DIAGRAMS AS COMMUNICATION IN MATHEMATICS DISCOURSE: A SOCIAL SEMIOTIC ACCOUNT

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What do the dotted lines in Figure 1 mean to you?

One answer would be that dotted lines are used to prove the theorems mentioned in the figure or to show the process of proving [1]. If you ask mathematicians or mathematics educators why they draw dotted lines while doing mathematics, their answers might include the use of dotted lines for highlighting and for expressing uncertainty, in addition to other conventional uses in mathematics, such as lines of reflection [2]. Yet the question remains valid: why dotted lines and not solid ones? Thinking about this question set me off on the journey I will describe here, looking at visual communication in mathematics discourse. Coming from a social approach to mathematical practice, I wanted to explore if there is any connection between this practice of drawing dotted lines and the practice of presenting mathematics in a way that conceals the human factor.

Temporality

I still remember my mathematics teacher in elementary school using dotted lines in geometry lessons. He would first write a 'template' or a specific structure for solving problems in geometry: the given information, the problem or the goal (what we need to prove/calculate, *etc.*) and then the proof itself, in a style similar to Greek mathematics (Netz, 1999). Most of the time, he would leave a space between the goal and the proof. This space, we later learned, was left for the actions (constructions) required to carry out the proof. Most of these constructions were represented by dotted lines. In other words, these actions occur over time. That is how I began to consider the notion of temporality in diagrams. Netz noticed a similar phenomenon in the practice of Greek proofs, in relation to the lettering of diagrams through what he called "the principle of baptism" and the process of drawing the diagram (pp. 83–85). He argued that points are 'baptised' according to the order in which they occur in the text of the proof, not their position in the diagram. A temporal ordering outweighs a spatial ordering.

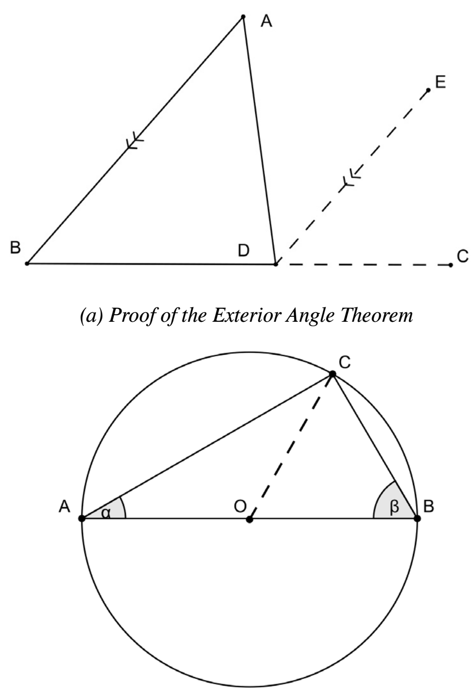


Figure 1. Why do mathematicians draw dotted lines?

Temporality means that there is a representation of time, within the diagram, that can be followed or traced. In other words, there is a timeline one can follow in order to read, make sense of or unfold the sequence of time. For instance, in Figure 1a, the side BD was extended to C, and the line DE was drawn parallel to BA, so that one can prove that the angle CDA equals the sum of the angle A and B. Similarly, the line CO, in Figure 1b, was drawn to show how one may prove Thales theorem.

Communication in mathematics discourse

This article is a continuation of the discussion about visual communication in mathematics from a specific point of view. For decades, scholars have engaged in a lively study of communication in mathematics discourse. As expected, they focused on language and its use in mathematics, starting from viewing mathematics itself as a language that students are required to 'acquire', and moving to analysing the use of language in mathematics (Austin & Howson, 1979). Halliday (1974) investigated the relationship between natural language and mathematical symbolism and specialist vocabulary, introducing the notion of a mathematics register. He extended how we view the use of language in mathematics beyond words to incorporate "meanings, including the styles of meaning and modes of argument" (p. 65).

The notion of a mathematics register found its way into mathematics education through two seminal works: *Speaking Mathematically* (Pimm, 1987) and *Writing Mathematically* (Morgan, 1996). Pimm explores the mathematics register in teaching and learning mathematics while Morgan focuses on the role of language in written mathematical texts. The basic notion here is not just to look at the use of language in learning and teaching mathematics, but rather to investigate closely the grammatical forms used in written mathematical texts and to analyse these forms and the way in which mathematics is represented as a social practice. That practice views mathematics as a discourse which employs special language and methods of endorsement (Sfard, 2008).

