# MULTIPLE VISIONS OF TEACHERS' UNDERSTANDINGS OF MATHEMATICS

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This conversation was spearheaded by some research we (Ann, as supported by Ralph) conducted, as mathematics educators interested in learning from elementary teachers (Kajander & Mason, 2007), particularly those we judged as experts in their field (Kajander, 2010). In this research, we were interested in unpacking and reflecting upon the mathematical thinking teachers found themselves enacting as they negotiated their daily mathematics classroom practices, and what particular mathematical ideas and understandings they drew upon as they did so. As mathematics educators, we wanted to delve into the debate about *what* teachers need to know as well as *how* they need to enact this knowledge while teaching. We hoped that this method of asking teacher-practitioners to document their (mathematics-related) thinking might help us build on what has been gleaned by observing and documenting teachers teaching (such as the research that underpinned the work of Ball and her colleagues, (*e.g.,* Ball,1990; Ball & Bass, 2003; Hill & Ball, 2004; Ball, Hill & Bass, 2005) or by posing particular mathematics questions to teachers for discussion in professional development contexts (for example, see Davis & Simmt, 2003). We felt that investigating the specialised (or not) nature of this 'knowledge' might better support our collected efforts in pre-service education and (in-service) professional development.

Not surprisingly, the reflections of the teacher-research-participants (in the form of meeting transcripts and journal entries) shared during the research project were highly contextualised in their daily work with students. Nevertheless, a number of examples emerged which appeared to support the notion that teachers need something 'more' than a deep knowledge of the curriculum, as has been argued by some researchers (*e.g.,* Hill, Sleep, Lewis, & Ball, 2007; Philipp, Ambrose, Lamb, Sowder, Schappelle, Sowder, Thanheiser, & Chauvot, 2007; Stylianides & Ball, 2008).

As I (Ann) began to craft these examples, gleaned from the teacher-practitioners' conversations and journals, into a paper (Kajander, 2010), I circulated some of the examples to mathematics and mathematics education colleagues (the other authors), because I found the teachers' reports and questions fascinating. What impressed me in particular were the number of domains in which the teacher-practitioners were simultaneously operating; while always situated in the contexts of their students' modes of thinking, the teachers drew on their curricular knowledge, their personal mathematical understandings, and sometimes what seemed to me to be "other" mathematical understandings. The plurality I observed aligns well with Ball and Bass's (2003) model of 'mathematics for teaching'. Silverman and Thompson feel "it is axiomatic that teachers' knowledge of mathematics alone is insufficient to support their attempts to teach for understanding" (2008, p. 499), and that a specialised knowledge is required. Indeed, Ball, Bass, Sleep and Thames (2005) argue that an underdeveloped understanding of the specialised nature of the knowledge needed to teach mathematics is problematic in the advancement of teacher education. These "other" understandings which I observed during the research project, such as facility with appropriate mathematical models, alternate approaches to concepts, and ways of thinking and reasoning conducive to students, seemed to me to fall into the domain termed "mathematics for teaching". I also (naively) assumed that my colleagues and I would generally find the same aspects of the examples worthwhile and interesting. I was, however, unprepared for the vigorous nature of the discussion which ensued, which clearly illustrated a breadth of viewpoints and interpretations. It is this discussion that has been 're-enacted' in the dialog to follow. We are (collectively), Ann, Ralph and Dan, mathematics educators, Peter, Wes and Edward, mathematicians with an interest in mathematics education, and Tom, a classroom teacher.

## Context
In the broader scheme, we are collectively interested in better supporting teachers as they learn mathematics. In some of our own institutions for example, mathematics courses for prospective elementary teachers are offered (if they are offered at all) by mathematics departments, not faculties of education, and thus we felt that working together is critical. Hence we are particularly interested in ultimately better informing the debate about by whom and where teacher development activities and courses should be offered and housed, and what mathematical experiences might be important aspects of such courses for teachers. It also emerged in the dialog to follow that a common understanding of the terminology would better support effective discourse.

