Towards an Instructional Theory: the Role of Student's Misconceptions*

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I. During the past decade we have witnessed a new trend in cognitive research emphasizing expert systems. A great deal of effort has been dedicated to the study of experts' performance in various fields of knowledge. My presentation today deals with the question: what kind of expertise is needed for instruction? Researchers in the field agree that the process of learning necessarily combines three factors: the student, the teacher and the subject to be learned. In addition, it seems obvious that to teach a given subject matter we need at least two kinds of expertise: the subject matter expert who can knowledgeably handle the discipline to be learned, who can see the underlying conceptual structure to be learned with its full richness and insights; and there is also, obviously, the expert teacher whose expertise is in successfully bringing the student to know the given subject matter by various pedagogical techniques that make him the expert in teaching. In this framework of experts' systems, what is, then, the role of the student? what does he contribute to the learning situation? And though it might seem absurd, I would like to suggest that the student's "expertise" is in making errors; that this is his contribution to the process of learning.

My talk consists of three main parts First, I will focus on the contribution of performance errors to the process of learning. I will, then, demonstrate that errors do not occur randomly, but originate in a consistent conceptual framework based on earlier acquired knowledge I will conclude by arguing that any future instructional theory will have to change its perspective from condemning errors into one that seeks them. A good instructional program will have to predict types of errors and purposely allow for them in the process of learning But before we reach such an extreme conclusion let me build the argument and clarify what these "welcomed" errors are.

II. In order to better understand the process of learning, I would like to make a digression here and learn something from scientific progress. Science involves discovering truths about our universe; it does so by forming scientific theories. These theories then become the subject matter for learning. Philosophers worried for a long time about these truths. How can one be sure that one has reached truth and not falsehood? Are there clear criteria to distinguish truth from falsehood? These philosophical discussions can also enlighten our understanding.

It was C. S. Peirce, the American scientist and philosopher (1839 - 1914), who brought to our attention that we all act most of the time according to habits which are shaped by our beliefs (and from the history of science we know that there have been many false beliefs). But we do not regularly question these beliefs; they are established in the nature of our habitual actions. It is only when doubts about our beliefs are raised that we stop to examine them and start an inquiry in order to appease our doubts and settle our opinions Thus, in Peirce's view, starting inquiry on a certain question is not an arbitrary act, but rather an unavoidable act when some doubt arises When do such doubts arise? When an expectation is not fulfilled because it conflicts with some facts On such occasions when one feels that something is wrong, only then does a real question arise and an inquiry become initiated, an inquiry that should settle our opinions and fix our beliefs [Peirce, 1877].

A similar, though not identical view was strongly advocated by K. Popper [1963]. In his book *Conjectures and refutations* he argues against an idealistic and simplistic view of attaining truths in science. He claims that "Erroneous beliefs may have an astonishing power to survive, for thousands of years" [Popper, 1963, p. 8], and since he does not believe in formulating one method that would lead us to the revelation of truth, he suggests changing the question about "sources of our knowledge" into a modified one — "How can we hope to detect error?" [Ibid p 25]. If we are lucky enough to detect an error we are then in a position to improve our set of beliefs. Thus for Popper science should adopt the method of "critical search for error" [Ibid, p. 26] which has the power of modifying our earlier knowledge.

In the systems of these philosophers which I have only touched upon here, there are several points relevant to learning in general that should be clearly stated:

1) Falsehood is adjunct to the notion of truth, or in the words of Russell: "Our theory of truth must be such to admit of its opposite, falsehood." [Russell, 1912, p. 70]
2) Though having a truth-value is a property of beliefs, this may be established by many different methods and is independent of whether our beliefs will ultimately become true or false (a point which I will take up again later).

3) We hold many beliefs that we are unaware of and which are part of our habits, yet once such a belief clashes with some counter-evidence or contradictory arguments, it becomes the focus of our attention and inquiry.

Is all this relevant to the child's learning? I believe it is. If I replace the terms "true and false" with "right and wrong" or "correct and erroneous" we will find ourselves in the realm of schools and instruction, where, unlike in the philosophical realm, "being wrong" and "making errors" are negatively connotated. The system, in fact, reinforces only "right" and "correct" performances and punishes "being wrong" and "making errors" by means of exams, marks, etc., a central motive in our educational system.