Language, images and mathematics: mathematics as multi-modal discourse

This article goes beyond arguing that the use of diagrams is inevitable and also beneficial (as Giaquinto, 2007, argues) to suggest a tool to analyse diagrams themselves and their role in communicating mathematics, including their role in proofs. I adopt a social semiotics lens to look at diagrams. Halliday (1985) looks at language as a social semiotic resource for making meaning and analyses its functional role in communication. He argues that every text fulfils three metafunctions: (a) Ideational: our ideas about the world are represented through our grammatical choices which can be analysed by the transitivity system; (b) Interpersonal: representing the social relations between the author and the reader. It can be analysed through the use of pronouns (first or second pronouns or their absence as in the passive voice); and (c) Textual: the focus is the whole text as a coherent text or message focusing on themes.

Kress and Van Leeuwen (2006) extended Halliday's work to analyse the use of images and suggested a grammar for reading images. For example, in the interpersonal metafunction, they considered that the presence of *gaze* in an image may suggest a demand from the viewer—this is in comparison to Halliday's notion of *offer* and demand in language, where in our communication with others either we offer information (a statement for example) or we demand something (ask a question). Later they developed the notion of multi-modality from a social semiotic point of view, in which different modes of communication are deployed to express meaning.

Diagrams as communication

In addressing the ideational metafunction of Halliday's Systemic Functional Grammar (SFG), Kress and Van Leeuwen argue that images are of two kinds, narrative and conceptual. The former has a direction in it that is realised by the presence of an arrow (vector), whether it is explicitly presented or can be traced implicitly. The latter, the conceptual, lacks such direction.

For Kress and Van Leeuwen, an image containing arrows, explicit or implicit, is a narrative image that tells a story. For example, they analysed "The British used guns" picture to demonstrate their notion of directionality. The left side of the picture shows two British soldiers walking stealthily in the dark, holding guns and approaching a group of aboriginal people who are busy around a fire on the right side of the picture. Kress and Van Leeuwen produced a schematic figure (2006, p. 49) with an arrow that emanates from the side of the British soldiers towards the aboriginal people, a direction in the image that is similar to the English sentence representing Actor and Goal.

I took that idea and applied it to geometrical diagrams. In order to do so, I adopted the notion of temporality instead of directionality. This enabled me to categorise diagrams as 'narrative' or 'conceptual'.

Narrative and conceptual diagrams

Using an iterative process, and following Halliday's SFG, I developed an analytic tool for reading geometric diagrams (shown in Table 1). I used 'bootstrapping', as Pratt (1998) discussed, to initiate the process of investigation, allowing a preliminary tool to emerge through exploration, and then reinvestigating to refine the tool. I validated the tool by testing its applicability to new diagrams. The result is a method for classifying geometrical diagrams as either narrative or

Table 1: An analytic tool for reading geometric diagrams

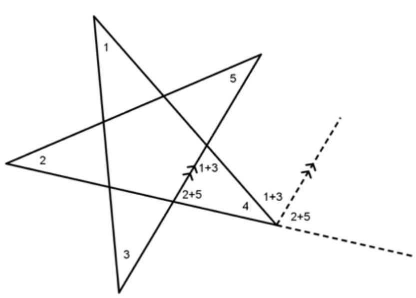
Ideational meaning Nature of mathematical activities	**Interpersonal meaning** Roles and relationships between author and viewer	**Textual meaning** Unity & Coherence
This meaning is realised by determining the structure of the diagram; whether it is narrative or conceptual: * *Narrative structures:* • Arrowed • Dotted • Shaded • Sequence of diagrams • Construction * *Conceptual structures:* • Classificational • Identifying (indexical & symbolic) • Spatial (positional & size)	The realisations of this meaning are: * *Contact:* • Demand diagrams • Offer diagrams (labels & colour) * *(Social) Distance:* • Neatness (neat vs. rough diagrams) • Labels (general vs. specific) • Colour and arrows and words * *Modality* • Diagrammatic modality markers (abstractness, natural or contextual, label, additional information, neatness).	Considering the whole text (the visual and the written). The realisations of this meaning are * *Information value* • Left and right (given and new) • Top and bottom (ideal and real) • Centre and margin * *Salience* • Colour, size, perspective, position * *Framing* • Separation (frame lines, white space, colour) vs. connection (visual links, lack of framing)

conceptual and then defining at least five different kinds of narrative diagrams and three different kinds of conceptual diagrams. I summarise this method in Table 1, but will focus the rest of this article on the first metafunction, the ideational, which will be presented in detail with illustrative examples. For more details and other metafunctions, see Alshwaikh (2011).

