ functions like the **b** in our framework) as vectors which are coordinated with one another via the actions of multiplication by the matrix A and multiplication by the scalar 2. In particular, Nico's transformation view draws on Zandieh *et al.*'s (2012) metaphors of input/output, machine (does something to), and morphing (twice as long).

In contrast, Karl takes a linear combinations view, arguing that the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ stretches the columns of the matrix A. This interpretation emerged in the context of his explanation of why he thought that the determinant of A must be two: "If the two sides equal, and the x, y vector does not change, then you have, you know, A has to equal two and A is a matrix, the two has to be a determinant." In elaborating this remark, Karl offered a creative (albeit incorrect) argument, as follows.

Karl sketched the graph shown in Figure 2 (overleaf) and argued that:

> by multiplying this original set of vectors, which would be matrix A, by x and y, you could potentially maybe increase the length of these vectors, thus changing the, the overall determinant, so overall area of the figure that is formed.

When asked, he clarified that the two vectors starting from the origin in his graph were meant to represent the columns of the matrix A. He wrote the matrix to the right of the graph (see Figure 2) and explained as follows:

> So these [Karl points to the two vectors he has drawn which start at the origin] would be the two vectors, the you know, a_1, b_1, and a_2, b_2, which is represented by the A matrix. And then when combining the A matrix with the x, y, vector, you could potentially increase the length or change, change the overall area, overall determinant of the space. Essentially uh, closed in by the vectors.

Karl's argument was based on the idea that the determinant of a 2×2 matrix is the (signed) area of the parallelogram created by the column vectors of that matrix, an idea he coordinated with the interpretation that x and y act as weights on the column vectors of the matrix (an LC interpretation). While incorrect, we highlight the fact that Karl's reasoning interpreted matrix multiplication as a weighting of the columns of the matrix, evidencing an LC interpretation.

Taken together, these interpretations of the equation **Ax**=2**x** illustrate how differently students interpret this kind of equation. The variety in students' interpretations is highlighted by their views of x: x as a solution (Charlene), x as a vector to be transformed (Nico), and x as a set of weights (Karl).

Example 2: One student's under-coordinated interpretations

In this example, Charlene exhibits a computational LC approach that is not coordinated with a geometric view. We see this in her response to the question

How do you think about $\begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$? Please explain.

She responded by representing this as a linear combination of the column vectors with weights of 2 and 3, respectively. However, when the interviewer asked her if she has a geometric way of thinking about the "situation" in Figure 3, Charlene responded:

> Not really, I mean you can draw it and everything, but I don't really remember how to do that. I don't really think of it geometrically, I see the numbers and I do it. But I don't really, we learned it in class, but I don't really remember it, because it wasn't something I understood very well.

In this way, we see that Charlene utilizes an LC computational strategy, but does not connect this with the corresponding LC geometric interpretation.

For the question

How do you think about $\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Charlene rewrote the linear combination in the question as a matrix multiplication, drew an equals sign, drew an arrow from the linear combination as if to indicate it to be part of the details of her computation, drew another equals sign, and then wrote the resulting vector.

Thus we see that for Charlene, a linear combination of vectors and the product of a matrix and a vector are computationally equivalent, and that she seems to see a linear combination of vectors as a step in the computation of the product of a matrix and a vector.

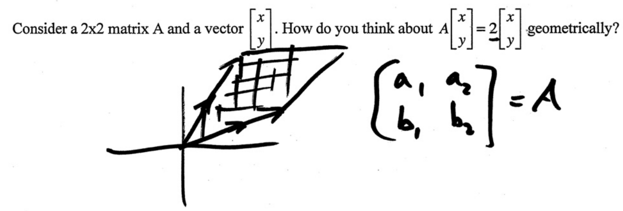


Figure 2. Karl shows how he thinks x and y could double the area of A.

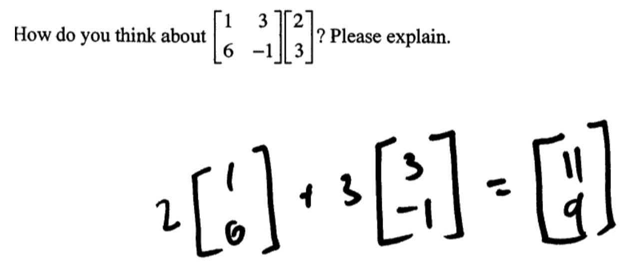


Figure 3. Charlene's product of a matrix and a vector.