I found it very refreshing when visiting a second grade class to hear the following unusual dialogue:

Ronit (second grader with tears in her eyes): "I did it wrong" (referring to her geometrical drawing).
"Never mind", said the teacher, "What did we say about making mistakes?"
Ronit (without hesitation) answered: "We learn from our mistakes".
"So", added the teacher, "Don't cry and don't be sad, because we learn from our mistakes".

The phrase "we learn from mistakes" was repeated over and over. The atmosphere in the classroom was pleasant and the use of this phrase was the way the children admitted making errors on the given task. At this point I became curious and anxious to know what children really did learn from their mistakes. I will first describe the task, and how the children knew when they made mistakes. Let us now observe a geometry lesson in which the students learned about the reflection transformation. The exercises consisted of a given shape and a given axis of reflection (see Figure 1); the children first had to hypothesize (or guess) and draw the reflected figure in the place where they thought it would fall, and then to fold the paper along the reflection axis and by puncturing the original figure with a pin to see whether their drawing was right or wrong.

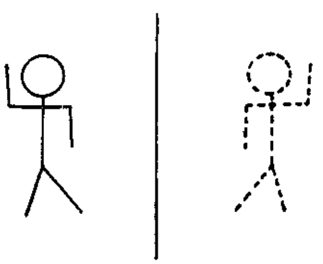

Figure 1

I would like to make it explicit that, from the child's point of view, he or she had to discover the "theory" of reflection. The teacher did not intend to serve as the authority for this knowledge, lecturing about the invariants of reflection, but instead supplied the child with a structured domain against which any erroneous conceptions could be checked. The line of dots created by the pin-puncture served both as an ideal reality for this kind of reflection, and as feedback for the child's conjectures. In my view this resembles in a nutshell scientific inquiry in several important aspects.

Delighted to find such a supportive atmosphere in the classroom, I became interested in the epistemological question: what did the children really learn from their mistakes? When each child who made an error was asked to explain to me what was learned from his or her mistake I could not elicit a clear answer. Instead the children repeated again and again that one learns from mistakes in a way that started to sound suspiciously like a parroting of the teacher's phrase. At this point it became clear to me that the teacher tolerated errors, but did not use them as a feedback mechanism for real learning on the basis of actual performance. I then drew on the blackboard three different errors:

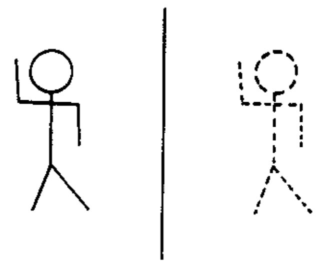

Figure 2

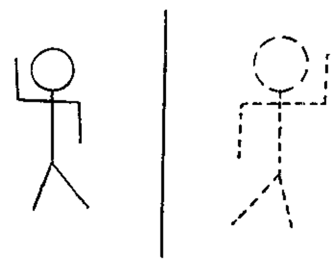

Figure 3

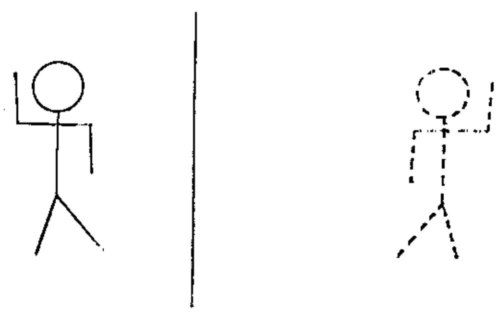

Figure 4

The first one, which I named Sharon's error, dealt with the property that a reflection is an opposite transformation; thus, what was right will become left in the reflection, and vice-versa (see Figure 2). The second error, named after Dan, was dedicated to the size property, i.e. that lengths are invariant under the reflection transformation (see Figure 3). This was also the basis for the third error, named after Joseph, that had to do with the distance from the reflection axis (see Figure 4). I asked the children whether one learns the same thing from each of the above errors? should Sharon, Dan and Joseph learn the same thing? or is there something specific to each error?