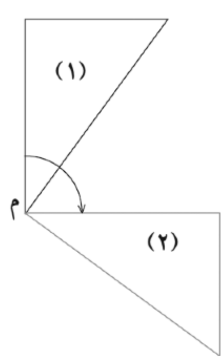
In order to analyse the way in which mathematical activities are represented visually, and following Kress and Van Leeuwen, I suggest classifying diagrams as either narrative or conceptual. The basic distinction is the presence of a timeline: narrative diagrams show temporality, but conceptual diagrams do not. This suggests that narrative diagrams show mathematical activity that may be seen as involving humans. Conceptual diagrams, on the other hand, show atemporal mathematical objects or relationships and, as a result, conceal the human role.

There are at least five types of narrative diagrams: Arrowed, dotted, shaded, sequence of diagrams, and construction. The dotted lines in Figure 1 and Figure 2a, suggest that these lines have been added to the original diagram. The time lapse is indicated by these dotted lines. In this case, these lines represent a mathematical activity that takes place, namely proving that the sum of the internal angles of a 5-pointed star is 180˚. Figure 2b shows an arrowed narrative diagram in which the direction of the arrow shows a rotation action of one triangle around a point in a clockwise direction.

In contrast, there are three types of conceptual diagrams: classificational, meaning the relationships between different polygons, as in Figure 3b; identifying, identifying a geometric object or its attributes, as in Figure 3a; and spatial, expressing relations between geometric objects within the diagram such as between points or points and lines. Figure 3a shows a circle, centre, radius, label and arrow pointing to that radius with the words (“Radius of the circle”). The goal here is mainly to introduce a mathematical object, the radius of the circle, with an identifying arrow that points to that object [3].

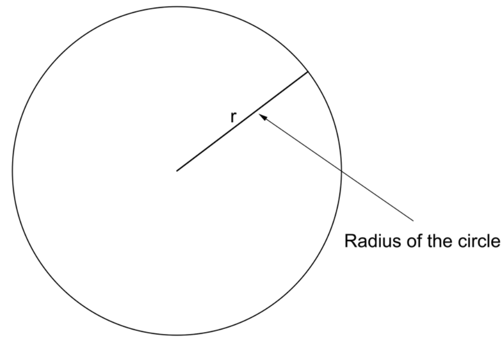


(a) Dotted narrative diagram

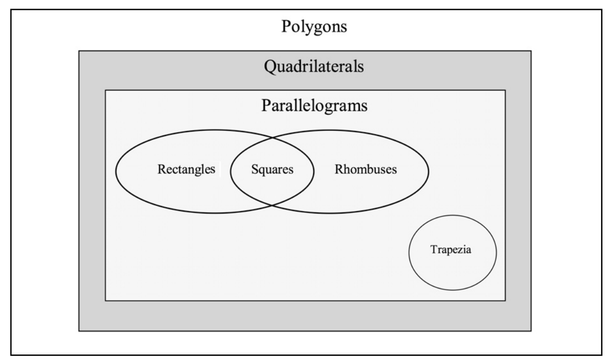


(b) Arrowed narrative diagram (MoE, 2012, Grade 9 – part 1, p.43)

Figure 2. Narrative diagrams



(a) Identifying conceptual diagram



(b) Classificational diagram (MoE, 2012, Grade 8 – part 2, p. 31, translated and redrawn by the author)

Figure 3. Conceptual diagrams

Two illustrative examples

“A proof is a story” (Stewart, 2007) to communicate mathematical objects and their relationships with other mathematicians, so that the community of mathematics may endorse it or may not. There are different ways to communicate that story. Some mathematical texts will tell the story in a visually dominant way; others will tell a verbally dominant story; a third kind is a story that is symbolically dominant and usually multi-modal. Here I look at two kinds of proof: one that is visually dominant (Figure 4) which is not common in mathematics textbooks and a second, multi-modal example, that is common in school mathematics textbooks (Figure 5).

Example 1: Proof as visual story, Area of the triangle

Figure 4 is a mathematical text that employed different narrative and conceptual diagrams in order to carry out a mathematical proof.

