A QUANTITATIVE AND COMBINATORIAL APPROACH TO NON-LINEAR MEANINGS OF MULTIPLICATION

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The *Colored Digits Problem* is intended to help eighth-grade students (aged 13-14) develop quadratic relationships:

> You have a deck of number cards. You draw a card, replace it, and draw a second card in order to make coordinate points, such as (1,1).
>
> a. Suppose the deck of number cards includes the numbers 1 through 7. The numbers are colored yellow. What color combination(s) are the coordinate points? How many coordinate points could you make?
>
> b. Suppose the deck of number cards has the numbers 1 through 21 on them. Every 7 digits is a different color (1 through 7 are yellow, 8 through 14 are red, 15 through 21 are green). How many color combinations could you make? How many coordinate points are in each color combination? How many total coordinate points could you make?

Figure 1 shows an array that Armando, an eighth-grade student in an interview study, produced as part of solving this problem. When solving part *a*, he created the lower left corner of the array, and determined that 7 digits produced 7^2 coordinate points, and that all coordinate points had a yellow first and second digit so the color combination for these coordinate points was yellow-yellow. He then expanded his array for part *b* of the problem where each square in the array shows a two-color combination, and the points (either drawn or imagined) show the coordinate points in each two-color combination. Armando symbolized his reasoning for part *b* as shown in Figure 2 where the symbolic statements refer to different ways of "seeing" the array. He then provided the following explanation for his notation in Figure 2:

T: You want to talk through that and talk through the notation.

A: It'd be this one [points to Figure 2]. Yeah, um, that, um, let's see. So I started out by putting twenty-one squared, and that equaled to seven times three in parenthesis multiplied by seven times three in parenthesis, as well.

T: Mm-hmm.

A: And that also gave me the answer that twenty one squared would be, and then another way to get that would have been seven squared multiplied by three squared, and in all that gave me four hundred forty one.

T: Yeah, in all, it gave you four forty one. Where do you see the seven times three and the seven times three in the picture [points to Figure 1]?

A: Um, the seven [puts his finger on the seven written on the horizontal axis] and in all how there is three colors [puts his fingers to span the three sevens he sees on the horizontal axis].

[…]

T: Um, and then where did you see seven squared times three squared?

A: Hmm, lets see. The seven squared would be right here as well [puts his pencil tip on the point (7,7) in Figure 1].

T: Do you mean just that one point?

A: Oh not just that one point. I am guessing all the points, like if you put down all the dots it would have been painful [gestures towards the square in the middle of the bottom row of Figure 1]. It would be there as well, and there, and there, and there [points to a blank box he has drawn in his array each time he says the word there, showing where each seven squared is].

T: Yeah. So like I guess my question is: is it just this point that is seven squared [points to the point (7,7)] or you're saying it's all the dots [circles all the dots in the lower left of Figure 1 with his pencil] or which one?

A: I guess all the dots.

T: Okay. Okay. And then where is the three squared?

A: The three squared would have to be the colors, and since how there are two of them [points to the vertical and horizontal axis] like two bars for them those two would be three times three or three squared [moves his fingers from the axes to the interior of the array where the color combinations are represented].

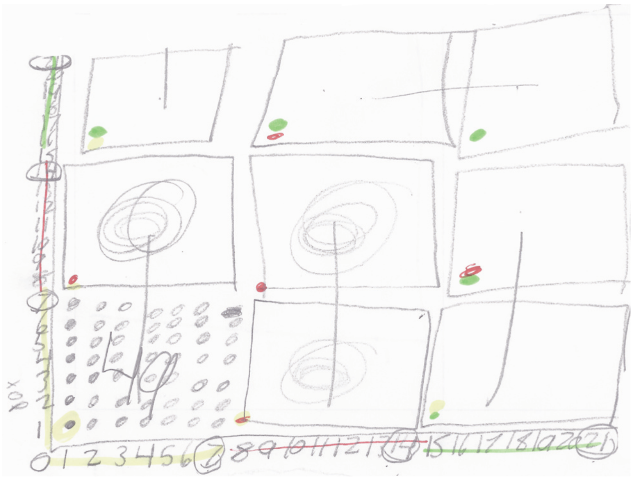

Figure 1. Array for the colored digits problem [1].

$$441 = 3^2 \times 7^2 = 21^2 = (\overset{d}{7} \times \overset{c}{3}) \cdot (\overset{d}{7} \times \overset{c}{3})$$

