The Curricular Shaping of Students' Approaches to Proof

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Proof has a multiplicity of meanings: as Tall [1989] observed, proof can imply "beyond reasonable doubt" to a jury; "occurring with a certain probability" to a statistician, and the result of empirical investigation to a scientist. Yet within the mathematical community, proof has a distinctive role. Proof lies at the heart of mathematics. It has traditionally separated mathematics from the empirical sciences as an indubitable method of testing knowledge which contrasts with natural induction from empirical pursuits. Deductive mathematical proof offers human beings the purest form of how to distinguish right from wrong [Wu, in press], a characteristic, some have argued, which has been responsible for the central role of mathematics in Western thought [Aleksandrov, 1963].

Amongst all the articles published on proof, many have discussed the nature of mathematical proof and the different frameworks adopted for thinking about its role in mathematical activity. Recently several mathematicians have been reassessing this role, and begun to recognise alternative methods for ascertaining the validity of mathematical statements including, for example, the need for social agreement and negotiation [see Hanna, 1995]. This reassessment has been influenced by developments in computer technology and the increasing number of computer proofs [Horgan, 1993] together with a growing emphasis on the role of proving in conveying and illuminating mathematical ideas as well as verifying them [see Hersh, 1993, Thurston, 1995].

At the same time, research in mathematics education suggests that proof is difficult for many students. This may be partly because of its ambiguous meaning but also because proof requires the coordination of a range of competences — identifying assumptions, organising logical arguments — each of which, individually, is by no means trivial. So, given the centrality of proof to the discipline of mathematics and the practice of mathematicians, considerable attention has been paid as to how best to enculturate students into a proving culture which respects this pivotal position while at the same time takes account of student views and ideas[1].

Many studies have classified students' approaches to proving along various dimensions: from pragmatic involving recourse to actions, to conceptual arguing from properties and relationships, [van Dormolen, 1977; Balacheff, 1988]; from weak to strong deduction [for example, Bell, 1976; Coe & Ruthven, 1994]; according to different modes — enactive, visual and manipulative [Tall, 1995], or proof schemes [Harel & Sowder, in press]. Despite differences in emphasis, this corpus of research evidence points to the fact that if the meaning of proof is taken only to be some kind of logical verification, proving in school mathematics is likely to be fraught with conceptual difficulties. Many students have a limited awareness of what proof is about. On the one hand, they show a preference for empirical argument over any sort of deductive reasoning and seem to fail to appreciate the crucial distinction between them: for example, many students judge that after giving some examples which verify a conjecture they have proved it. Yet, on the other hand, students tend to assume that deductive proof provides no more than evidence, with the scope of the proof's validity being merely the diagrams or examples in the text. Finally, other findings point to the difficulty many students have in identifying the premises of a proof and following through a logical argument from these premises to a conclusion [for evidence on all these points, see for example, Williams, 1979; Fischbein & Kedem, 1982; Balacheff, 1988; Martin & Harel, 1989; Porteous, 1990; Chazan, 1993; Finlow-Bates, 1994].

A common interpretation of these findings has been to argue that students' understandings of proof are organised along a hierarchy: with empirical "proof" or procedural validation by action at the bottom, and rigorous deductive argument or relational validation based on premises and properties at the pinnacle. But are there other, equally plausible, interpretations? In order to open an alternative window on to the situation, I will sketch out a fictional study in mathematics education that focuses on its potential limitations.

The study sets out to investigate students' understandings of proof and the proving process in mathematics. The sample of students is drawn from a school local to the researcher or from a class of students in the researcher's university or college. Usually, the mathematical background and experience of the students are briefly described but rarely is this description used as an explanatory variable in the interpretation of the results or in any discussion of how "representative" the students might be. The empirical core of the study comprises the identification and analysis of students' written responses to a range of questions concerning proof. The meaning of what is required as a proof is not made explicit; neither is it clear what students have been taught, what has been emphasised and what forms of presentation have been deemed to be acceptable. The influences of the content and sequencing of the curriculum are ignored in an analysis which takes the individual student and their constructions of proof as the object of attention — an analysis that leads almost inevitably to some kind of hierarchical classification.

This uniformity in the research methodologies employed in the international mathematics education community stands in stark contrast to the huge variation in *when* proof is introduced and *how* it is treated in different countries — as evident from even a cursory glance at textbooks and examination questions. In some curricula, the nature of mathematical proof is discussed explicitly in terms of premises, definitions and logical deductions and the acceptable forms of presentation of proofs are made apparent. In others, definitions and criteria for proving are either implicit or negotiated during the activity. Informal discussions with teachers in one country [the U.K.] reveal a multitude of opinions about how proof should and would be introduced and judged. Some teachers declare that they would be comfortable with an informal explanation while others would judge this to be hopelessly inadequate and would require a logical argument with each step explicitly justified and formally presented — rather like what is known as a two-column proof.