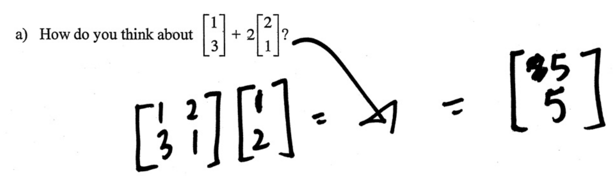


Figure 4. Charlene's linear combination of vectors.

Although Charlene claimed that she does not have a geometric way of thinking about the inscriptions in Figure 3 (the product of a matrix and a vector or the linear combination of vectors), recall that she offered a strongly geometric systems interpretation in the previous example. In that example, Charlene interpreted **x** as a solution that is unknown and is represented geometrically by an intersection point.

Taken together, these examples show how Charlene has some correct interpretations of **Ax**=**b**, but does not connect her LC computation to the corresponding LC geometric view and does not connect her systems geometric view to a symbolic representation. This illustrates just how challenging it can be for students to coordinate symbolic and geometric views within each interpretation of **Ax**=**b**.

Example 3: One student's incorrectly coordinated interpretations

The previous example showed how a student can draw on two different, uncoordinated but correct interpretations of **Ax**=**b**. This example serves to illustrate how students can have blended and/or partial interpretations.

Melissa revealed a mix of systems and linear combination interpretations in her response to the question

How do you think about $\begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$? Please explain.

Computationally, Melissa showed two different, correct ways of performing this computation as shown in Figure 5 (top and center), without the x and y, which she added later,

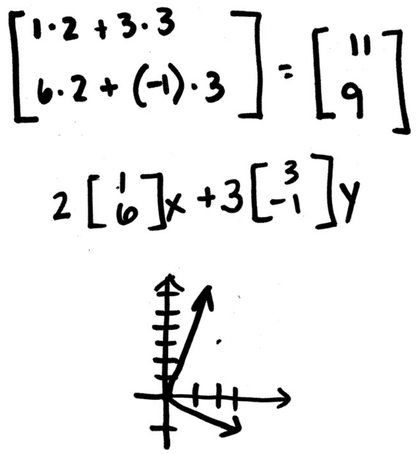


Figure 5. Melissa shows how she thinks about the product of a matrix and a vector.

as explained below. She explained her second method of computation, which is consistent with an LC computational strategy:

Melissa: You could write it separately, you'd have the vector 1, 6. Plus the vector 3, -1. Times 2 and times 3. And then you just multiply, you do it across, you do 2 times 1 plus 3 times 3. If this was x and this was y. [At this point, Melissa added the x and the y to the inscription shown in Figure 5, center]

Interviewer: If this was x, and that was y?

Melissa: You don't even have to do that, I don't know why, but sometimes I think of it with x and y, as if it's an equation.

Melissa elaborated, expressing a reasonable connection to a systems of equations interpretation, "When I think about it, I guess it makes me think of [...] as this is x, like one x plus three y [pointing to top row of the matrix], and this is six x minus y [pointing to the bottom row of the matrix], as equations." This statement indicates that when she looks at the matrix A, Melissa sometimes interprets the entries of the matrix as coefficients in a system of equations. However, as she continues to explain her thinking, it becomes apparent that the she is blending her systems interpretation and her LC interpretation in a way that is potentially problematic:

> I mean, say you had started out with x plus three y [writes $x + 3y$] as one, and six x minus y [writes $6x - y$] as another, and the vector two, three was acting on these [points at the expressions she had just written]. If you want to know what, I think if you wanted to know what the result would be if the vector 2, 3 acted on this, would be two x plus nine y, and twelve x minus three y [writes $2x + 9y$ and $12x - 3y$].

In the language of our A**x**=**b** framework, we see Melissa simultaneously interpreting the x as weights 2 and 3 (per her LC interpretation) and as unknowns x and y (per her systems interpretation). However, rather than viewing the weights and unknowns as interchangeable, she integrates them *both* into her explanation, evidence of a problematic blending of the two views.