Figure 2. Notation for the colored digits problem

We considered that supporting students to develop the relationships that Armando described, along with relating these relationships to part *a* of the problem, provided the opportunity for students to see that three times the number of digits produced 3^2 times the number of coordinate points, a quadratic relationship. We have designed problems like this in order for students to work on the identity $(nx)^2 = n^2x^2$ and, with development, to work on the solution of quadratic equations of the form $(nx)^2 = n^2x^2 = a$. More generally, we have been engaged in work with sixth-, eighth-, and tenth-grade students on establishing non-linear meanings of multiplication using a quantitative and combinatorial approach. In this article, we provide a conceptual analysis of combinatorics problems that we have used in this work. We ground our conceptual analysis of these tasks in prior work on students' quantitative (Smith & Thompson, 2008) and multiplicative reasoning (Steffe, 1992, 1994) in order to identify what novel ways of operating combinatorics problems can entail, and how these novel ways of operating connect to and extend prior research.

Studying non-linear meanings of multiplication: rationale and definition

Researchers have identified that students have difficulty establishing non-linear meanings of multiplication across a broad range of ages, and in a broad range of different mathematical domains (Van Dooren, De Bock, Janssens & Verschaffel, 2008; Greer, 2010). One prominent aspect of this difficulty is students' tendency to overgeneralize *linear* meanings of multiplication to situations that involve *non-linear* meanings of multiplication. Given the prominence of this difficulty, Van Dooren *et al.* (2008) have called for the development of novel approaches to studying students' understanding of non-linear meanings of multiplication.

The observation that students overgeneralize linear to non-linear meanings of multiplication raises the question: what is the difference between linear and non-linear meanings of multiplication? Van Dooren *et al.* (2008) define linear meanings of multiplication as any situations that involve a direct proportional relationship between two variables and adhere to the mathematical properties that $f(kx) = kf(x)$ and $f(x + y) = f(x) + f(y)$. Non-linear meanings of multiplication then, are any multiplicative situations that do not involve a direct proportional relationship or do not adhere to one or both of these properties.

While we appreciate these definitions of linear and non-linear meanings of multiplication because they provide a way to mathematically classify situations, they were not the starting place we took for our work. Instead, we started by designing tasks we thought could involve students in reasoning about quadratic and cubic relationships. Then we analyzed the tasks in relation to Steffe's (1992, 1994) prior models of students' whole number multiplicative reasoning, in order to identify what novel ways of operating our tasks might include that had not been identified in Steffe's earlier models of students' reasoning.

As part of the design process, we used a quantitative approach (Smith & Thompson, 2008) because of examples from prior research of students successfully establishing non-linear meanings of multiplication in the context of reasoning about quantities and quantitative relationships (*e.g.*, Ellis, 2011; Lobato, Hohensee, Rhodehamel & Diamond, 2012). We therefore begin by outlining what we mean by a quantitative approach, and then provide a brief summary of Steffe's (1992, 1994) prior work on students' whole number multiplicative reasoning.

Quantitative approach to non-linear meanings of multiplication

Following Thompson (2011), we consider a quantity as a person's establishment of a measurable quality of an object, an appropriate unit for measurement, and a process for assigning a numerical value to the quality. So, quantitative reasoning entails a person in establishing quantities and reasoning with relationships among them. As Figure 1 shows, we planned to have students represent problems like the Colored Digits Problem with arrays and to reason with *relationships* between one- and two-dimensional quantities.

Battista (2007) commented, in a review of research on spatial and geometric reasoning, that researchers have not yet provided careful analyses of how students establish *relationships* between one- and two-dimensional quantities. One reason Battista made this comment is that research on students' two-dimensional reasoning has frequently investigated how elementary students use a unit square to tile (or imagine tiling) a rectilinear figure. This approach to area does not necessarily involve students in establishing the basic *relationship* that one two-dimensional unit (*e.g.*, an area unit) can be created from two one-dimensional units (*e.g.*, two length units). Instead, the basic square unit is given to students, who may not constitute it as produced from two length units (Outhred & Michtelmore, 2000;

Simon & Blume, 1994; Thompson, 2000). From a quantitative perspective, this issue is problematic because it suggests that students may not establish the appropriate unit of measure that teachers or researchers are intending when they work with students on relationships among one- and two-dimensional units, even if students assign the "correct" numerical value in the solution of a problem (Battista, 2007). From the perspective of helping students establish non-linear meanings of multiplication, this issue is important because it is unlikely students would be able to establish more advanced relationships between one- and two-dimensional units without first establishing this basic relationship. For example, in the Colored Digits Problem, it is unlikely students would establish that (7 × 3), the number of digits, could produce (7 × 3) × (7 × 3), the number of coordinate points, without first establishing the basic relationship between one- and two-dimensional units [2].