Two "content" examples in particular from the initial research with teachers (Kajander, 2010) framed our subsequent discussions as academic colleagues. The first revolved around the mathematical topic of "smaller and smaller" or "closer and closer", as it emerged from several different classroom contexts provided by the teachers. Examples shared by the teacher-participants grouped loosely under this theme included discussions of discrete versus continuous data, as well as the circle area model in which smaller and

smaller wedges are assembled into a shape which eventually begins to approach a rectangle. A second theme which emerged from the teachers' discussions involved debates about various models used by the teachers to support their students' construction of understandings of operations with integers.

## Our conversation

### Views of 'understanding'

We begin with initial thoughts, focused mainly on the notion of 'understanding' in mathematics.

*Dan:* As a mathematics educator, I really like your approach of "getting down in the trenches," so to speak, and actually talking to teachers in the field. What I think I'm hearing you say is that this proximity to daily practice is in some ways highlighting the inability of teachers to know what it is they are supposed to/want to know in terms of conceptual understandings – and that the approach of stumbling upon one or two rich examples via research studies or the occasional professional development interaction is less than ideal for a carefully-organized, reform-oriented teacher development program.

*Tom:* As a teacher-participant in the research process described above, I think that the teacher-participants' examples (mentioned above, and further discussed in Kajander, 2010) suggest the rich flavour of the discussions shared by the teacher-research-participants. I have to say that from my perspective as an elementary classroom teacher, I firmly agree that there is indeed something very different than a strong (functional) understanding of procedural math that is required to successfully teach an entire class of students. Students learn differently from one another, and *many* do not become fluent at procedures that they don't understand. Using "steps" to determine "answers" (which many teachers still teach, for example in the standard algorithm for long division), does not lend itself to "understanding". Many students will be stumped for example, if you stop them half way into such an algorithm, and ask them *why* they are doing whatever step they happen to be doing at that point. "Because it's what you do next", is a common answer. Alternative strategies, however, foster understanding by incorporating simpler, more familiar operations that the students know and understand, such as place value concepts.

*Peter:* I certainly agree with Tom that it's important for teachers to be able to take such algorithms apart in a way that allows us to understand what the different steps are doing. Presumably that's what he means by "something very different than a strong understanding of procedural math." That's fair enough though I'd probably use "understanding" in a broader sense and say that understanding what the steps are doing is exactly what one could mean by an understanding of procedural math. No matter what, however, I think we all agree that this type of understanding is hard to come by. I tried it out myself with the division algorithm. I had to think pretty hard.

*Wes:* I am a mathematician also interested in learning more about mathematics education. I've been thinking a bit about this idea of "special knowledge" that teachers might need. My view would be that the mathematics *content* they need is essentially the same as the content they teach, but that they do need an extra level of mastery – a familiarity with the subject that allows them to move about freely. This is perhaps what the teacher-participants perceived as "special". What I mean by this is, not only do teachers have to be familiar with the subject matter, they must be confident enough to recognize and generate equivalent interpretations without aid and this is quite possible without going beyond the subject matter at hand.

*Ann:* I wonder then, why the teacher-participants in our research, chosen as "experts" based on our knowledge of them from their participation in a number of other projects (*e.g.,* Kajander & Zerpa, 2006; Kajander & Mason, 2007), as well as by observing their practice using a standard observation protocol (Horizon Research, 2003) were so adamant that what they needed to know about mathematics was somehow "more" than what students (and maybe even mathematicians) needed to know, as has also been argued elsewhere (Ball & Bass, 2003).

*Peter:* Good question. I'm guessing that this "more" would be things they might learn by tracking down answers to the questions they come up with when they try to "move about freely" to use Wes's phrase. There's perhaps nothing there that a good student mightn't already ask: why do we bring down the next number?, why is 2 × 3 the same as 3 × 2? I guess I want to sell the notion of math (at the elementary level) as being available and accessible to the teachers in their own professional activity given that they are curious, persistent, patient, and are prepared to be creative and think laterally (all wonderful qualities for a teachers and things we (at university) should be promoting *much* more than courses in differential equations). Of course the main thing they need for this model is more time for in-service professional learning. The area model of multiplication (and the extra insight when the diagram is drawn to scale) is a good example of this. The experience with the model is just as important and relevant for the student as for the teacher (and it's a good example, by the way, of the relation between abstract and concrete).