When asked how she thought geometrically about the matrix times the vector in this question, Melissa began by plotting the column vectors of the matrix as shown in Figure 5 (bottom):

> I wouldn't normally, but I mean, if I was asked to think of it geometrically, I would think of these as vectors, the vector 1, 6 and the vector 3, -1. [Creates inscription shown in Figure 5, bottom] Don't know where 2, 3 would come in, oh no, I don't, maybe I do, that was 2, 3. [Adds a dot at the point (2, 3) in Figure 5, bottom] I guess, I'm trying to get 2, 3. I don't know where 2, 3 would come in. I thought maybe that it might be where you're trying to shrink the one, and move it around so you get the 2, 3, but I don't know why you'd want the vector 2, 3.

In a correct LC interpretation, one would use 2 and 3 as weights on the column vectors in order to construct the output vector. Using our framework for interpreting the equation A**x**=**b**, we can see that Melissa is drawing on a partial LC interpretation in that she is attempting to use a weighting of the column vectors of the matrix A as part of a geometric construction. However, she does not specify the source of the weights of the column vectors, and she conflates the roles of **x** and **b** in trying to identify which vector is being constructed.

Example 4: One student's effectively coordinated interpretations

Our final example comes from a 90-minute problem solving interview in which the student was given a single, open-ended problem-solving activity, known as the car rental problem, that was selected as an unfamiliar problem context (Larson, 2010; Larson & Zandieh, 2011). This example shows a student coordinating two interpretations in productive ways. More specifically, we see a student beginning with a systems interpretation and shifting to a transformation interpretation in a way that affords him computational efficiency.

In the problem, students are presented with a scenario in which there is a car rental company that has three locations in a city. Patrons of the company are allowed to return cars at any of the three locations and the problem describes what percentage of cars from each location are returned where (see Figure 6, overleaf). For example, each week, 95% of the vehicles rented from the Airport location are returned at the Airport location, 3% rented at the Airport are returned Downtown, and 2% of the cars rented from the Airport location are returned at the Metro location. Students are given an initial distribution of cars (500 at Airport, 250 Downtown, 200 at Metro), and asked to describe the long-term distribution of the cars if the cars are returned at the described rates.

In order to interpret the problem situation and find a mathematical way to represent it, Matthew first drew on a systems view. He initially wrote a system of three expressions that used a, d, and m to represent the values of the number of cars at the airport, downtown and metro (see Figure 7). He then continued in the systems view by rewriting the system of expressions as a system of equations that captured the role of time. In other words, he wrote three expressions, each of which detailed a computation for determining the number of cars at a specific location, based on the number of cars at each location the *previous week.*

About 28 minutes into the 90-minute interview, Matthew rewrote the system as the product of a matrix and a vector. The recursive nature of the relationship initially served as a source of challenge for Matthew: he recognized that he could find the configuration of cars based on the data from the previous week, but struggled to see how this was helpful in determining the long term behavior given only the initial configuration of cars. In discussing the possibility of using matrix multiplication to compute the number of cars during an arbitrary week x, Matthew commented, “To do that, we’d have to know what the week before it was. And we only know the initial value.” His comment that knowing the week before would yield the configuration during week x is consistent with a transformations view; in fact, he uses function notation to indicate this relationship. In terms of dealing with the recursive issue, Matthew eventually recognized that he could use his calculator to repeatedly multiply the initial configuration of cars by the matrix to determine the configuration of cars after many weeks had passed.

In this example, we see a student begin computationally with a systems approach, and strategically shift to a transformation approach for computational efficiency. In the language of our A**x**=**b** framing, after writing a systems-like set of expressions, Matthew recognized that given **x**, he can determine **b**. By rewriting the coefficients in his set of expressions as a matrix, Matthew treated the matrix A as a whole, conceptualizing it as something that can act on **x** (the configuration of cars at any point in time) to compute **b** (the configuration of cars one week later), instead of relying on a system of computations with *a*, *d*, and *m.*

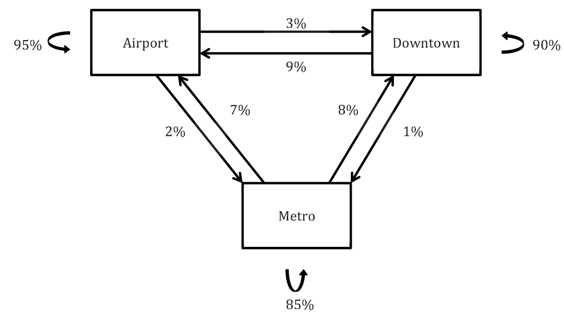


Figure 6. Redistribution rates in the car rental problem.