At this point we turned from the psychological support and tolerance of errors to discover the epistemological and cognitive value of errors in the process of learning. From errors like these a child could learn distinct properties of reflection that he or she was not aware of before. (If they had been aware, they would not have committed this kind of error.) Committing an error, however, revealed the incompleteness of their knowledge and enabled the teacher to contribute additional knowledge, or lead them to realize for themselves where were they wrong. The clash between their expectations, demonstrated by their drawings, and the "reality" as shown by the pin-puncture, created a problem, an uneasiness (up to tears), that they had yet to settle. The solution to this problem in fact involved the process of learning a new property of the reflection transformation not known to them until then. As Popper [Ibid. p. 222] wrote:

> Yet science starts only with problems. Problems crop up especially when we are disappointed in our expectations, or when our theories involve us in difficulties, in contradictions; and these may arise either within a theory, or between two different theories, or as the result of a clash between our theories and our observations. Moreover, it is only through a problem that we become conscious of holding a theory. It is the problem which challenges us to learn; to advance our knowledge; to experiment and to observe. [Ibid. p. 222]

I think that if we use the word "theory" in not too rigorous a manner, and substitute the word learning for science, then Popper's description is most pertinent to our issue.

III. In the title of this presentation, I did not use the word "error" or "mistake" but rather "misconception". The notion of misconception denotes a line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic, unconnected and non-systematic errors. It is not always easy to follow the child's line of thinking and reveal how systematic and consistent it is. Most studies, therefore, report on classification of errors and their frequency, though this does not explain their source and therefore cannot be treated systematically. Or, when dealt with, it is on the basis of a mere surface-structure analysis of errors, as in the case of "Buggy" [Brown and Burton, 1978; Brown and VanLehn, 1980], where we end up with a huge, unmanageable catalogue of errors. It seems that this lack of parsimony could be avoided if one looked into the deeper levels of representation in which a meaning *system* evolves that controls the surface performance. When an erroneous principle is detected at this deeper level it can explain not a single, but a whole cluster, of errors. We tend to call such an erroneous guiding rule *a misconception*.

I would like to describe now two detailed examples of misconceptions (out of many others) that demonstrate how errors do not occur randomly but rather have their roots in erroneous principles. Moreover, these misconceptions are not created arbitrarily but rely on earlier *learned* meaning systems; and again, although seemingly absurd, they are actually derived from previous instruction. These examples are chosen because they are each based on extensive research programs which deal with unveiling students' misconceptions and focus on plausible explanations for their erroneous performance.

The first example is taken from a series of studies about the nature of errors made by elementary school children in comparing or ordering decimal numbers. In these studies an attempt was made to trace the sources of the students' systematic errors. The findings which emerge, following studies in England, France, Israel and USA [Leonard and Sackur-Grisvald, 1981; Nesher and Peled, 1984; Swan, 1983] show that in all these countries there is a distinct and common system of rules employed by those who fail in comparing decimals.

Consider for example the following tasks which were administered to children of grades 6, 7, 8, and 9. The subjects had to mark the larger number in the following pairs:

case I	0.4	vs.	0.234
case II	0.4	vs.	0.675

Jeremy marked in case I that 0.234 is larger than 0.4; and in case II he marked that 0.675 is the larger one. Does he or does he not know the order of decimal numbers? In our study in Israel the data was gathered in individual interviews so that the children could explain their choices. This helped us understand their guiding principles. In both cases Jeremy said that the number with the *longer* number of digits (after the decimal point) is the larger number (in value). Jeremy had one guiding principle as to the order of decimals and, accordingly, in case I Jeremy was wrong while in case II he was right. Although his guiding principles was a mistaken one, he succeeded in correctly solving all the exercises similar to case II. It is also not hard to see that his guiding principle was one that served him well up to this point, having been imported from his knowledge of whole numbers where the longer numbers really are larger in value. And, unless something is done, Jeremy's "success" or "failure" on certain tasks is going to depend on the actual pairs of numbers given to him. This, of course, blurs the picture of his knowledge in any given test.

Now consider Ruth, who decided in both cases I and II (in the above example) that 0.4 is the larger number, i.e. in each case she pointed to the shorter number as the larger one in value. Ruth gave the following explanation: "Tenths are bigger than thousandths, therefore the shorter number that has only tenths is the larger one" Ruth does not differentiate between case I and case II either She will be correct in all cases similar to case I, but wrong in all cases which are similar to case II. We can understand this kind of erroneous reasoning in the light of what is learned in fractions. Ruth has a partial knowledge of ordinary fractions and cannot integrate what she knows about them with the new chapter on decimal fractions and their notation In particular she finds it difficult to decide whether the number written as a decimal fraction corresponds to the numerator or the denominator. She cannot corrdinate the size of the parts with their number in decimal notation.