### The first mathematical example

We turn to a particular example, namely the derivation of the circle area formula using a parallelogram model, and debate the mathematical understanding required and the implications for teacher education.

*Ralph:* Using this example, I'd like to talk for a moment about the wedge model of the area of a circle – imagine students cutting a circle into 12 equal wedges, and rearranging the wedges to compare the area of the original circle with a rectangle or parallelogram. I know that there is significant mathematics-for-mathematics-learning in this model. I have seen it enabling students to think about π in fresh ways, and to think about their understanding of formulas in fresh ways. It is part of a package that has enabled students to shift gears in their algebraic reasoning when it comes to formulas, variables, and relations. In our research, the magic is when the students extrapolate from what they can do in a tangible finite world to an infinite number of wedges of infinitesimal width. (This happens when the students realize that 12ths wedges are more visually convincing than 6ths, and

that 24ths wedges are more visualizing convincing than 12ths. And then they think, what if we went smaller again? and again? and, wow! And no, they haven't taken an analysis course. Neither had Archimedes when he thought this way.

*Edward:* I think that the parallelogram argument is just one component of a whole long history of reasoning about the area of a circle, and that most people notice that there is something incomplete and unsatisfying about the argument, even if it is presented as something that is supposed to be complete and convincing. Teachers who have not been taught the background, and who are short of time with a hundred other things to do, and who may be unwilling to display their discomfort and lack of certainty to students (or parents or to their peers) for social reasons, might be tempted to present the argument as something that is supposed to be complete and convincing, a "topic", then move on. They would leave a sense of discomfort in many of their students, a discomfort not necessarily that something isn't known, but that something isn't even being addressed. That to me is a strong argument about the benefits of teachers knowing something of the history and deeper details of the math they teach, although I wouldn't go so far as to say there is an absolute "need" for teachers to know all of this. Some mathematicians don't know most of the story, and furthermore there is a back story to each and every item on the curriculum. It might take the equivalent of a degree in the history of mathematics to fill the stories all in. To have a teacher able to say, "But this isn't the whole story; you can learn the rest of it in university, or if you read this book, or whatever," is the base line from which we should start. We have no trouble saying such things in English, or history, or just about any other subject; math teachers should approach their subject in the same way. Just as an English teacher will be able to do a better job, the more they know about the background of a text and what else has been written about it, so too a math teacher should be able to do a better job, the more they know of the background and alternative arguments and objections and resolutions and so on. It's not a need or a requirement, but a matter of degree.

*Peter:* This is a great comment. Stories are a wonderful part of teaching, but good stories are hard to come by. History is an excellent source.

*Ann:* Knowing more about important aspects of teachers' mathematics learning is important for mathematics educators, and anyone else charged with supporting teachers' growth in mathematical fluency. To be practical, we need to know what is critical to provide prior to entry into teacher professional certification programs, during teacher preparation programs, and what growth should be supported via professional development at the in-service level. We also need to know whether courses that teachers take at university in mathematics should be offered by mathematics departments and should be similar to courses other students might be interested in, or whether such courses should be specific to the processes of teaching. Peter has already suggested that high quality mathematics courses that encourage investigation and reflection should be fine for prospective teachers, and these need not be different for teachers than for other students. He also mentioned that reflective practitioners should be able to seek out what they needed subsequently via

improved professional development, and that might suffice.

*Dan:* I also think it's important that teachers are able to recognize and to facilitate these concrete/abstract and cross-topic/cross-discipline/cross-grade connections for their students (*i.e.,* not make them for them, but provide opportunities via activities in which students are encouraged to make these connections themselves). I'm left unsure, however, as to where teachers' knowledge of appropriate mathematics will ultimately come from, if not from an external source (*i.e.,* an expert in the field, written information, or other professional development).

