A FRAMEWORK FOR EXAMINING MATHEMATICS TEACHER KNOWLEDGE AS USED IN ERROR ANALYSIS

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Research on teacher knowledge has developed significantly over the past two decades, prompted in large part by Shulman's (1986) proposal that, besides subject-matter knowledge (SMK) and curricular knowledge, teachers need 'pedagogical content knowledge' (PCK). His proposal has sparked a number of focused studies, which can be generally classified into two groups: one devoted to examining the special kind of mathematical knowledge that teachers need in order to teach well (Adler & Davis, 2006; Ball, Hill & Bass, 2005), and the other concerned with identifying and describing the knowledge required to teach specific content areas (Even, 1993; Peng, 2007; Tirosh, 2000). A few studies within the two categories discuss how teachers deal with students' mathematical errors.

There is a long history of error analysis in mathematics education (Radatz, 1979) and teachers and researchers have long recognized its value. As Brown and Burton (1978) suggested, "one of the greatest talents of teachers is their ability to synthesize an accurate 'picture,' or model, of a student's misconceptions from the meagre evidence inherent in his errors" (pp. 155–156). However, despite a wealth of results about mathematics teacher knowledge, there is a lack of detailed understanding regarding mathematics teacher knowledge as used in error analysis. Sfard (2008) made this point when interpreting the thinking of Gur, who "does not understand what the formula and the table are all about, what is their relation, and how they should be used in the present context," noting that "although certainly true, this statement has little explanatory power" (p. 21).

Error analysis is a basic and important task in mathematics teaching, and it is more challenging for mathematics teachers (Luo, 2004). The absence of a framework of knowledge to inform error analysis could render mathematics programs for both teachers and students less effective. Thus, we address the development and validation of such a framework for examining mathematics teacher knowledge as used in error analysis.

Theoretical considerations

Teacher knowledge frameworks

In Shulman's (1986) categories of teacher knowledge, SMK refers to the knowledge of facts and concepts, and understanding the structure of the subject. PCK is first described as

> an understanding of what makes the learning of specific topics easy or difficult.... If those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners. (pp. 9–10)

Many categories of teacher knowledge have subsequently been proposed and investigated, among these are the subcategories of SMK and PCK developed by Hill, Ball and Schilling. Under the category of SMK, they described three sub-categories – namely, common content knowledge, specialised content knowledge, and knowledge at the mathematical horizon. Under the category of PCK, there are knowledge of content and teaching, knowledge of content and students, and knowledge of curriculum. They further describe knowledge of content and students to include the ability to anticipate student errors, to interpret incomplete student thinking, to predict how students will handle specific tasks, and what students will find interesting and challenging (Hill, Ball & Schilling, 2008).

An elaboration on error analysis in mathematics education

The variety of students' mathematical errors and the range of researchers' interests have contributed to the formation of many theories about the nature of mathematical errors, their interpretation and their remediation (Gagatsis & Kyriakides, 2000). With the widely recognized conceptual change framework, errors initially conceptualised negatively are now seen as a natural stage in knowledge construction and thus inevitable (Vosniadou & Verschaffel, 2004). In the following, studies related to error analysis, respectively from the perspectives of 'student' and 'teacher', will be scrutinized. The former is closely related to the nature of mathematical error and the latter highlights the ways that teachers' engage in error analysis.

Focusing on the student's cognitive process, Davis (1989) proposed two kinds of regularity. The first regularity refers to certain errors made by different students that are extremely common, and the second refers to the wrong answers given by one person in response to a sequence of questions. Brousseau (1981) used historical elements to explain pupils' errors in decimal factions, and found that pupils make the same errors independently of the teaching methods used and thus concluded that some errors could be attributed to pupils' epistemological foundations. Leron and Hazzan (1997) suggested that when analyzing students' productions filled with confusion and loss of meaning, affective

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and social factors are as much a part of students' thinking and behavior as cognitive factors. The plentiful exercises tactic is a popular phenomenon in China. According to many years of research experience into how some Chinese students solve problems, Luo (2004) pointed out that mathematical errors had both logical and strategic bases.

From the perspective of 'teacher', Borasi (1994) reported a case study designed to explore how secondary school students could be enabled to capitalize on the potential of errors to stimulate and support mathematical inquiry. Tirosh, Even and Robinson (1997) found that experienced teachers and novices differ in their awareness of students' tendency to conjoin or 'finish' open expressions. Chamberlin (2005) found that teachers find it challenging to interpret their students' thinking based upon working on non-routine, thought-revealing mathematical tasks. Furthermore, Hill, Blunk and others (2008) identified that responding to students inappropriately – that is, the degree to which teacher either misinterprets or, in the case of student misunderstanding, fails to respond to student utterance – is a key aspect of the phrase (or type) of the mathematical instruction.