These considerations lead me to question the existence of a universal hierarchy of "proving competencies". I have argued elsewhere [see Noss & Hoyles, 1996] that hierarchies of this sort (e.g. concrete/abstract or formal/informal) are largely artifacts of methodology — if we restrict our terms of reference simply to the interaction of epistemology and psychology, and ignore the social dimension, then it is inevitable that mathematical learning will be perceived as the acquisition of context-independent knowledge within a hierarchical framework. Thus starting from such a position of epistemology/psychology locks research and its findings into a tautological loop.

There seem to be two ways out of this dilemma. One is to search for patterns of reasons for differences in student response that stretch beyond the purely cognitive — encompassing considerations of feelings, teaching, school and home. Another is to ensure that the goals for including proof in the curriculum and how these are operationalised are clarified and taken into account. Clearly proof has the purpose of verification — confirming the truth of an assertion by checking the correctness of the logic behind a mathematical argument. But at the same time, if proof simply follows conviction of truth rather than contributing to its construction and is only experienced as a demonstration of something already known to be true, it is likely to remain meaningless and purposeless in the eyes of students [see, for example, de Villiers, 1990; Tall, 1992; Hanna & Jahnke, 1993]. Hanna has argued for an alternative approach based upon what she calls explanatory proofs — proofs that are acceptable from a mathematical point of view but whose focus is on understanding rather than on syntax requirements and formal deductive methods [Hanna, 1990, p. 12]. Maybe school proofs, where the content is "given", should aim to provide insight as to why a statement is true and throw light upon the mathematical structures under study rather than seek only to verify correctness. One way to operationalise this approach that also encourages student engagement and ownership of the proving activity has been to add a social dimension to explanatory proving; that is, to insist that students explain their arguments to a peer or a teacher as well as to convince themselves of their truth. It is this sense that has been taken up in U.K. and it is to this innovation that I now turn.

Proof in the U.K. National Curriculum

In the U.K., the main response to evidence of children's poor grasp of formal proof in the 60's and 70's was the development of a process-oriented approach to proof. Following Polya [1962] many argued [for example, Bell, 1976; Mason, Burton, & Stacey, 1982; Cockcroft, 1982], that students should have opportunities to test and refine their own conjectures, thus gaining personal conviction of their truth alongside the experience of presenting generalisations and evidence of their validity.

Clearly, there are potentially considerable advantages in this approach in terms of motivation and the active involvement of students in problem solving and proving. Indeed many prominent researchers at the present time [see, for example, de Villiers, 1990] are arguing for just such a shift in emphasis, suggesting that students develop an inner compulsion to understand *why* a conjecture is true if they have first been engaged in experimental activity where they have "*seen*" it to be true. But before other countries follow this route it would be useful to learn some lessons from what has happened in the U.K. What the mathematics education reform documents failed to predict was how teachers, schools, and the curriculum would act upon and re-shape this "*process*" innovation: in fact, the deliverers of the innovation ignored just the same potential influences on student response as alluded to earlier in my description of a fictitious mathematics education research study. How will the goals and purposes of the different functions of proof be conceived and how will these functions be organised when they are systematised and arranged into a curriculum? What will be the implications of this choice of organisation? How will the changes be appropriated and moulded by teachers and students?

Answers to these questions can be sought by an analysis of the present situation in the U.K. following the imposition of the National Curriculum. The National Curriculum in Mathematics for children aged 5-16 years is organised into four attainment targets [Department for Education and Employment Education, 1995][2].

AT1 Using and applying mathematics
AT2 Number and algebra
AT3 Shape, space, and measures
AT4 Handling data

Rather oddly, proving and proof are to be found in the target named "Using and applying mathematics" (AT1). This may well have implications for their meanings but has had an immediate consequence in that almost all of the functions of proof are separated from other mathematical content[3]. Many textbooks written for the National Curriculum are now divided into sections according to attainment targets. Rather than construction, justification, and proof[4] working together as different windows on to mathematical relationships, students are expected to *use* results of theorems in, for example, "Shape, space, and measures" — Pythagoras' theorem will be stated and students asked to apply it to calculate a length of a side of a triangle — while proofs may be encountered elsewhere in AT1. What will be the status of these proofs and what is likely to be the reaction of students when they have already used the results as facts?

A second consequence of this fragmentation of the curriculum has been that work under the banner of AT1 has become strangely transformed into an "investigations curriculum" dominated by data-driven activity during which students are expected to spot patterns, talk about, and justify them. Rarely if ever are students required to think about the structures their justifications might illuminate — it is the process that counts.

The third consequence of the imposition of the National Curriculum is the division of all of its attainment targets into 8 levels of supposed increasing difficulty. In AT1, the sequence in the proving process is given below.