*Peter:* I believe that a perfectly fine understanding of the ideas for an elementary teacher can indeed come "through reflective practice." This example of the wedge approach to circle area is certainly a nice and important example. You say in your paper Ann that teachers need:

> … a deep level of understanding of real numbers, again something that is unlikely to develop during the course of daily classroom experience alone. … Understandings of explicit connections between more and less sophisticated mathematical ideas must be constructed, while remaining grounded in concepts that might be assessable to a typical 12-year-old. (Kajander, 2010, p. 12)

I fear that this is exaggerated and runs the risk of making the example seem beyond the level of most teachers unless they take a university analysis course. It is true that the ideas at the heart of this example are challenging and subtle, in much the same way as our historical struggle to understand the limit was challenging and subtle, but I believe that there are ways of packaging these examples that are elementary and accessible.

*Ann:* I agree that there may be fairly effective ways of organising this learning for teachers. I think the point I was making is that such learning opportunities have to be explicitly offered somewhere during teacher development. I haven't seen evidence in working with teachers that reflective practice alone is sufficient to support teacher development.

*Peter:* Reflective practice is in fact hard to come by, and it might not even be natural; it might just have to be taught. (Maybe that's what you mean by "the other understanding.") When you assemble the wedges you do get an approximation to a parallelogram whose area is the area of the circle. I was objecting (mildly) to the idea you mentioned that this needed connections across vast domains of math. But I do agree that it is hard to attain through reflective practice. The mathematical question is whether this sequence [of wedges of decreasing size] gets arbitrarily close (whatever that means) to the circle area. And it's making that last step precise that was the principal historical work. However, it's of course possible to give teachers an elementary intuitive approach to this step that does not require the technical understanding of limit, but which will give them a good start on their journey to such understanding.

### A second example

Turning to a second mathematical example, related to integer operations, we continue to discuss curriculum interpretation

as well as the role of modeling in developing generalisations.

*Ann:* Let me talk about the modeling of integer operations, as well as other operations, that appear to be important topics in the elementary curriculum, at least in our province (Ontario Ministry of Education, 2005). How are teachers to decide which operations should be modeled, and which merely derived from some previous generalisation? The (Ontario) Grade 8 curriculum, for example, contains an *Expectation* as follows: "represent multiplication and division of integers using a variety of tools, *e.g.,* if black counters represent positive amounts and red counters represent negative amounts, you can model $3 \times (-2)$ as three groups of two red counters" (Ontario Ministry of Education, 2005, p. 111). The next *Expectation* goes on to say "solve problems involving a variety of tools such as 2 colour counters, virtual manipulatives, and number lines" (*ibid.*). Are you saying teachers should teachers stop the modeling process after $3 \times (-2)$? And how should they know to do that? And what about $(-3) \times 2$?

*Peter:* $(-3) \times 2$. Hmm. That's interesting. I wouldn't have treated that any differently than $3 \times (-2)$.

*Ann:* Conceptually, I think it's completely different.

*Peter:* So it is. And I can see that finding a good model for each of these that allows us to understand why they ought to come out to be the same, would be an important activity for the teacher, or at least, if not finding a model, at least knowing how to search for one.

*Ann:* But my pre-service teacher candidates feel a lot of confusion and tension around knowing when to investigate and model an idea conceptually with students, and when to simply "tell" them what to do. How are teachers supposed to know what else might be important to do with students when the curriculum says they are to model the ideas? The 60's 'new math' was supposed to make mathematics easier to learn by simply showing kids the generalisations – which might seem the 'easiest' way to think about something *once you understand what they are generalisations of.* Richard Skemp might have argued that this is not how mathematics is learned or initially understood. But I can see how as a mathematician you would be finding these models unnecessarily cumbersome. The question is what is better for the learner?