We selected combinatorics problems in the design of tasks because we considered them to be one site where students could establish the basic relationship between one- and two-dimensional *discrete* units. Combinatorics problems can support students in this endeavor because they have the potential to involve a pairing operation—a mental operation students can use to, for example, establish one two-dimensional unit from two one-dimensional units (Tillema, 2013). We use an analysis of the *Flag Problem* to discuss this potential:

> You are designing a two-striped flag for a new country. You can choose from three colors. How many possible flags could you make? (Assume order matters and the colors of the stripes do not have to be different).

To solve the Flag Problem, a student might envision assigning a color to each stripe of the flag. If, in this process, the student takes the correspondence she establishes between the two colors as one unit (*i.e.*, the flag is a unit composed of two stripes), then we consider her to be using a pairing operation. The units that result from a student using her pairing operation are units that contain two units, but are counted as a single unit (Figure 3) (Behr, Post, Harel & Lesh, 1994; Vergnaud, 1983). We consider these units to be discrete two-dimensional units, and we call them pairs.

One reason to use combinatorics problems with students to help them develop relationships between one- and two-dimensional units, then, is that creating pairs can arise directly out of the goal of making, for example, a flag in problems like the Flag Problem. This means that students' solutions of such problems have the potential to involve an *explicit* operation in which they transform two discrete one-dimensional units into one discrete two-dimensional unit. This observation is not to suggest that problems involving area cannot involve such work (see Simon & Blume, 1994; Thompson, 2000), but simply that we considered this issue important in the design of tasks, and that combinatorics problems have the potential to support the establishment of this basic relationship.

In addition, we anticipated that establishing quadratic relationships (*i.e.*, non-linear meanings of multiplication) would entail students in establishing more complex relationships between one- and two-dimensional units than the one we just described. To understand these more complex relationships, we began with Steffe's (1992, 1994) work on students' multiplicative reasoning with whole numbers. We outline this work below, but note that it has not been used to understand how students create *relationships* between one and two-dimensional units. Therefore, we considered that it would need to be modified in order for it to be useful for our purposes.

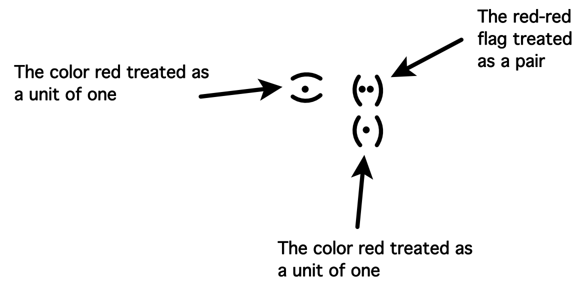

Figure 3. A two-dimensional unit created from two one-dimensional units.

Multiplicative reasoning

Steffe (1992, 1994) has identified three qualitatively distinct multiplicative concepts that students construct (see also Hackenberg, 2013; Ulrich, 2015). Each multiplicative concept involves units coordination as a central cognitive operation. At its most basic level, a units coordination involves the insertion of one composite unit into the units of another composite unit (Steffe, 1992, 1994; see also Kamii & Housman, 2004). Here we use the *Colored Digit Problem II* to discuss each of the three multiplicative concepts Steffe has identified:

> You have a deck of number cards. Every 7 digits in the deck were written in a different color (starting at 1 and continuing onward). The teacher tells you that she used 3 colors to create the deck of cards. How many total cards are in the deck?

Students operating with the first multiplicative concept (MC1 students) can engage in a units coordination *in activity*, meaning they insert the units of one composite unit into the units of another composite unit as part of their activity. To solve the Colored Digits Problem II, a student using MC1 might reason as follows: "7 that is one color; 8, 9, 10, 11, 12, 13, 14, that's two colors; 15, 16, 17, 18, 19, 20, 21, that's three colors, so there are 21 digits". The student inserts seven units into each of the three units to create a unit of 7 units in activity.

Students operating with the second multiplicative concept (MC2 students) have interiorized (von Glasersfeld, 1995) this units coordination, which means they no longer need to carry it out as part of their activity. Instead, they can take a unit of 7 units as given prior to operating, which enables them to use strategic reasoning to solve the Colored Digits Problem II. For example, a student might reason "7 and 7 is 14 because 7 and 3 is 10, and 4 more would be 14". This kind of reasoning involves a student disembedding 3 and 4