The Mathematics National Curriculum of England and Wales Attainment Target 1: Using and applying mathematics[5]

Level 3	Students show that they understand a general statement by finding particular examples that match it.
Level 4	They search for a pattern by trying out ideas of their own.
Level 5	They make general statements of their own, based on evidence they have produced, and give an explanation of their reasoning.
Level 6	Students are beginning to give a mathematical justification for their generalisations; they test them by checking particular cases.
Level 7	Students justify their generalisations or solutions showing some insight into the mathematical structure of the situation being investigated. They appreciate the difference between mathematical explanation and experimental evidence.
Level 8	They examine generalisations or solutions reached in an activity, commenting constructively on the reasoning and logic employed, and make further progress in the activity as a result.
Exceptional performance	Students use mathematical language and symbols effectively in presenting a convincing reasoned argument. Their reports include mathematical justifications, explaining their solutions to problems involving a number of features or variables.

First, it is worth noting that the division into levels and the stipulation of eight as the number of levels applies to all subjects in the National Curriculum. This decision was not based on any analysis of stages of progression in subject areas but rather arose from the need to impose uniformity over the whole curriculum — in order that levels could serve as a mechanism to measure and compare the achievement of students, teachers, and schools. What was not anticipated, however, were the far-reaching implications of this levelled classification on the subject disciplines themselves and on how they would be experienced by students. For example, the levels of AT1 mean that the majority of students will engage in data generation, pattern recognition, and inductive methods while only a minority, at levels 7 or 8, are expected to prove their conjectures in any formal sense. The imposition of this strictly prescribed hierarchical organisation has therefore meant that most students have little chance to gain any appreciation of the importance of logical argument in whatever form and few opportunities to engage in formal discourse requiring any linguistic precision.[6] In a nutshell, it is now official that proof is very hard and only for the most able.

Clearly, the shift in emphasis to a process-oriented perspective is an understandable attempt to move away from the meaningless routines that characterised what was largely geometrical proof in an earlier period. While some students managed to undertake the routines of Euclid correctly, far fewer understood more about geometry as a result. But in trying to remedy one problem, others have come to the surface: the meaning of "to prove" has been replaced by social argumentation (which could mean simply giving some examples); justifying is largely confined to an archaic "investigations curriculum" separated from the body of mathematics content; and proof is labelled as inaccessible to the majority.

But what are the consequences for student attitudes to and understanding of proof following this massive change in the treatment of proof? What are the consequences for student learning of a curriculum that now contrasts sharply with that adopted elsewhere — in the U.S., France, Germany, and countries on the Pacific Rim to name but a few. Some recent research by Coe & Ruthven [1994] into the proof practices of students who have followed this curriculum suggests, rather unexpectedly, that nothing appears to have changed and students remain locked in a world of empirical validation. Even more surprisingly, given the emphasis in the curriculum reforms, the researchers also report that students show little attempt to explain why rules or patterns occur, or to locate them within a wider mathematical system.

How far are these findings generalisable? As yet this is not known, but far more influential than any research study is the pervasive belief amongst influential groups in the U.K., that students' understanding of the notion of proving and proof in mathematics has deteriorated. There has been a huge outcry, mainly amongst mathematicians, engineers and scientists in our universities, complaining about the mathematical incompetence of entrants to their institutions. The argument is that the National Curriculum only pays lip-service to proof with the result that even the more able students who go on to study mathematics after 16 years have failed to grasp the essence of the subject. The debate culminated in 1995 in a publication spearheaded by the London Mathematical Society, a powerful group of mathematicians in the U.K. Points 4 and 5 of the summary of their report, known as the LMS Report, are reproduced below:

> 4. Recent changes in school mathematics may well have had advantages for some students, but they have not laid the necessary foundations to maintain the quantity and quality of mathematically competent school leavers and have greatly disadvantaged those who need to continue their mathematical training beyond school level.
>
> 5. The serious problems perceived by those in higher education are:
>
> (i) serious lack of essential technical facility — the ability to undertake numerical and algebraic calculation with fluency and accuracy;
>
> (ii) a marked decline in analytical powers when faced with simple problems requiring more than one step;
>
> (iii) a changed perception of what mathematics is — in particular of the essential place within it of precision and proof.
>
> (London Mathematical Society, 1995, p. 2)