People who are familiar with geometry would recognise that the goal of this figure is to show that the area of a triangle is half its base multiplied by the height or, symbolically, $\frac{1}{2}(b \times h)$.

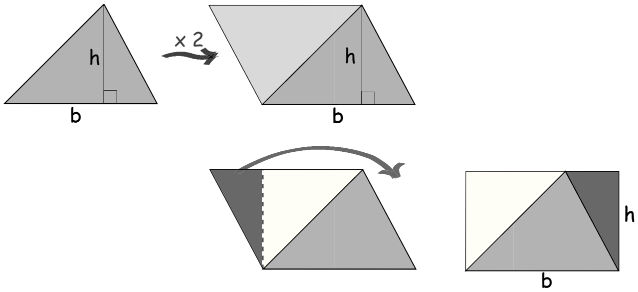


Figure 4. *Visual proof (Pierce, 2017)*

The original triangle is a conceptual triangle showing base (*b*) and height (*h*) that is perpendicular to the base (indicated by the little square right-angle symbol). The arrow with '×2' on it represents an action (a mathematical activity) suggesting that the original triangle is doubled or copied. In the following shape, that new copy (drawn with a lighter shading than the original) is positioned relative to the original triangle to produce a parallelogram (this can be shown by congruence). The shape underneath the parallelogram is narrative and shows three actions or mathematical activities. The first activity is marked by the dotted line dividing the copied triangle into two triangles with different sizes. The second activity is shading the small triangle. The third activity is moving the shaded triangle to the other side of the original triangle as indicated by the arrow. The final shape is the product of the actions carried out on the previous shape and it can be shown to be a rectangle with area $b \times h$.

But, is this a valid generalisation? Can we, in other words, generalise from diagrams? Giaquinto states that

> generalizing is always a step from premisses to a conclusion that is not deductively entailed by those premisses, hence cannot occur in a mathematical proof. Generalizing occurs in mathematical argument when one reasons about an arbitrary instance of a class of things in order to draw a conclusion about all members of the class. (2007, p. 78)

It would be interesting exercise to apply Giaquinto's statement to this example because it seems that we will need words and symbols. The whole notion here is that the rectangle consists of two congruent triangles (the original and the copy) and, hence, the area of the original triangle is half the area of the rectangle. In words and symbols, that activity could be expressed as follows:

> To find the area of a triangle with base *b* and height *h*, double the triangle itself to make a parallelogram, *i.e.*, the area of this parallelogram is twice the area of the original triangle. Now transform that parallelogram into a rectangle with the same area. Hence, the area of the original triangle is half the area of the rectangle. Thus, Area of triangle = ½ Area of parallelogram = ½ Area of congruent rectangle = $\frac{1}{2}(b \times h)$

These are different representations of the activity of finding the area of a triangle. The argument here is that the visual representation is functional and communicates the activity of proof using narrative diagrams. Giaquinto (2007) considers diagrams that carry information not conveyed in the written part of the proof 'non-superfluous'. As this proof has no written part, the diagrams in this example are non-superfluous.

Example 2: Proof of the Central Angle Theorem

Example 1 is purely visual, which is not common in mathematical texts. In Figure 5 a common mathematics school text with different (multi-modal) representations is presented. Here, I analyse only the ideational function, but elsewhere (Alshwaikh, 2015), I also considered the other metafunctions mentioned above (the interpersonal and the textual).

One can see in Figure 5 that the written text (words and symbols) state the theorem or the given information and what is required to be proven. The theorem itself is in the upper *Ideal* part of the text (Kress & Van Leeuwen, 2006) and its wording is of a timeless nature, presenting the mathematical objects (two angles share an arc in a circle) and the relationship between them, without any reference to human agents. Moreover, the theorem is separated from the rest of the text by a frame.

The proof itself has two components, words and diagram. It starts with the written part on the right in the original Arabic text (in the left in English), followed by a narrative diagram left of the text (right in English). The diagram actually adds information to the written text and thus *complements* it (Kress & Van Leeuwen, 2006) in a *non-superfluous* part of the presentation of the proof. The additional information is added by the dotted line and by naming the resultant angles (1, 2, *a, b*) to which the written text refers, but which it does not introduce, relying instead on the diagram. In addition, the written text makes use of the diagram in the reasoning of the proof. The use of the symbol ∴ is interesting and shows a complement relationship, as if saying "*since the dotted line in the diagram divides the* $\angle AOB$ *into* $\angle 1$ *and* $\angle 2$, *therefore* $\angle 1 = 2a$" and providing a