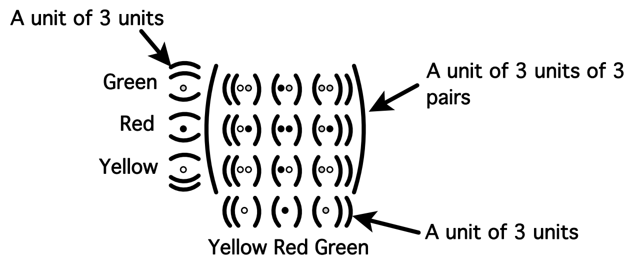

Figure 4. *A multiplicative relationship among a unit of 3 units, a unit of 3 units, and a unit of 3 units of 3 pairs.*

from the second 7, where disembedding means that they can treat 3 and 4 as a part of, but also independent from 7. To finish solving the problem, an MC2 student may continue this kind of strategic reasoning. Once they determine the result, 21, MC2 students can also engage in a units coordination by inserting the result, 21 into a containing unit, to establish it as a unit of 3 units each containing 7 units (a three-levels-of-units structure). Thus, students using MC2 have interiorized two levels of units and are able to create a third level of unit in activity.

Students operating with the third multiplicative concept have interiorizied three-levels-of-units, which allows them to operate on a three-levels-of-unit structure. To solve the Colored Digit Problem II, students using MC3 might reason that three 7s is equal to three 5s and three 2s, evaluate this by multiplying 3 times 5 to get 15 and 3 times 2 to get 6, and then combine 15 and 6 to get 21. Operating in this way provides an indication that they can treat the three 7s as a unit of 3 units of 7 units, and operate on this by disembedding a unit of 3 units of 5 units, and a unit of 3 units of 2 units, evaluate each of these three-level-of-unit structures, and then combine them.

Steffe (1994) conjectured that these multiplicative concepts could be a basis from which students establish more complex multiplicative reasoning. We see our conceptual analysis as aligned with this conjecture, and now return to analyzing the Colored Digits Problem. In our analysis, we highlight how a pairing operation and units coordination are likely involved in establishing robust quantitative relationships between one- and two-dimensional units, and how this has the potential to enable students to use combinatorics problems to develop non-linear meanings of multiplication.

Conceptual analysis of a two-dimensional combinatorics problems

We first note that to solve the Colored Digits Problem and produce the array shown in Figure 1, a student could begin by creating the color combinations in the array, which has the potential to involve establishing a multiplicative relationship among a unit of 3 units (*i.e.*, 3 colors represented on the horizontal axis), a unit of 3 units (*i.e.*, 3 colors represented on the vertical axis), and a unit of 3 units of 3 pairs (*i.e.*, the 3^2 possible color combinations) (Figure 4). To create this relationship, a student might assimilate part *b* of the Colored Digits Problem with a unit of 3 units (the number of colors), disembed a second unit of 3 units (to create a second

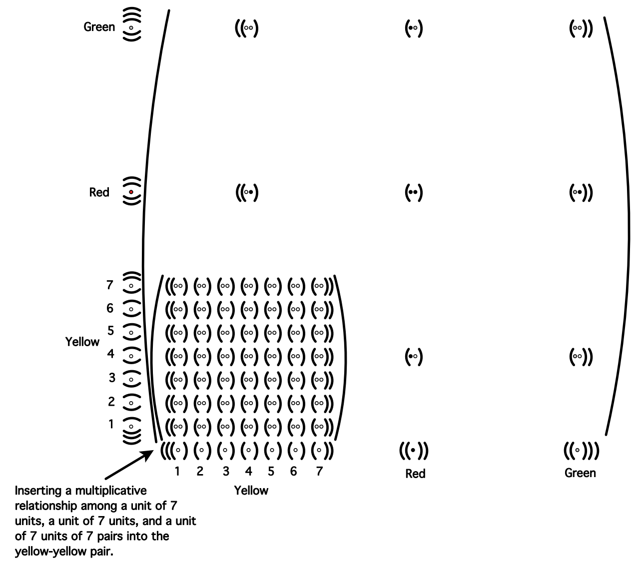

Figure 5. *A units coordination using $MC3_{2D}$.*

set of three colors), and then pair the first color with each of the three colors, the second color with each of the three colors, *etc*. As part of establishing this conceptualization, a student could treat the first three pairs (yellow, yellow), (yellow, red), and (yellow, green), as a unit of 3 pairs, the second three pairs (red, yellow), (red, red), and (red, green), as a unit of 3 pairs, *etc*., and treat all three units of three pairs as itself a unit. By doing so, the student could create a unit of 3 units of 3 pairs. Two important features of this conceptualization are that a student establishes the two-dimensional units as a three-levels-of-units structure and establishes the one-dimensional units as independent of, but related to the two-dimensional units (*vis-à-vis* a disembedding operation). Tillema (2013) has called this conceptualization $MC3_{2D}$.