It is interesting to note that about 35% of the sixth graders in Israel who completed the chapter on decimals acted like Jeremy and were, in fact, using the abovementioned rule which relies heavily on the knowledge of whole numbers, and about 34% of the Israeli sample of sixth graders made Ruth's type of mistake. Even more interesting is the fact that while Jeremy's rule declines in frequency in higher grades, Ruth's rule is more persistent and about 20% of the seventh and eighth graders still maintain Ruth's rule [Nesher and Peled, 1984]

As I remarked before, these misconceptions are hard to detect. This is because on some occasions the mistaken rule is disguised by a "correct" answer That is, the student may get the "right" answer for the wrong reasons Thus, for the student who holds a certain misconception, not all the exercises consisting of a pairs of decimal numbers will elicit an incorrect answer For example, decimals with the same number of digits are compared as if they were whole number and, therefore, these questions are usually answered correctly. In fact this also related to a method taught in schools: add zeros to the shorter number until it becomes as long as the longer one and then compare them.

An interesting question then emerged: if the teacher is not aware of the cases that discriminate between various types of misconceptions and those cases that do not discriminate misconceptions at all, what is the probability that he or she will give a test (or any other set of exercises) that detect systematic errors? Irit Peled, my former student, in her Ph.D thesis dealt with precisely this question [Peled, 1986] She built a series of simulations that made it possible to evaluate quantitatively the probability of getting discriminating items on a test

Let me return to the question of a discriminating item for a certain error. For example, consider the following item, "Which is the larger of the two decimals 0.4 and 0.234?" If the student answers 0.234 we may suspect that he holds Jeremy's misconception. But, if he answers 0 4 we cannot know whether he knows how to order decimals, or if he is holding Ruth's error but happened to get lucky numbers and be correct on this particular item Thus this item can discriminate between those holding and not holding Jeremy's misconception, but cannot discriminate between those holding Ruth's misconception and experts (i.e those who really know the domain). Along these lines, in the same task, the pair of numbers 0.4 and 0.675 can discriminate those holding Ruth's misconception from the rest, but cannot discriminate between those holding Jeremy's misconception and experts Comparing the numbers 0.456 and 0 895 cannot discriminate students holding either Jeremy's or Ruth's misconception (whether the answers are correct or not).

So, if a teacher composes a test (or any other assignment) without looking intentionally for the discriminating items, there is little chance that such items will be included. In Peled's simulations it was found that when pairs of numbers are randomly selected from all the possible pairs of numbers having at most three digits after the decimal point, the probability of getting items that will discriminate Jeremy's error was 0.10, and Ruth's error 0 02. Thus both Jeremy and Ruth can score up to 90% on a test composed by their teacher if she is not aware of this problem. It is not surprising, then, that teachers are usually satisfied with the performance of children holding Jeremy's or Ruth's misconceptions, and they should not be blamed On the basis of one item answered wrongly it is impossible to discover the nature of the student's misconception.

The teacher could of course increase the difficulty of the test by allowing only pairs of numbers with unequal lengths (up to three digits after the decimal point) This raises the probability of getting discriminating items on the test, but will not insure correct diagnosis of a specific misconception (see Appendix B for a sample test). The probability is that on such a random test Jeremy will get 58% correct and Ruth will get 48% With awareness of the problem, the teacher can design a test to intentionally diagnose and discriminate the known misconceptions to a proportion and distribution already determined

Teachers, however, are hardly aware of such an analysis of misconceptions. Some of them listening to our report could not believe the existence of Ruth's type of misconception at all until they returned to their classes and found it for themselves. Teachers do not generally build such knowledge into their instruction and evaluation of the student's performance Frequently the teacher completes the section of instruction on comparing decimals, gives a final test, and believes that the children know it perfectly well, not noticing that many of them still hold important misconceptions such as Jeremy's and Ruth's, as we and others have found in our studies. In such a classroom it will also be very difficult for Jeremy or Ruth to give up their misconceptions since they are daily rewarded for their erroneous guiding principles by correctly answering non-discriminating items.

Several lessons can be learned from these studies:

a) In designing the instruction of a new piece of knowledge it is not enough to analyze the procedures and their prerequisites — which is, in many cases, done We must know how this new knowledge is embedded in a larger meaning system that the child already holds and from which he derives his guiding principles.

b) It is crucial to know specifically how the already-known procedures may interfere with material now being learned. In the case of decimal knowledge a fine analysis will show the similarity and dis-similarity between whole numbers and decimals, or between ordinary fractions and decimals.