The message of the LMS report is clear. Students now going on to tertiary education in mathematics and related subjects are deficient in ways not observed before the reforms: students have little sense of mathematics; they think it is about measuring, estimating, induction from individual cases, rather than rational scientific process. Clearly we might argue that the evidence of "decline" is not sound or that its putative causes are hard to pinpoint given the complexity of the educational process — not least the massive expansion in the university population in the U.K. But this argument is difficult to sustain in the absence of systematic evidence. In fact, the conclusions concerning proof are eminently plausible: given that there are so few definitions in the curriculum, it would hardly be surprising if students are unable to distinguish premises and then reason from these to any conclusion. But rather than pointing out what students "lack", it would seem to be more fruitful and constructive to find out what students *can* now do and understand following the reforms of the curriculum and the different functions of proof they have experienced. What is needed is a comprehensive study of students' views of proving and proof and the major influences on them. Having followed the new curriculum, what do students judge to be the nature of mathematical proof? What do they see as its purposes? Do they see proving as verifying cases or as convincing and explaining? Do they forge connections between the different functions of proof or do these functions remain fragmented and isolated? What are their teachers' views? Although we have a national curriculum, are there variations in how it is delivered and experienced and, if so, why and what are the implications for student learning?

These questions take me to a discussion of a research project, Justifying and Proving in School Mathematics, which I have been undertaking with Lulu Healy at the Institute of Education in London since 1995. In this research, we aim to answer some of these questions by surveying student views of proof and trying to explain these against a landscape of variables and influences that extends beyond a simple description of students' mathematical competencies. In the next section, I will describe some aspects of the project in more detail.

A nationwide research project

The first phase of the research has been to conduct a nationwide survey of the conceptions of justification and proof in geometry and algebra amongst 15-year-old students[7]. Our aim is to open a multiplicity of windows on to students' conceptions of proof in order to find out what they think it involves, what they choose as proofs and how they read and construct proofs. We are only mildly interested in discovering what students cannot do. Rather, we seek to describe profiles of student responses in order to identify strengths as well as weaknesses. We also want to tease out how student conceptions might have been shaped — by the curriculum, teachers, and schools. Given that it is only high-attaining students who have any acquaintance with proving in our curriculum, the sample surveyed is drawn from this group. The findings from the survey will form the basis for thinking about how we might introduce students to proof in the future — to capitalise on the outcomes of the reforms in the curriculum that are positive while seeking to reduce those that are negative. In fact, the survey is only the first phase of our project. In the second phase, following our analysis of student and teacher responses, we will design and evaluate two computer-based microworlds for introducing students to a connected approach to proving and proof.

We spent many months reviewing existing literature and discussing with teachers, advisers, and inspectors in order to come up with two survey instruments: a student and a school questionnaire. For the former, we wanted the mathematical content to be sufficiently straightforward for the proofs to be accessible, familiar and in tune with the U.K. National Curriculum, yet sufficiently challenging so there would be differentiation amongst student responses. In our questionnaire, proofs and refutations were to be presented in a variety of forms — exhaustive, visual, narrative and symbolic — and set in two domains of mathematics — arithmetic/algebra and geometry[8].

The questionnaire was pre-piloted by interviews with 68 students in 4 different schools aiming to find out how far the questions were at an appropriate level and engaging for students. Following the pre-pilot, items were removed that were too easy or modified if too hard. We also wanted to be able to make comparisons between responses in algebra and geometry, so revised the format to make its presentation in each domain completely consistent.

Simultaneously with the development of the student questionnaire, we designed a school questionnaire to obtain information about the schools — the type of school, its organisation generally and the hours spent on mathematics, the textbooks adopted and examinations entered, and specifically the school's approach to justification and proof. We also sought teacher data to provide information on their background, qualifications, their reactions to the place of proof in the National Curriculum and the approaches they adopted to proof and the proving process in the classroom.

We piloted both questionnaires with 182 students in 8 schools after which we were able to iron out any remaining ambiguities and to specify the time required to complete the survey (70 minutes) and the instructions for its administration. The questionnaires were completed between May and July 1996 by 2459 students in 94 classes from 90 schools in clusters throughout England and Wales. We had originally planned to use 75 schools but more requested to take part in our survey — a reflection we believe of the interest teachers have in this topic, their recognition of its importance, and their concern about the changes that have taken place. The questionnaires were administered by members of the project team or mathematics educators in different parts of the country who had volunteered to help us. This process ensured consistency in administration procedures and a 100% return of questionnaires. While the students answered their questionnaires, their teacher filled in parts of the student questionnaire (see later) as well as completed the school questionnaire.

Schemes for coding the questionnaires were devised and all the coding undertaken and checked during July and August 1996. We are now producing descriptive statistics based on frequency tables and simple correlations as well as modelling student responses against all our teacher and school variables using a multilevel modelling technique [see Goldstein, 1987]. The purpose of this paper is not to report the findings of this statistical analysis but rather to provide a flavour of how students in the U.K. see proof through the presentation of a selected sample of questions together with some student responses.

Windows on students' approaches to proof

The first question of the student questionnaire asks students to write down everything they know about proof in mathematics. A rather typical answer is given below:

> "All that I know about proof is that when you get an answer in an investigation you may need some evidence to back it up and that is when it is proof. You have to prove that an equation always works.