This conceptualization already involves a three-level-units structure, but we think more is involved in establishing the relationships symbolized in Figure 2. In particular, once a student has established the color combinations using $MC3_{2D}$ he may engage in a units coordination by inserting 7^2 coordinate points into, for example, the yellow-yellow color combination (Figure 5). This way of operating would entail inserting a multiplicative relationship among a unit of 7 units, a unit of 7 units, and a unit of 7 units of 7 pairs into the yellow-yellow color combination. A student might then envision that such an insertion is possible for any of the remaining color combinations, to give a total of 3^2 times 7^2 coordinate points. Once a student has made this insertion, he or she may switch to seeing other three-levels-of-units structures. For example, a student might see the digits represented along each axis as a three-levels-of-units structure, a unit of 3 units of 7 units, since each of the three colors contains seven digits, and then quantify the total number of coordinate points as $(7 \times 3)^2$.

We have conjectured that envisioning the equivalences notated in Figure 2 involves a pairing operation, units coordination, and switching between different three-levels-of-units views of the array. One important distinction in the conceptual constructs we have introduced is that a *pairing operation* is, for us, a multiplicative operation that produces higher dimensional units (*e.g.*, two-dimensional units) from lower dimensional units (*e.g.*, one-dimensional units), while a *units coordination* is a multiplicative operation that students use within a particular dimension.

This point is highlighted in our analysis in that to establish the color combinations and the coordinate points entailed a pairing operation, whereas to produce the total number of coordinate points per color combination entailed a units coordination. We see the use of a pairing operation in concert with a units coordinating operation to be a conceptual foundation for the establishment of non-linear meanings of multiplication. We think that this kind of distinction in conceptual operations provides a foundation for analyzing when and how students establish non-linear meanings of multiplication.

Conceptual analysis of a three-dimensional combinatorics problem

We now turn to an analysis of the *Three Suit Card Problem,* a problem we designed to support tenth-grade students establish cubic relationships:

> You have the 2, 3, and king of diamonds, a friend has the 2, 3, and king of hearts, and another friend has the 2, 3, and king of spades. A three-card hand consists of one card from each person's hand (order does not matter).
>
> a) How many different three-card hands is it possible to make?
>
> b) How many three-card hands have no face cards, exactly one face card, exactly two face cards, and exactly three face cards?

The aim of this problem is to help students develop the equivalence that $3^3 = (2 + 1)^3 = 2^3 + 3 \times (2^2 \times 1) + 3 \times (2 \times 1^2) + 1^3$. We conjectured that this equivalence could grow out of reasoning that there are a total of 3^3 possible three card hands and that this total could be quantified as $(2 + 1)^3$ because each person has 2 non-face cards and 1 face card. Then, this total could be quantified as $2^3 + 3 \times (2^2 \times 1) + 3 \times (2 \times 1^2) + 1^3$ because there are 2^3 three-card hands that have no face cards, since each person has two non-face cards, there are three ways to have one face card with each way having $(2^2 \times 1)$ three-card hands in it, there are three ways to have two face cards with each way having (2×1^2) three-card hands in it, and there is 1^3 three card hands that have three face-cards in it, since each person has one face card. One goal of designing problems like the Three Suit Card Problem was to eventually work with students on the cubic identity that $(x + y)^3 = x^3 + 3(x^2 \cdot y) + 3(x \cdot y^2) + y^3$.

Given that we aimed to engage students in quantitative reasoning, we considered how they might modify the conceptual operations they used in establishing relationships among one- and two-dimensional units to support their establishment of relationships among one-, two-, and three-dimensional units. In order to establish the most basic relationship—the relationship that three one-dimensional units produce one three-dimensional unit—we considered that problems like the Three Suit Card Problem could entail students using their pairing operation recursively (Vergnaud, 1983). For example, a student could pair the two of diamonds with the two of hearts to establish a discrete two-dimensional unit (Figure 6, left), and then pair it with the two of spades to establish a discrete three-dimensional unit, which we call a triple (Figure 6, right). From a quantitative perspective, we considered engaging in these operations crucial because it meant that students were establishing an appropriate unit of measure for three-dimensional units—a unit of measure constituted from three one-dimensional units (Battista, 2007). However, we considered that more complex conceptual operations would be involved in students' establishing a three-dimensional array to represent these problems.