Some of the elements of earlier knowledge may assist in the learning of decimals, but some of them are doomed to interfere with the new learning, because of their semi-similarity (see Appendix A).

c) All the new elements, which resemble but differ from the old ones, should be clearly discriminated in the process of instruction, and the teacher should expect to find errors on these elements. Needless to say, although they elicit more erroneous answers, such elements should be presented to the children and not avoided

My second example is taken from a series of studies by Fischbein et al. [1985]. In their study Fischbein's group claimed that in choosing the operation for a multiplicative word problem (let's say, choosing between multiplication and division) students tend to make specific kinds of mistakes derived from their implicit intuitive models that they already have concerning multiplication Thus identification of the operation needed to solve a problem does not take place directly but is mediated by an implicit, unconscious, and primitive intuitive model which imposes its own constraints on the search process. The primitive model for multiplication is assumed to be "repeated addition".

The data supporting their hypotheses is based on the following findings Multiplication word problems in which according to the context, the multiplier was a decimal number (e.g. 15 × 0 75) yielded 57% success, while those consisting of a decimal number in the multiplicand (0.75 × 15) yielded 79% success. Fischbein's group attributed this to the fact that the intuitive model of multiplication as repeated addition does not allow for a non-integral number as a multiplier

Similarly, in division contexts when the numbers presented in the word problem were such that the students had to divide a smaller number by a larger one, they reversed the order and divided the larger one by the smaller, so that it would fit their previous notions of division. It also became apparent in this series of studies that students hold the misconception that "multiplication always makes bigger" [Bell et al., 1981, Hart, 1981] Fischbein's research paradigm has been repeated several times with different populations, always yielding the same results. [Greer and Mangan, 1984; Greer, 1985; Tirosh, Graeber and Glover, 1986; Zeldis-Avissar, 1985]

This set of misconceptions, again, is not easy to detect This is where research can directly affect school teaching. The probability of the occurrence of multiplication and division word problems in the textbooks that detect such misconceptions is low. In the absence of items or problems purposely directed to detect misconceptions we are shooting in the dark We are likely to put too much emphasis on trivial issues while overlooking serious misconceptions.

There is another lesson from these studies which is harder to implement. We can trace the sources of major misconceptions in prior learning. Most of them are overgeneralizations of previously learned, limited knowledge which is now wrongly applied. Is it possible to teach in a manner that will encompass future applications? Probably not. If so, we need our beacons, in the form of errors, that mark for us the constraints and limitations of our knowledge.

IV. So far what I have said suggests that teachers should be more aware of their students' possible misconceptions and incorporate them into their instructional considerations But this is not sufficient, and I would like to return to the example of the second graders working on the reflection transformation.

Let us suppose that in designing the pin-puncture booklet the teacher was aware of the possible misconceptions and included all the discriminating items she could think of. However, another significant characteristic of the booklet is that it enables the child to decide for himself whether he is right or wrong and in what respect is wrong. This is possible because the rules by which the pin-puncture behaves are dependent only on mathematical reality and not on the learner's beliefs. The fact that the rules of mathematics and one's set of beliefs are independent allows for discrepancies between them. Therefore when the student holds a false belief, or a false conjecture, it clashes with "reality" as exemplified in the booklet. This kind of instructional device enables the child to pursue his own inquiry and discover truths about the reflection transformation, and at the same time make errors resulting from his misconceptions, some of which could not have been anticipated by the teacher. He is working within what I call a *Learning System,* a conception to which I will devote the rest of my talk.

A Learning System (LS) is based on the following two components:

1) an articulation of the unit of knowledge to be taught, based upon expert knowledge, which is referred to as the *knowledge component* of the system, and
2) an illustrative domain, homomorphic to the knowledge component, and purposely selected to serve as the *exemplification component*

Although "microworld" may seem a natural choice of term for a *Learning System,* I prefer to use a different term since "microworld" is sometimes identified with the exemplification component only, and sometimes with the entire Learning System I have therefore introduced the term "Learning System" to ensure we understand that a microworld encompasses both components. Various concrete materials employed in the past, such as Cuisenaire Rods, or Dienes' Blocks [Gattegno, 1962; Dienes, 1960] serve as illustrative aspects of Learning Systems. Moreover, I believe that the rapid progress of computers in the last decade, with their tremendous feedback power, will lead to the development of many more such Learning Systems.