The proposed framework

The literature does not provide a coherent picture of mathematics teacher knowledge in error analysis, and the categories of teacher knowledge investigated to date are not closely related to error analysis. Considering its complexity and significance, a more comprehensive framework describing mathematics teacher knowledge as used in this special task is necessary. In Table 1 we formalize many of the suggestions from the literature, described above, as elements of the framework. Using the findings from the literature from the perspectives of the 'student', four keys for the nature of mathematical error were identified – namely, mathematical, logical, strategic and psychological. Using the findings from the perspectives of 'teacher', four key phrases (types) of error analysis were identified: identify, interpret, evaluate, and remediate.

Compared with the category of SMK and PCK, the nature of mathematical error and the phrase (type) of error analysis are the corresponding categories, which consist of teacher knowledge as used in error analysis. The two dimensions are treated separately for analytic purposes, although it is recognized that they are closely linked in a complex way. Given the acknowledged differences between mathematics teaching and error analysis, the proposed framework enables a focus on descriptions of teacher knowledge in the special teaching task.

Method

In the following, the framework is highlighted with empirical examples. The two examples were from China, which might be described as an examination-driven cultural context. The examples were selected from professional journals, and the criteria for selection were representativeness and controversy.

The examples were analyzed in several phases. In the first phase, student's error was analyzed. Then, teacher's analysis was re-analyzed. After that, mathematics teacher knowledge as used in error analysis was matched with the proposed framework. Two researchers analyzed the examples independently. Disagreements concerning the analysis were negotiated until joint agreement was established.

Dimension	Analytical categorization	Description
Nature of mathematical error	Mathematical	Confusion of concept and characteristics, negligence of the condition of formulas and theorem
	Logical	False argument, rearrange concept, improper classification, argue in a circle, equivalent transform
	Strategic	Couldn't distinct from pattern, lack of integral concept, not good at reverse thinking, couldn't transform the problem
	Psychological	Deficiency of mentality, improper mental state
Phrase (Type) of error analysis	Identify	Knowing the existence of mathematical error
	Interpret	Interpreting the underlying rationality of mathematical error
	Evaluate	Evaluating students' levels of performance according to mathematical error
	Remediate	Presenting teaching strategy to eliminate mathematical error

Table 1. Proposed framework for examining mathematics teacher knowledge as used in error analysis

Empirical examples

Case 1

Table 2 describes the error analysis of the first example, including the student's answer and teacher's analysis. The task was as follows:

If $\lim_{n\to\infty}(3a_n + 4b_n) = 8$ and $\lim_{n\to\infty}(6a_n - b_n) = 1$, calculate $\lim_{n\to\infty}(3a_n + b_n)$.

Student's answer	Teacher's analysis
From $\lim_{n\to\infty}(3a_n + 4b_n) = 8$ and $\lim_{n\to\infty}(6a_n - b_n) = 1$, we can get the following equations: $\begin{cases} 3\lim_{n\to\infty} a_n + 4\lim_{n\to\infty} b_n = 8 \\ 6\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = 1. \end{cases}$ Solving this system of equation, we can get $\lim_{n\to\infty} b_n = \frac{15}{9}$, $\lim_{n\to\infty} a_n = \frac{4}{9}$. $\therefore \lim_{n\to\infty}(3a_n + b_n) = 3\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$	It's wrong. According to the rules of limit calculation, only if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$. It is not right for the contrary, since it couldn't conclude that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist, from $\lim_{n\to\infty}(3a_n + 4b_n) = 8$ and $\lim_{n\to\infty}(6a_n - b_n) = 1$

Student's answer	Teacher's analysis
$= \frac{12}{9} + \frac{15}{9} = 3.$	The right argument is: $\lim_{n\to\infty}(3a_n + b_n) = \frac{1}{3}\lim_{n\to\infty}(3a_n + 4b_n) + \frac{1}{3}$, $\lim_{n\to\infty}(6a_n - b_n) = \frac{8}{3} + \frac{1}{3} = 3.$ There are preconditions for some mathematical theorems or rules. If students practice more, it's easy to neglect the precondition, which leads to the misunderstanding about rules and theorems. This is why the teacher should always emphasize so as to help students understand it.

Table 2. The transcript of example 1 (Zhao, 2001)

When analyzing the student's answer, the teachers might have noticed that he could not only use the given conditions but could also think about the rules of limit calculation. It was a pity that he acquiesced in the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ and couldn't use the rules as a whole, from which it can be deduced that the mathematical, logical, and psychological error coexisted. Mathematical error is shown in these aspects: using the rules of limit calculation without validating the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$, couldn't prove the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$, couldn't use the rules of limit calculation in an alternative way; logical error is shown in 'couldn't deduce': using the rules of limit calculation directly while neglecting the necessary precondition of the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$; psychological error is shown in 'potential hypothesis': acquiescing in the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$.