*Peter:* I don't really find any of it cumbersome. And I agree with you about the new math. What we give the learner should be simple and concrete and insightful, at the beginning at any rate. Numbers themselves are simple and concrete …

*Ann:* [smiling] Concrete?

*Peter:* Well maybe not. I guess they do require some sort of realization, which I probably take for granted. And some realizations wouldn't support negative numbers as well as others. I guess that's what we're talking about.

*Tom:* Your example of $(-3) \times 2$ is one I work with in my Grade 8 class in which we examine multiplication as "a number of groups" of something or another. These groups can be added or removed, or we can think of less than zero groups of something. It would be very easy to just let kids use the commutative property of multiplication to model $(-3) \times 2$, but it would leave the students with absolutely no conceptual understanding of what $(-3) \times 2$ means. The kids who memorize well, and learn procedures well would be fine. The students who struggle with those skills would be left, as they so often are, alone struggling and thinking they are no good at math … I believe that that is why so many adults who reflect back on their school years, remember being "no good at math", or "hating math", or "thinking that they were stupid in math".

*Wes:* You've got me wondering if perhaps traumatic experiences in mathematics are not created by the inability to memorize a procedure but by the confusion created by a set of contrived examples meant to facilitate understanding. If a teacher had told me about 'less than zero groups of something', I probably would have accepted life as a hermit to distance myself from this confusing subject!

*Ann:* Other teacher-participants in fact preferred variations on this model. For example, while the idea of "removing" 3 groups of 2 was proposed to model $(-3) \times 2$, there are also the "chips models" and the "number line" models. The debate over which model was most helpful to students – as well as most mathematical – was highly important to the teachers as they attempted to move from particular examples to generalisations.

*Wes:* Certainly there is an important question here – when is it appropriate to abandon concrete interpretations of mathematical concepts and journey into abstraction? Might a teacher say: "Okay class, we remember what $3 \times 2$ is, now when we see $(-3) \times 2$, we use the same answer but put a negative sign in front. Granted, this approach is a leap from what the students have learned up to this point, as there is no concrete reason why we should do such a thing, but I believe it is on par with thinking of $32 \times 14$ as $32 \times 10 + 32 \times 4$. And an important concept is introduced inadvertently: $-3$ is the same as $-1 \times 3$, which may facilitate the understanding of multiplication of two negative numbers later on. I'm wondering why has it become taboo to teach a student what is done first and what is done second? Perhaps if a student becomes good at doing an operation they will become less intimidated by it and more willing to engage in a creative struggle to understand it.

*Ann:* In my experience it is harder to get students (and teachers) to investigate models and their meanings once "rules" are in place.

*Edward:* One day I made a quick, unscientific survey of my colleagues, asking them why the product of two negative numbers is positive. Some just looked at me weirdly, as if it were simply a stupid question to ask, and all of the others said "to satisfy the distributive law", as if they would get arrested if they slipped in an extra negative sign while the math police were looking. My response to that is that we are in control, and we can choose to satisfy the distributive law or not. If we want a system which is useful in a certain application, or a system which is reflective of a particular "reality", whatever that may mean, perhaps it is better to have a distributive law, but we have to argue that it is a good thing, not simply to invoke it as a requirement from on high.

*Wes:* I totally agree. When we "impose" the distributive law or the commutative law, we need to make it clear to the students just why these laws are so critical to what we want arithmetic to do for us.

*Ed:* I think it is important and valuable to be able to address questions such as "why is the product of two nega-