We conjectured $MC3_{2D}$ could be one constructive resource for students to establish these more complex relationships. For example, a student might establish all possible two-card hands with the spades and diamonds by creating a multiplicative relationship between a unit of 3 units (the three spades), a unit of 3 units (the three diamonds), and a unit of 3 units of 3 pairs (all two-card hands with the diamonds and spades), and then *pair* this concept ($MC3_{2D}$) with a unit of one (*e.g.*, the two of hearts). This pairing produces a multiplicative relationship among a unit of one (*e.g.*, the two of hearts), a unit of three units (*e.g.*, all three spades), a unit of three units (*e.g.*, all three hearts), and a unit of three units of three triples (*e.g.*, all triples containing the two of hearts), which can be represented as a cross section of a three-dimensional array. Figure 7 at the top shows this cross section, which represents all possible triples that contain the two of hearts. A similar cross section of a three-dimensional array can also be made for all possible

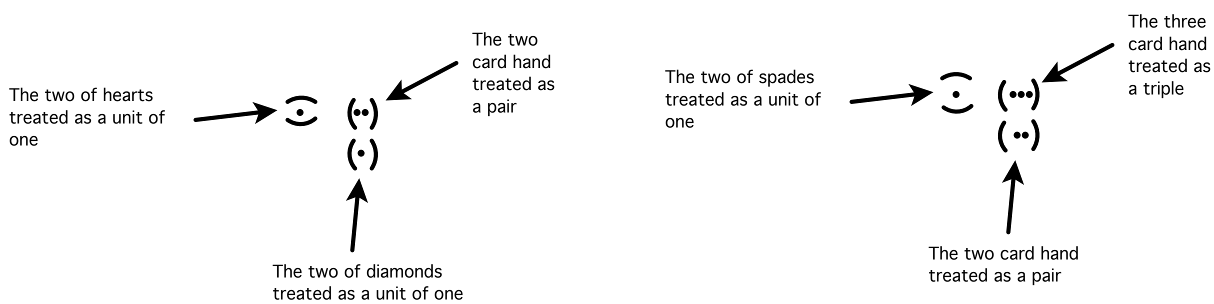

Figure 6. Creating a triple through a recursive use of a pairing operation.

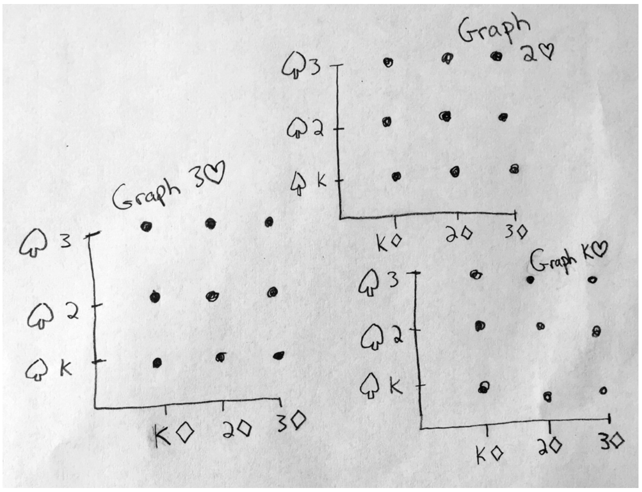

Figure 7. Three two dimensional arrays for the Three Suit Card Problem.

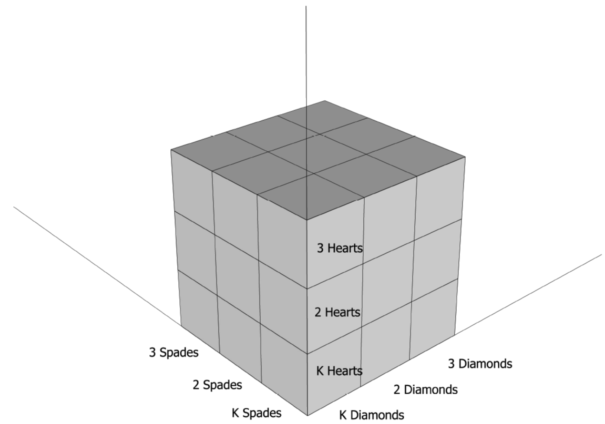

Figure 8. A three dimensional array for the Three Suit Card Problem [3].

triples that contain the three of hearts (Figure 7, bottom left), and king of hearts (Figure 7, bottom right).

We have designed tasks so that students use cross-sections of three-dimensional arrays to establish the array itself. This design decision was made because we wanted to allow students to use their understanding of how two-dimensional arrays are spatially organized as a basis for establishing a spatial organization for three-dimensional arrays. For example, a student has the opportunity to observe that to change which spade is in a triple requires moving vertically in each array just as moving horizontally changes the diamond in a triple. This observation opens the way for students to consider in which direction they could move in order for the heart to change. One way to think about this is moving out of the page, which can lead to the creation of a three-dimensional array like the one shown in Figure 8.