When looking back to the teacher's analysis, it can be found that his arguing about the existence of $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ is not right. In fact, with the undetermined coefficient principle, the result can be gotten not only for $\lim_{n\to\infty}(3a_n + 4b_n)$ but also for $\lim_{n\to\infty}(\alpha a_n + \beta b_n)$. Furthermore, the teacher's lengthy argument has no pertinence.

Combining the analysis of the student's answer and the re-analysis of the teacher's analysis, we can examine the teacher's mathematics knowledge. With respect to the nature of mathematical error, mathematical, logical, and psychological error coexisted, but the teacher couldn't recognize them, which showed the deficiency of teacher knowledge in this aspect. For the phrase (type) of error analysis, the teacher could identify student's error, but interpreted it with wrong mathematical knowledge, which led to meaningless evaluation of the student's performance and unspecific presentation of teaching strategy.

Case 2

Table 3 describes the error analysis of the second example using the framework in example 1. The task of example 2 is as following:

> Find out all of the roots with real number for the equation $x^2 - 2x\sin\frac{\pi x}{2} + 1 = 0$.

Student's answer	Teacher's analysis
According to the task, we can get $\Delta = \left(-2\sin\frac{\pi x}{2}\right)^2 - 4 \geq 0$, namely $\left(\sin\frac{\pi x}{2}\right)^2 - 1 \geq 0$. Since $\left(\sin\frac{\pi x}{2}\right)^2 - 1 \leq 0$, $\therefore \sin\frac{\pi x}{2} = \pm 1$, $\therefore x = 4n \pm 1 (n \in Z)$.	Since it is a transcendental equation, it is ridiculous to think $\sin\frac{\pi x}{2}$ as an invariable and use the discriminant without any bases. The right answer should be: According to the method of completing the square, there holds $\left(x - \sin\frac{\pi x}{2}\right)^2 + \cos^2\frac{\pi x}{2} = 0.$ Therefore we can get that $x - \sin\frac{\pi x}{2} = 0$ and $\cos\frac{\pi x}{2} = 0$. $\therefore x = \pm 1.$

Table 3. The transcript of example 2 (Chen, 1999)

To check the student's answer, we can try $n = 0$ and substitute for the original equation, then, we get $x = \pm 1$ as roots; trying $n = 1$ and substituting again, we find the left of the equation is not equal to the right, which shows $x = 4n \pm 1$ ($n \in Z$) is not the right root. Why, then, is it a necessary condition but not a sufficient condition?

Turning back to the teacher's analysis, there is no definite interpretation for it, although the right answer is given. In the teacher's arguing about the absurdity of student's answer, there are two statements: one is that the discriminant couldn't be used in a transcendental equation, the other is that the variable $\sin\frac{\pi x}{2}$ shouldn't be recognized as a coefficient of equation with degree two. In fact, the two statements are ambiguous if we think about the task from the perspective of functions rather than from the perspective of equations.

Based on the above analysis, let's examine the teacher knowledge as used in error analysis. With respect to the nature of mathematical error in the second example, there is rationality of the student's error. This is because 'the sufficient' has not been validated, while both the mathematical and logical error aids the confusion between 'a necessary condition' and 'a necessary and sufficient condition'. The teacher however attributed it in an extremely simple way. For the phrase (type) of error analysis, the teacher could identify the student's error, but could not explain the underlying rationality. Furthermore, the teacher did not recognize the relationship between the use of the discriminant and the method of completing the square.

Discussion and conclusions

The importance of understanding the knowledge needed for teaching has become an important topic for the teaching and leaning of mathematics. Our framework for examining teacher knowledge as used in error analysis provides specific elaboration for it by giving a holistic and structured picture of the complex phenomena.

In the framework, we focus on two dimensions of the nature of mathematical error and the phrase (type) of error analysis. Empirical examples presented demonstrate the potential of the current framework to shed light on teacher knowledge as used in error analysis. Considering error analysis as an inseparable part of the routine of mathematics teaching, this framework can also be used as a tool in organizing instruction and refining error analysis. Despite the potentials of the framework, it still needs further clarification and development. For instance, there may be some aspects that are conspicuously absent for different settings. Nevertheless, this work provides the preparatory step to investigate teacher knowledge as used in error analysis and will hopefully stimulate new ideas and further development.

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How would you respond to a learner who concludes:

$$3 - (-8) = -5$$

and who argues: "I'm saying the answer is minus five because looking at the sum you can see that the signs are not the same. Three is greater than zero and negative eight is less than zero. Therefore you have to subtract and take the sign of the bigger number. Eight minus three equals five."

(submitted by Craig Pournara)