> Another student wrote:

> "all I know is the proof in mathematics is that, if say you are doing an investigation, and you find a rule, you must prove that the rule works. So proof is having evidence to back up and justify something".

These responses clearly echo our curriculum structure. It is only a special type of mathematical activity — the "investigation" — that requires proof, where this means the provision of some sort of evidence.

Following this open-ended question, the questionnaire is divided into two sections, the first concerned with algebra and the second with geometry. The first question of each section is in a multiple-choice format, as illustrated in Figures 1 and 2.

The purpose of having a multiple-choice question at the beginning of each section is to introduce students who may not be acquainted with the meaning of "to prove" to a range of possible meanings — remember that our students are not introduced to definitions, nor generally required to produce logical deductions in mathematics[9]. Almost all the student responses used as options for these questions were derived from our pre-pilot and pilot studies or from school textbooks, so we could be fairly sure that some at least would be familiar. These questions (and others with a similar multiple-choice form) were designed to help us ascertain what students *recognised* as a proof. These responses could then be compared and contrasted with what students actually *produced* as proofs later in the questionnaire. Clearly these two processes are related but not identical — constructed proofs require specific knowledge to be accessible. As well as presenting different types of proof (here empirical or analytic), the choices in the questions ranged over different forms to enable us to analyse how far students might be influenced by the form as well as the content of a proof. The "proof types" shown in Figures 1 and 2 can be categorised as: empirical, enactive, narrative, visual[10] or formal, with two examples of formal proof, one correct and one incorrect.

A1. Arthur, Bonnie, Ceri, Duncan and Eric were trying to prove whether the following statement is true or false:

When you add any 2 even numbers, your answer is always even.

Arthur's answer
a is any whole number
b is any whole number
$2a$ and $2b$ are any two even numbers
$2a + 2b = 2(a + b)$

So Arthur says it's true.

Bonnie's answer
$2 + 2 = 4 \quad 4 + 2 = 6$
$2 + 4 = 6 \quad 4 + 4 = 8$
$2 + 6 = 8 \quad 4 + 6 = 10$

So Bonnie says it's true.

Ceri's answer
Even numbers are numbers that can be divided by 2. When you add numbers with a common factor, 2 in this case, the answer will have the same common factor.

So Ceri says it's true.

Duncan's answer
Even numbers end in 0 2 4 6 or 8. When you add any two of these the answer will still end in 0 2 4 6 or 8.

So Duncan says it's true.

Eric's answer
Let x = any answer whole number, y = any whole number
$x + y = z$
$z - x = y$
$z - y = x$
$z + z - (x + y) = x + y + 2z$

So Eric says it's true.

From the above answers , choose **one** which would be closest to what you would do if you were asked to answer this question.

From the above answers, choose the **one** to which your teacher would give the best mark.

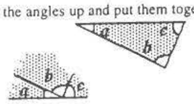
Figure 1
The first algebra question

G1. Amanda, Barry, Cynthia, Dylan, and Ewan were trying to prove whether the following statement is true or false:

When you add the interior angles of any triangle, your answer is always 180°.

Amanda's answer

I tore the angles up and put them together.

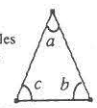


It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

So Amanda says it's true.

Barry's answer

I drew an isosceles triangle, with c equal to 65°.



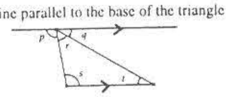
Statements	Reasons
$a = 180° - 2c$	Base angles in isosceles triangle equal
$a = 50°$	$180° - 130°$
$b = 65°$	$180° - (a + c)$
$c = b$	Base angles in isosceles triangle equal

$\therefore a + b + c = 180°$

So Barry says it's true.

Cynthia's answer

I drew a line parallel to the base of the triangle



Statements	Reasons
$p = s$	Alternate angles between two parallel lines are equal
$q = t$	Alternate angles between two parallel lines are equal
$p + q + r = 180°$	Angles on a straight line

$\therefore\ s + t + r = 180°$

So Cynthia says it's true.

Dylan's answer

I measured the angles of all sorts of triangles accurately and made a table.

a	*b*	*c*	*total*
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

They all added up to 180°.

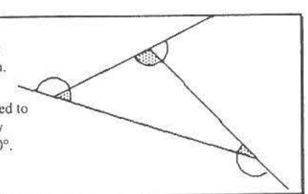
So Dylan says it's true.

Ewan's Answer

If you walk all the way around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360°.

You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line. This makes a total of 540°. 540° - 360° = 180°.

So Ewan says it's true.



From the above answers, choose **one** which would be closest to what you would do if you were asked to answer this question.

From the above answers, choose the **one** to which your teacher would give the best mark.