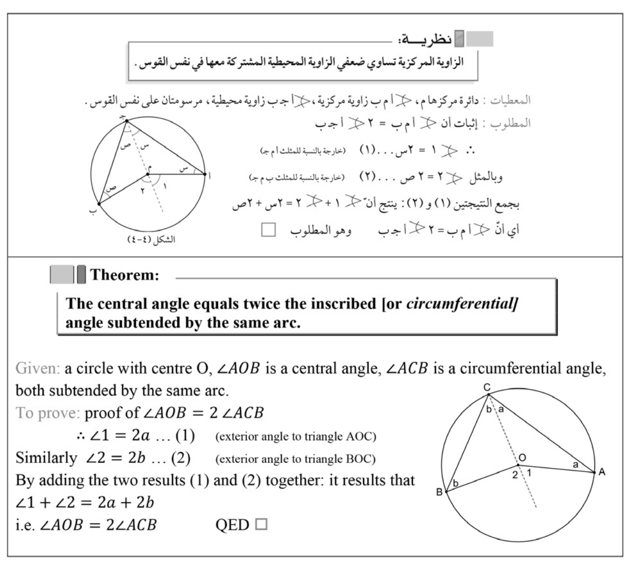


Figure 5. *Multi-modal mathematical text [4].*

reason for that statement (exterior angle to triangle AOC). The use of "Similarly" and "By adding" contributes to the coherence of the text.

Finally, it is worth noting that the final statement in the proof is presented in symbols ($\angle AOB = 2\angle ACB$). One interpretation could be that words and diagram work together in order to move the proof into the language of symbols, which are accepted and endorsed by the mathematical community, which tends to view mathematics as abstract, symbolic and formal.

Concluding remarks

In this article I have tried to reveal the human role in doing mathematics by looking at geometrical diagrams. In the process of exploring mathematical activities represented in geometric diagrams, I paid special attention to dotted lines. The result of this focus was the notion of temporality and its realisation in diagrams. The use of dotted lines (and other features) reveals a timeline that can be traced in a diagram. This means, I argue, that some geometric diagrams tell a narrative while others just present mathematical objects or concepts. In representing the nature of mathematical activities, I introduce two types of diagrams; narrative (Figure 1 and Figure 2) and conceptual (Figure 3). The former reveals the story of mathematical activities, which the latter tries to conceal.

This attempt, furthermore, can be seen as a contribution to extending the semiotic landscape beyond language (and symbolism) to include visual communication and representation in mathematics discourse. Adopting a social semiotic theoretical approach to language, images and mathematics, and zooming in on geometry enabled a targeted focus on diagrams.

Understanding what we communicate and what we want to communicate in diagrams is particularly important, given what a powerful tool diagrams are to represent spatial logical relations that language cannot do on its own, as de Freitas and Zolkower (2015) showed us in students work on word problems. The use of diagrams can, they claim, give students access to more complex problems.

Focusing on diagrams as visual representations, as well as distinguishing between different types of diagrams and analysing what they communicate is useful for teachers, educators and textbook developers. For example, mathematics is often presented in classrooms and textbooks as abstract, symbolic and devoid of human agency, a view which affects students' access to mathematics. If we understand what is communicated in diagrams—whether they tell a story and include human agency or whether they are conceptual and 'timeless'—we can better design textbooks and better understand what they communicate to learners. Perhaps different kinds of diagrams can help break down stereotypes about mathematics that influence students' ability to access it. Improving access to mathematics also has implications for issues of equity (see for example, Adler & Pillay, 2017). The better we understand how different student populations see themselves in all aspects of mathematics, the better we can design textbooks and other teaching tools to be more inclusive, including through the diagrams they present.

Notes

[1] For example, Giaquinto (2007) presents the proof of Thales Theorem as an illustrative example to argue against the claim that diagrams are always superfluous. In doing so he uses the dotted lines to background the information of the diagram and the solid lines to bring into focus what he is referring to (pp. 75–77).

[2] Another conventional use of dotted lines expresses an 'unseen' side in 3D diagrams.

[3] Unlike the arrows that represent action or narrative, they in some way come from 'outside' the diagram, referring to a specific part of the diagram in order to identify it with words.

[4] Taken from MoE, 2011, Grade 9 - part 1, p. 75; translated and redrawn by the author.

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