tive numbers positive" in a convincing way. Doing so removes a distraction from students, and restores their faith and trust in their teacher and the system; this is particularly important with adults and with Aboriginal students, both of whom may have reasonable grounds to be suspicious of the "system". I am currently teaching "Adult Mathematics 001", a course starting with integers and fractions and ending with graphs of linear inequalities and systems two linear equations, to a group of adults, mostly Aboriginal. When I told them that the product of two negative numbers is positive, about half simply accepted the statement (most of those students have irregular attendance, by the way), and about half, including some of the best students, asked me "Why??" in a rather irritated manner. One student shouted "But WHY??" and was nearly in tears. The others I suspect were just as perplexed but didn't let it "get to them" so much. I sensed a watershed moment. Fortunately I was prepared with a model which has worked well for me in the past. I took some coins from my pocket, and using the pennies as "negative counters" and nickels as "positive counters", I showed the students that a) giving them a positive counter results in a net increase in their wealth, b) giving them a negative counter resulted in a net decrease, c) taking from them a positive counter resulted in a net decrease, and d) taking from them a negative counter resulted in a net increase. "That makes sense", said one of the students. Then I modeled $3 \times (-2)$ as giving 2 negative counters 3 times, and $(-3) \times 2$ as taking 2 positive counters 3 times, *etc*. After a few rounds of that game I think everyone was quite satisfied that the rules were "right", in the sense that they reflected something in "reality" or were useful (in some context). And, perhaps more important for my mission, the students seemed to trust me more after that, both for my knowledge and for my willingness to address their concerns in a constructive fashion.

*Peter:* That's a nice example. I certainly believe that teachers ought to be able to develop the capacity to search for models or heuristics for such procedural examples as $(-3) \times 2$. Maybe that's what Tom and Ann mean by "something different than a strong understanding of procedural math." Notice that I said "the capacity to search." It is possible that a teacher will fail to find a good model. That does not make that teacher a failure as a teacher. Indeed, I found it hard to get a good model for that question. Brent Davis has written a good deal about multiplication and has a number of nice models that could be brought to bear on this (transformations, rotations, reflections, *etc*.) but certainly very few elementary teachers would have encountered Brent's papers.

*Ralph:* I do not believe that all persons should know all of the various contexts and formalisms for integer arithmetic. I suspect I don't! But we should be experienced with some of them, and if we want kids to understand integer operations, they should experience some of them. It's not a one-to-one mapping: students need experiences in more than one context to understand $(-3) \times (2)$. And probably the multiple contexts should not be encountered all at once. Like so much of the math content we teach, integer operations are a perfect opportunity for kids and teachers to negotiate different kinds of understandings (contextual, formal, algorithmic) and interrelate those understandings as their experiences with them are enriched. This prepares learners and their classroom communities for even richer engagement in mathematical thinking and learning with other content. That makes $(-3) \times (2)$ huge, in terms of math education significance.

## Implications for mathematics courses for teachers

We turn now to the specific question of teacher education in mathematics, and what can and should be done.

*Edward:* Teachers knowing more helps deliver a better experience to the students, as with the circle/parallelogram example. On the other hand, there is probably a law of diminishing returns in effect. Having one or two or three models of multiplication on hand for different contexts is very useful, but having a whole whack of them might not be that much more useful. As teacher educators, our job ought to include sorting it all out to make the most useful model or two available to teacher candidates, and leaving the others for more specialized training. The same goes for historical knowledge. Archimedes and Diophantus and Cardano didn't "believe" in negative numbers, and they were truly great minds, and there is a whole long, agonizing history of the appearance of negatives and questions about the way they should be handled. Ultimately it was convenience, rather than necessity, that brought people to accept negative numbers and the rules for managing them. The more of that history a teacher knows, the potentially richer the experience they can make for their students, but the process could be streamlined and "optimized" if someone (*i.e.,* you or me or someone like us) could select the most important parts of the historical thread, and leave the rest as potential enrichment in the teacher education experience.

Negative numbers are disquieting, and mathematicians were not completely happy until they had a way of constructing them rather than just tossing negative signs around. (The method of construction is just an "official" way of managing positive and negative counters, as ordered pairs; the official construction and the model in terms of counters actually play off one another, one giving structure and the other meaning.) Maybe that could be part of a good math teacher education program.

Perhaps math education attracts the kind of students who are willing to accept, memorize, and apply rules without much further thought. They come to math because that is what they see in it, and then they teach and replicate their vision of math to another generation. That may be a caricature but I believe there is some truth to it. I would like to try to break that cycle and show that there is a *method of inquiry* at the heart of mathematics, not just a bunch of rules, and not even just knowledge. Math is not just about what we know, but is also about how we know what we know.