Figure 2
The first geometry question

We are seeking to investigate the influence of the teacher in various ways — through their responses to the school questionnaire but also through the eyes of the student in the student questionnaire. In the last part of each multiple-choice question, the student is required to choose the proof to which they think their teacher would give the best mark. Responses here will help us to see how students interpret what is rewarded by their teacher. The teachers are also asked to complete these same questions — to write down what *they* would choose as a proof, as well as what they think their students will choose as the one given the best mark. The analysis to date reveals a picture that is by no means simple, although in both domains, formal presentation (correct or incorrect) is highly favoured for the best mark, while narrative in algebra is the favourite for individual choice. What is interesting too is the sizeable minority of students whose personal choice bears no resemblance to the one they believe will receive the best mark (only 21% overall made the same choice for both[11]) and the small group who choose, for the latter, a formal proof that is incorrect!

Following each multiple-choice question in both algebra and geometry are questions seeking to find out how students evaluate each of the choices previously presented. Do they think it is correct? Do they believe that the proof holds for all cases or simply for a specific case or cases? Do they judge it to be explanatory or convincing? An example of the format used as it applies to Bonnie's "proof" is shown in Figure 3 below.

Bonnie's answer

2 + 2 = 4	4 + 2 = 6
2 + 4 = 6	4 + 4 = 8
2 + 6 = 8	4 + 6 = 10

So Bonnie says it's true.

Bonnie's answer:			
Has a **mistake** in it	1	2	3
Shows that the statement is **always true**	1	2	3
Only shows that the statement is true for **some** even numbers	1	2	3
Shows you **why** the statement is true	1	2	3
Is an easy way to **explain** to someone in your class who is unsure	1	2	3

Figure 3
Student evaluations of Bonnie's answer

By analysing all the responses to this question, we will find out if students are convinced by a list of empirical examples. Do they judge that these examples help them to explain the result? Do they recognise that "the proof" only shows the conjecture is true for the given examples even though they might have chosen it as their response or as the one to which the teacher would give the best mark[12]? Does it make a difference if the proof evaluated was the one chosen by the student? Preliminary analysis in algebra suggests that students tend to choose for themselves "proofs" that they evaluate as general *and* explanatory, while the proofs they think will be assigned the best mark are evaluated as general but not necessarily explanatory.

The second multiple-choice question in each section of the survey concerns an incorrect conjecture which is proved by some choices and refuted by others. Again, preliminary results show the same strong preference for formal language in the choices for the best mark, even in these circumstances where a simple empirical counter-example would suffice. It seems that despite our investigations culture, the students have picked up what the game *should* be!

A rather different picture emerges from students' own proofs where the influence of the investigations curriculum is again very evident, as illustrated in Fig. 4 below. It must be emphasised that great care was taken to choose proofs that would be familiar to or at least accessible to most students who had followed our curriculum.

A4. Prove whether the following statement is true or false. Write down your answer in the way that would get you the best mark you can.

When you add any 2 odd numbers, your answer is always even.

My answer

Statement -
When you add any 2 odd numbers your answer is alway even.
Hypothesis - I believe that the above statement is absolutely correct.
My aim - I will prove the statment is correct by conducting some further work and calculations :-

$1 + 1 = 2$
$3 + 3 = 6$
$5 + 5 = 10$

There is clearly a pattern between the 3 addictions I have just carried out. Now lets see if I can use algebra instead to find the true answer to the statement

$n = x + y$ the n th number $= x$ - the first number plus y the second.

Conclusion -
I can see that the statement is correct from the working I have carried out.

Figure 4
An "investigations" response to adding 2 odd numbers

Note the careful presentation of data in tabular form — the heart of an investigation —, the spotting of a pattern and the adding-on of letters to add status to "the proof". Similar phenomena are illustrated by another student's response to the second, rather harder, algebra proof construction:

My answer

p	q	$(p+q)\times(p-q)$	$\div 4$
~~1~~	~~3~~		
~~3~~	~~5~~		
5	3	8 × 2	16 ÷ 4 = 4
7	5	12 × 2	24 ÷ 4 = 6
9	7	16 × 2	32 ÷ 4 = 8
11	9	20 × 2	40 ÷ 4 = 10
13	11	24 × 2	48 ÷ 4 = 12

p = odd number
q = odd number
r = multiple of 4

$$(p+q)\times(p-q) = \frac{r}{4}$$

Figure 5
An "Investigations" response to a harder algebra question

Even in geometry, rarely the site for a school "investigation", the discourse of investigations is evident in the form of many of the student responses and the explanations given, as illustrated in Figure 6.

G4. Prove whether the following statement is true or false. Write your answer in a way that would get you the best mark you can.

If you add the interior angles of any quadrilateral, your answer is always 360°.

My answer

angle 1.	angle 2.	angle 3.	angle 4	
90°	90°	90°	90°	360
60°	90°	90°	120°	360
80°	80°	90°	110°	360
70°	70°	100°	120°	360

∴ this shows that when you add the interior angles of a quaderlateral you get 360°.