We considered that starting with cross sections of the array could also support students to identify regions in the array that represented three-card hands that contained zero, exactly one, exactly two, and exactly three face cards. We considered this because to produce the three-dimensional array from cross sections could involve seeing that the bottom layer contains all three-card hands that have the king of hearts (Figure 9, left), the middle layer contains all triples that have the two of hearts, and the top layer contains all triples that have the three of hearts. A student might, then, switch views of the array to see that all the triples that contain the king of spades are in the front-most face of the 3-D array (Figure 9, middle), and all triples that contain the king of diamonds are in the left-most face of the 3-D array (Figure 9, right). Once students had identified regions that contained a particular kind of king we considered that they could use these regions to identify, for example, the region that has triples with all three kings because it is the region that is common to all of the regions in Figure 9. Similarly, the triples that contain two kings contain exactly two of the three regions in Figure 9, the triples that contain one king contain exactly one of the regions in Figure 9, and the triples that contain no kings cannot contain any of the regions in Figure 9. Figure 10 (overleaf) shows these different regions: the triples with no kings (medium grey in Figure 10, not all triples are visible), one king (dark grey in Figure 10, not all three regions are visible), two kings (white in Figure 10), and three kings (light grey in Figure 10).

In establishing these regions of the array, we considered that students could establish cubic relationships (*i.e.*, non-linear meanings of multiplication) because of the potential for them to recursively use their pairing operation in concert with their units coordinating operation. We envisioned that students could establish each of the eight regions as a triple: in each person's hand there were two possible types of

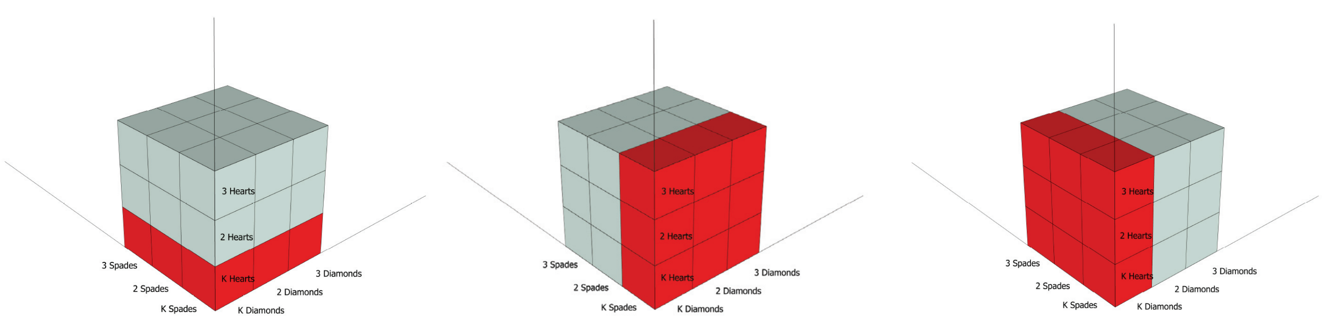

Figure 9. Highlighting where triples are that contain a king.

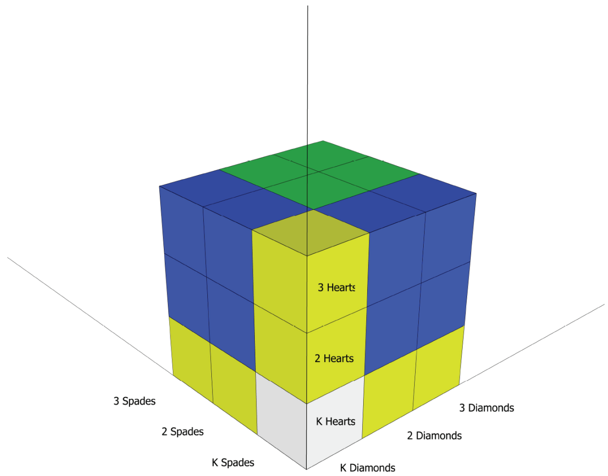

Figure 10. Three-dimensional array with different regions identified.

cards, face or non-face; the options of face or non-face from the first person could be paired with these options for the second person, yielding four possible outcomes; and these four options could be paired with the two options from the third person's hand yielding a total of 2^3 or eight possible triples (Figure 11, top). Each of these triples corresponds to a geometric region of the array. We then envisioned that students could engage in a units coordination by inserting the specific number of three-card hands into each region. For example, for the region that was a diamond face card, spade non-face card, and heart non-face card, a student could insert the four triples into the region by creating these triples from the one face card paired with the two non-face cards, and then paired again with the two non-face cards (Figure 11, bottom). This process would entail the student in establishing the region as a multiplicative relationship among a unit of 1 unit (the one diamond), a unit of 2 units (the two spades), a unit of 2 units (the two spades), and a unit of 2 units of 2 triples (the four three-card hands). The student might then establish other regions of the array as three-level-of-unit structures similarly to how they established this particular region. We note here again that a pairing operation, a units coordinating operation, and switching among different three-level-of-unit structures, is what we have conjectured is central to students producing non-linear meanings of multiplication in this task, and the other tasks we have designed.