*Ralph:* Is it not inherently interesting to those interested in cognition, that multiplicative operations on integers are cognitively non-commutative? (That is, if we reposition the positive and negative in $(+3) \times (-2)$, it's not as easy for learners or teachers to envision the operation as sensible within a given context.) For me it is a fundamental premise (a mathematics education axiom, if you wish) that to understand multiplication of integers, a person benefits from multiple experiences that generate facility within and across

multiple sets of images, both contextual and not, symbolic and not. And experience has taught us that it isn't easy to have people develop imagery for multiplication across multiple contexts. In other words, it's likely to be a potent environment for teacher inquiry and mathematics educator research. Given the possibility that mathematics exists nowhere but in the cognition of those learning it or doing it, we can celebrate the additional complexity, beauty, and elegance of math-in-the-learning, as we do for any flower when it is in bud.

Integers (along with the operations on them) are on the cusp of numbers (by which I mean quantities, measures, and/or relative positions of things and actions in the world, represented by numerals of some kind) which can make sense (I am using "sense" literally – we can see, taste, and touch the quantities, and we enact the processes that the operations summarize). They may be the perfect place to blend the contextual modeling of operations (addition and subtraction as moves forward and backward on number lines; as changes in temperature on a thermometer; as financial transactions entered and removed in a financial record, …) with the elegance of models within formal mathematics such as sequences and inverses.

*Tom:* I feel strongly that the lack of such deep conceptual understanding of math concepts and models by many teachers is largely responsible for much of our students' limited success. Further, I feel that teachers also need to be able to accurately assess "where students are" with respect to conceptual understanding and skill development.

*Edward:* So if we were to create a new mathematics course for teacher candidates at one of our institutions, what would it look like?

*Ralph:* I like Edward's suggestion, for moving the discussion towards our imagery of what we'd do with our students in a mathematics-for-teaching undergraduate course. I see in the wording of Ed's question, an invitation to say, "What we would *do* with our students?" or rather "What would they do with us?" He's inviting us to say what the present-tense activities would be, rather than fall into the outcomes trap. Let's see if we can keep our contributions to the bonfire-to-come on a conceptual scale that respects the early progress we made with (–3)(2). For middle years teachers, is (–3)(2) just an integer times-table fact, an instance of the 'signs-different, product is negative' rule? Perhaps for mathematicians-to-be, that's more than enough. But I'd like teachers to have more, to (following Ed's suggestion) do more with it in a math-for-teaching course that we might design.

I'd like teachers-to-be to encounter (–3)(2) in multiple contexts, while they're making sense of stuff. As an example, I suggest a pure math context where they might find meaning. It relates to creating concrete representations of numbers including those represented as binomials.

Imagine a rectangular array for (20)(20). You could even construct it out of grid paper or hundreds blocks. Now, *adjust* that array, to show each of these new arrays: what rows and columns do you need to add or subtract to (20)(20) = 400 in each case, to make an array that matches the new question? (And please note the (–3)(2) in the last example.)

(21)(20) = 410
(21)(21) = (20 + 1)(20 + 1) = 400 + 20 + 20 + 1 = 441
(23)(24) = (20 + 3)(20 + 4) = 400 + (3)(20) + (4)(20) + 12 = 552
(19)(19) = (20 – 1)(20 – 1) = 400 – 20 – 20 + 1 = 361
(17)(22) = (20 – 3)(20 + 2) = 400 – 3(20) + 2(20) + (–3)(2) = 374

Students could also be asked to multiply these binomials using any number of methods:

    i)   (20 – 3)(20 + 2)
    ii)  (y – 3)(y + 2)

How are these examples the same, and how are they different?