Check

100° + 100° + 100° + 60° = 360°

∴ the above statement is true

Figure 6
An "investigations" response to a geometry question

Here data are invented to fit the given pattern. There is even a set of data given special status to fulfil the investigation's requirement of providing a check!

What is evident from these responses is that the students connect the requirement to prove with the investigations part of the curriculum where they have learned a format and a language of presentation. They have appropriated some structures to help them make sense of the situation and to assist in developing a language for proof. But, the limitations of this scaffolding are very apparent. Students appear to be imposing a "type" of proof on every question regardless of whether it is appropriate; for example, a proof must involve data. Additionally, all too easily students seem to have shifted their notion of proving from one ritual to another — from a *formal* ritual to a *social* ritual — something added on to the end of an investigation. This new ritual is likely to be equally meaningless, empty of mathematical illumination, and missing any mathematical point. My student examples show how little connectivity there appears to be with the structure of odd numbers in the first two cases or to the geometric nature of a quadrilateral in the last one.

I have selected student responses to illustrate my point and it is important to guard against over-generalisation. Yet, despite this cautionary note, it has been salutary to trace the extent of the curriculum influences, either intended or unintended. Student responses cannot simply be "blamed" on the student. Their behaviours cannot be ascribed to properties of age, ability, or even individual interactions with mathematics. The meanings they have appropriated for proof have been shaped and modified by the way the curriculum has been organised. For example in our survey, responses in geometry are very different — and much worse from a mathematical perspective — from those in algebra. This finding is hardly surprising if one is aware of the almost complete disappearance of geometrical reasoning in the curriculum. Nonetheless, it casts doubt on how far proof can be considered as a unitary mathematical "object" with its own hierachy separated from any domain of application. Many mathematics educators have shown how we must take seriously the influence of the teacher — and our teacher data sheds light on this. But, surely, we have now to look for interpretations which also take account of curriculum organisation; in the case of the U.K., its separate targets and the straightjacket of its levelled statements.

Our survey certainly points to the curricular shaping of students' approaches to proof. But there is also considerable variation among student responses. We are in the midst of generating complex and sophisticated models using the variables from both student and school questionnaires, to tease out how these variables interrelate and to describe the range of contributory "causes" of any differences between student response profiles. Is it curriculum, textbook, examination board, school, or teacher that shapes response or is it a combination of these? Are students' responses consistent across or even within domains? The starting point of our research was the belief that proving in mathematics need not be restricted to "the exceptional" and that the organisation of the U.K. National Curriculum seriously underestimates the potential of our students. Responses to the survey are proving to be rather promising in this respect. Alongside the ritualised responses described above and the all too numerous solutions that simply resort to empirical examples, there are some fascinating and ingenious proofs which provide a pointer to the wealth of resources that might be tapped and built upon in the process of building a proving culture — of starting conversations with students about proof.

Take, for example, the following proof of the first algebra example which combines a visual approach with a narrative to indicate the generality of the argument.

There is a coherence in the explanation apparent in the diagram, the written text as well as the language describing the generalisation, which stems from the mathematics — here from the structure of odd numbers. Even in geometry where responses are in general disappointing, we find several examples of creative proofs, as illustrated below.

A4. Prove whether the following statement is true or false. Write down your answer in a way that would get you the best mark you can.

If you add any 2 odd numbers, your answer is always even.