Concluding remarks

We see conceptual analyses like the one presented in this article to be a crucial aspect of studying whether and how students establish non-linear meanings of multiplication, because they help to identify key ways that students might operate to produce non-linear meanings of multiplication. These kinds of conceptual analyses are particularly important for mathematical domains like non-linear meanings of multiplication where researchers have reported widespread student difficulties (Van Dooren *et al.*, 2008) because they provide a starting point from which to investigate novel curricular approaches to this domain. Moreover, because the conceptual analysis we have presented builds from prior models of students' reasoning (*e.g.*, Battista, 2004; Steffe, 1992, 1994), it provides us the opportunity both to delineate the conceptual operations that students might use to establish non-linear meanings of multiplication, and how these conceptual operations may differ from those that students use to establish linear meanings of multiplication. By doing so, they make explicit our conjectures about how students might operate, which can subsequently be tested in interactions with students, a key component of design research (Confrey & LaChance, 2000; Lobato, 2008). While combinatorial approaches to algebraic identities have been used in reform curricular materials (*e.g.*, Peressini *et al.*, 1998), we have not seen curricular materials that use combinatorics problems with the aim of engendering students' quantitative reasoning to support non-linear meanings of multiplication. In the case of non-linear meanings of multiplication, many of the quantitative structures have the potential to engender combinatorial reasoning, and combinatorics problems have the potential to involve key operations that support students to develop these quantitative structures (Vergnaud, 1983). We therefore think the integrated approach we have outlined here is promising.

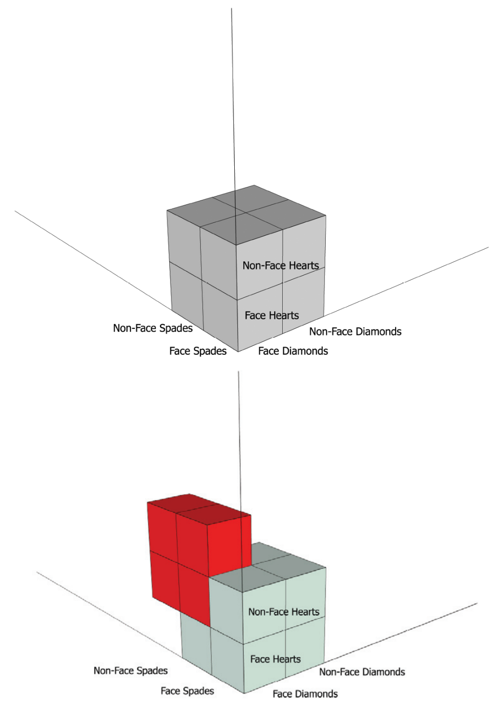

Figure 11. Eight regions of face and non-face cards.

Notes

[1] Armando used a highlighter to color the first seven digits on each axis yellow, the next seven red, and the third seven green. He then used the markers to create the color combinations in the lower left corner of each box. The dot in lower left square was yellow-yellow. The square just above it was yellow, red. The square above that was yellow green, and so on.

[2] Here we are not suggesting that a tiling approach to "area" is inappropriate for elementary grade students. Rather we are suggesting that as students develop more sophisticated ideas about area it is important to

consider whether they are establishing the basic relationship between one and two-dimensional units because this issue would have a significant impact on whether and how they establish more complicated relationships between one- and two-dimensional units.
[3] When working with students, we have used cubes to represent the three-dimensional discrete arrays because of the difficulty of having a discrete representation in three dimensions.

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367. A mental image is the image which is described when someone describes what he imagines.

370. One ought to ask, not what images are or what goes on when one imagines something, but how the word "imagination" is used. But that does not mean that I want to talk only about words. For the question of what imagination essentially is, is as much about the word "imagination" as my question. And I am only saying this question is not to be clarified—neither for the person who does the imagining, nor for anyone else—by pointing; nor yet by a description of some process. The first question also asks for the clarification of a word; but it makes us expect a wrong kind of answer.

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