In the above examples, you will note that I am using arithmetic and algebraic pure-math as my spaces for having my students encounter (–3)(2) and make sense of it as part of a variety of pure-mathematical structures. That's some (but maybe not all) of the things I'd want my students to do with (–3)(2), if they were in a math-for-teaching course for future middle-years teachers.

*Ann:* I am relieved to finally see some common ground emerging. You have provided us with some "pure math" examples that might illustrate instances of a "deep" conceptual understanding that everyone can live with. While we have not yet collectively resolved whether we believe that mathematics-for-teaching is entirely contained within the domain of a more traditional mathematical culture, or whether we have two overlapping sets neither of which is a subset of the other, I believe you have helped us create an example of something we can build on collectively. Although some of us still feel that what teachers need to know sometimes extends beyond the domain of academic mathematics, it may be possible to construct other such examples which at least overlap the domain of pure mathematics. This is a starting point from which to move forward in future conversations.

### In reflection

We've been exploring the challenges of conversing across disciplines about mathematics for teaching. Our discussion process has been fraught with greater challenges than we initially expected. Moreira and David (2008) suggest that the values and forms of conceptualising objects in academic mathematics may differ from, and even collide with, the demands of school teaching practice. Indeed, when we have tried in our discussion to move toward a synoptic position the conversation seems to have lost generativity, possibly exacerbated by our tendency to have different understandings of the terminology. We may be bringing some preconceptions into play here, and we have to believe that our preconceptions (that is, our taken-as-shared implicit understandings that aren't necessarily shared), the things that make this conversation so hard for us to focus and move forward, are the very conceptions that our broader community are finding problematic. At that level, our discourse becomes a single turn in the broader community's sincere search for common ground for discussing mathematics for teaching.

We (Ann and Ralph) would like to close with some suggestions for where we might continue to collectively find common ground. The need remains for us to be patient as we

search for commensurability among our perspectives and our different discursive approaches. We believe the interplay of different perspectives has enriched our conversation, even though it has at times made participation difficult because of seemingly incommensurable perspectives, priorities, and cross-purposes. We find reason for hope in such discourses in the ongoing shifts in the manner of that discourse, such as the emergence of 'mathematics for teaching' as a term for pointing at what math teachers should experience/understand, rather than what knowledge they should receive. We continue to feel that the contextuality present in discussions about math for teaching means the conversation must privilege observations about the students, the classrooms, and the teachers. We notice in our collective discussions the loss of privilege of the "content" or "knowledge" alone as an outcome of our emerging views. Instead, what get privileged are the processes: the tasks that become activities, which become experiences, through engagement and reflection. The value of such mathematics must be determined by its relevance for the work of the teachers and students, not its acceptance by a community of academics alone. It must be negotiated based on what understandings students already have, and not draw on mysterious higher-order mathematical properties of which students might not yet have developed an understanding. We firmly believe that the genesis of possibilities (and needs) must arise from within the thinking spaces of mathematics classrooms, and honour teachers' immediate needs rather than arising from the working repertoire of mathematics departments.

Accepting the possibility that math for teaching is a particular mathematics sub-discipline, like math for engineers or carpenters or citizens, requires the dismissal or at least questioning of the premise that the math that teachers need will be a subset of mathematics that mathematicians value. If we think of two realms, one the realm of academic mathematics and one the realm of mathematics teaching in schools, we find ourselves a little closer to understanding what manner of discourse could overcome some of the incommensurability of the two realms' perspectives. Can mathematics for teaching be viewed simply as the overlap of the two realms, or is it something that has yet to emerge, something that will encompass the full realities of both of its parent realms? In either case, our next steps must privilege the voices of practitioners, with the voices of mathematicians acknowledged as a resource provider but not as an over-arching authority of the nature of the sub-discipline.

The challenge of finding common ground for discourse about content of mathematics courses for future and current teachers (about what content experiences should be offered, and even about what learning processes should take place in those courses by which learners experience that content) is daunting. We believe a framework for sharing and aligning our diverse perspectives about the kinds of experiences by which future teachers should develop the mathematics they will take with them to teaching might better support our future conversations.

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