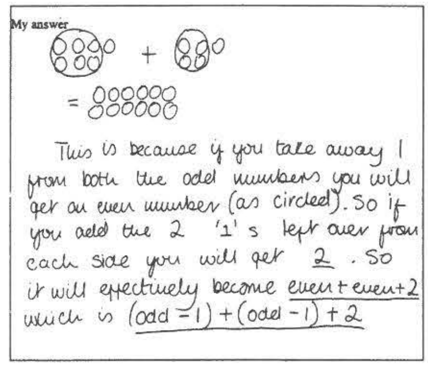


Figure 7
Developing a language for proving

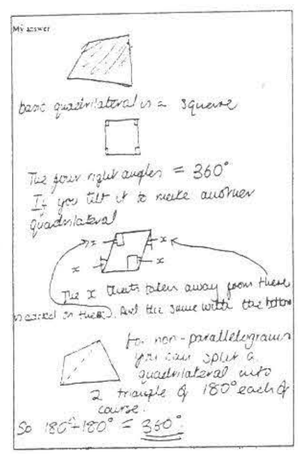


Figure 8
From a special case towards a generalisation

Here the set of quadrilaterals is divided into two subsets, parallelograms and non-parallelograms, and a different proof applied to each. The first "proof" hints at the invariance of the angle sum of parallelograms under carefully contrived parallel deformations, while the second (the obvious one — note the "of course") takes care of all other quadrilaterals by applying a known fact about the angle sum of a triangles. Both of these responses are likely to be influenced by school factors which we intend to investigate through follow-up interviews with teachers in the schools concerned. It may be, for example, that the geometry proof shown above has roots in prior experimentation with dynamic geometry software. Note how both of these proofs point to an approach to proof where the starting point is *not data* but rather a specific and special case where the conjecture is known to be true and from which the road to the general case is suggested — in the former case by language and image and the latter by means of "adding a bit in one place" and "taking the same from another". This calls into question the whole notion that students' development of mathematical justification *has* to proceed from inductive to deductive processes and maybe suggests a different route for proof in school. This will one of the many conjectures generated by the survey that we will investigate to explore in Phase 2 of our project.

Conclusions

We have a long way to go to unpick all the factors that together underpin student conceptions of proof in the new scenario we now face in the U.K. It is almost certain that many of the influences on student responses, interactions in classrooms and institutions were not anticipated. Nonetheless, we can learn from this experience. The main message of this paper is that mathematics educators can no longer afford simply to focus on student and teacher if we are to understand the teaching and learning of proof and if we seek to influence practice. We cannot ignore the wider influences of curriculum organisation and sequencing if we are to avoid falling into the trap illustrated by the U.K.'s ubiquitous "investigation".

The challenge remains to design situations that scaffold a coherent and connected conception of proof while motivating students to prove in *all* its functions. We must resist the temptation to assume that situations that engage students with proof *must* follow a linear sequence from induction to deduction. As Goldenberg (in press) has argued, we must aspire to develop "ways of thinking" not their "products" and use these as guides to curriculum organisation, but not neglect to recognise how these ways of thinking are deeply connected with content domain and that there are serious implications if this link is ruptured. To do this effectively, we must exploit all the resources to hand: our collective knowledge from research, much of it undertaken under the influence of a very different set of curriculum restraints; the findings of our present survey; and the opportunities opened up by the new tools now available — tools that will change the landscape of assumptions underpinning proof as well as the strategies available for proving. If we fail in this endeavour, there is a real danger that the pendulum will simply reverse in the face of opposition to the reforms and we will return to the failed approaches of the past. In the U.K., we now stand at this turning point. Will the curriculum "swing backwards"? Or will we seize the opportunity opened up by all the discussion around proof to find out how to shape student conceptions in new directions, so they come to see proof as generative not merely descriptive, negotiable but also mathematical.

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Notes

1 There have, for example, been special editions on proof in many of the major journals in mathematics education.

2 The National Curriculum has been through several changes, each time with a different number of attainment targets. Nonetheless, the basis of its organisation has remained the same. The structure described here was put in place in 1995 where some attainment targets only appear for children of certain ages.

3 Teasing out all the reasons for this compartmentalisation would make a fascinating story of the demise of geometry intertwined with political intrigue — but, unfortunately, this is beyond the scope of this paper.

4 In the remainder of this paper, I will take justifying to mean an explanation which convinces oneself and is communicated to others. I will leave the term "proving" to convey the more formal sense of logical argument based on premises.

5 Children age 11-14 years should be within the range of Levels 3 to 7. Level 8 is available for very able pupils.

6 A similar trend in North American has been noted by [Hanna, 1995] who has argued that the gradual decline of the position of proof in school mathematics and its relegation to heuristics can be attributed partly to the "process orientation of much of the reforms in mathematics education since 1960s". She also suggests that another contributing factor is the persuasiveness of constructivism — or at least the way it is operationalised in the classroom. Whilst I have some sympathy with Hanna's remarks, this paper suggests that she underestimates the effect of the curriculum.

7 The project is funded by the Economic and Social Research Council, Grant number R00236178.

8 We organised a small invited international conference on proof in order to share frameworks and present our first ideas for the student questionnaire [Healy and Hoyles, 1995].

9 Before the students started to respond to the questionnaire, it was pointed out to them that for this type of question several options could be "correct".

10 The visual option appears on another page of the questionnaire and is not shown here.

11 In fact, there is a significant relationship between these two choices but the correlation is low.

12 Preliminary analysis suggests this latter choice may be subject to interesting gender differences.

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It is sometimes unreasonably required by persons who do not even themselves attend to such a condition, that experimental information should be submitted without any theory to the reader or scholar, who is himself to form his conclusions as he may list. Surely the mere inspection of a subject can profit us but little. Every act of seeing leads to consideration, consideration to reflection, reflection to combination, and thus it may be said that in every attentive look on nature we already theorise. But in order to guard against the possible abuse of this abstract view, in order that the practical deductions we look to should be really useful, we should theorise without forgetting that we are so doing, we should theorise with mental self-possession, and, to use a bold word, with irony.

J. W. von Goethe