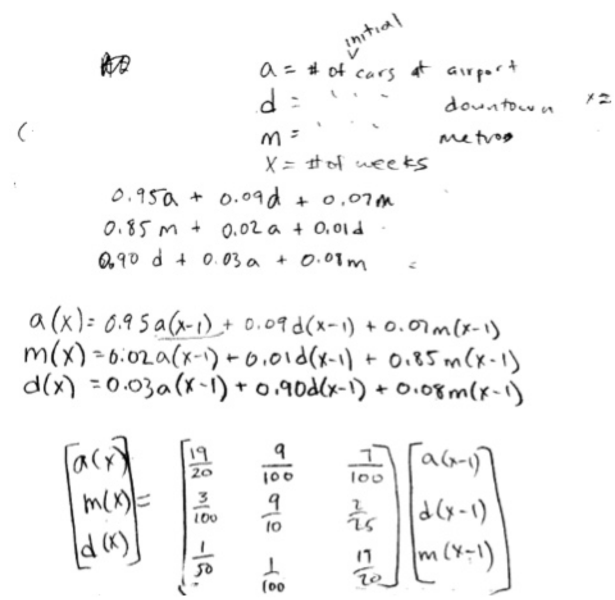


Figure 7. Matthew’s written work.

Conclusion

In this article, we have presented a framework detailing three symbolic and geometric ways in which students make sense of the matrix equation A**x**=**b**. By detailing the specific ways the literal symbols A, **x**, and **b** are viewed in each interpretation, we have shown how this framework can serve as a diagnostic tool for making sense of student thinking. The first example showed how, on a given question, different students may gravitate toward any of the three interpretations in our framework. The second example showed how students can draw on isolated aspects of particular interpretations (LC & systems) without coordinating symbolic and geometric aspects within interpretations. The third example showed how students can hold multiple correct interpretations (LC & systems) that are incorrectly blended. The final example showed a student effectively first developing a systems interpretation of a real-world problem, and then shifting to a transformation view in order to achieve computational efficiency.

Our framework highlights the complex nature of the interpretations and coordinations asked of students at even the most introductory level in linear algebra. While the examples used to illustrate the framework in this paper are situated in 2 and 3 dimensional contexts, we argue that the interpretations presented here have the potential to offer insights for higher dimensional settings. First, there is evidence in the literature that geometric interpretations in lower dimensional settings are utilized by both students and research mathematicians for intuition in higher dimensions (see, for example, Wawro, Sweeney & Rabin, 2011). Second, the symbolic views of each interpretation generalize to higher dimensions using standard mathematical notation. The invertible matrix theorem offers an example of this type of generalization and consolidation of notation. Assuming A is an invertible $n \times n$ matrix, three equivalent statements highlight the importance of the three interpretations: column vectors of A span $\mathbf{R}^n$ (definition of span is highly consistent with LC view), A**x**=**0** has the unique solution **x**=**0** (emphasizes systems interpretation), and the transformation $x \mapsto Ax$ maps $\mathbf{R}^n$ onto $\mathbf{R}^n$ (emphasizes transformation interpretation). Establishing equivalence of any of these statements entails coordination of interpretations in the framework.

In research, teaching, and instructional design, anticipation of student thinking is needed to help students develop a set of rich, well-connected interpretations. We offer this framework to help teachers, researchers, and curriculum designers better understand ways of supporting students in developing the ability to move flexibly among interpretations to powerfully leverage the analytic tools of linear algebra.

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Any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form.

The structure of any domain of knowledge may be characterized in three ways, each affecting the ability of any learner to master it: the mode of representation in which it is put, its economy, and its effective power. Mode, economy, and power vary in relation to different ages, to different "styles" among learners, and to different subject matters.

Any domain of knowledge (or any problem within that domain of knowledge) can be represented in three ways: by a set of actions appropriate for achieving a certain result (enactive representation); by a set of summary images or graphics that stand for a concept without defining it fully (iconic representation); and by a set of symbolic or logical propositions drawn from a symbolic system that is governed by rules or laws for forming and transforming propositions (symbolic representation).

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