The *knowledge component* in a Learning System is articulated, not by experts who are scientists in that field, but rather by those who can tailor the body of knowledge to the learner's particular constraints (age, ability, etc) and form the learning sequence. In order for the *exemplification component* to fulfil its role, it must be familiar to the learner. He should intuitively grasp the truths within this component It is necessary that the learner while still ignorant about the piece of knowledge to be learned, be well acquainted with the exemplification so that he can predict the results of his actions within this domain and easily detect unexpected outcomes the familiar aspects of the Learning System provide an anchor to which to connect an understanding of the new concepts and new relations to be learned.

Familiarity, however, is not sufficient The selection of the exemplification component should ensure that the relations and the operations among the objects be amenable to complete correspondence with the knowledge component to be taught For example, in the case of teaching and reflection transformation, the exemplification by the pin-puncture corresponds more to the knowledge component than a mirror does which enables reflection of only one half of the plane on the other (There are some other advantages as well which I will not go into here.)

The gist of the Learning System is that we have a system with a component *familiar* to the child, from previous experience, which will be his stepping stone to learn *new* concepts and relationships, as defined by the expert in the knowledge component. A system becomes a *Learning System* once the *knowledge component* and the *exemplification component* are tied together by a set of well-defined correspondence (mapping) rules. These rules map the objects, relations and operations in one component on to the objects, relations, and operations of the other component

Functioning as a model, the exemplification component of a Learning System must fulfil the requirements described by Suppes [1974], i e. it must be simple and abstract to a greater extent than the phenomena it intends to model so that it can connect all the parts of the theory in a way that enables one to test the coherence and consistency of the entire system This forms the basis for the child's ability to judge for himself the truth-value of any given mathematical conjecture in a specific domain. It provides the learner with an environment within which he can continuously obtain comprehensible feedback on his actions, as was apparent from the second graders' behavior

I believe that arriving at mathematical truths is the essence of what we do in teaching mathematics This brings me back to the question I raised at the beginning of my talk about mathematical truths. This is a deep philosophical question that I will not delve into here, recalling instead Russell's formulation on the correspondence theory of truth. Russell [1959/1912] clarifies the fact that truth consists in some form of a *correspondence* between belief and fact Thus, though the notion of truth is tied to an expressed thought or belief, by no means can be it determined by it. *An independent* system of facts is needed against which it is tested. This, however, is not the only theory of truth. In the same chapter Russell also mentions a theory of truth that rests on *coherence* He writes that the mark of falsehood is a failure of coherence in the body of our beliefs

How children arrive at truths is problematic Clearly the child cannot reach conclusions about the truths of mathematics with such rigorous methods as those applied by a pure mathematician While mathematicians can demonstrate the truth of a given sentence by proving its coherence within the entire mathematical system, young children cannot If a young child is to gain some knowledge about truths in mathematics not based on authoritative sources, he must rely on the correspondence theory of truth rather than on the coherence theory. Thus he should examine the correspondence between his belief and the state of events in the mathematical world. In our example this correspondence is between his conjecture — where to draw the image of a reflection — and the result of his pin-puncture, representing mathematical reality

But this approach is not without its difficulties. Employing exemplifications as the source of verification commits one to introducing mathematics as an empirical science rather than a deductive one. On the other hand, I believe that young children and even many not so young will be unable to reach mathematical truths merely by chains of deduction without first engaging in constructing and feeling intuitively the thrust of these truths Therefore I think that constructing a world in which the learner is able to examine the truth of mathematical sentences *via* an independent state of events is the major task for any future theory of mathematical instruction. Such a world, which I have labelled a Learning System, is one in which all our knowledge about true conjectures as well as of misconceptions are built in as its major constraints. Limited by the System's constraints, the child will learn by experimentation and exploration the limitations and the constraints of the mathematical truths in question. On this basis can he later attend to the more rigorous demands of deductive reasoning

V. In summary I would like to recapitulate several points touched on today. At the moment, unlike the promise in the title of this presentation, my remarks do not look like a theory at all; rather they specify some assumptions that, in my view, will underlie any future instructional theory.

a) The learner should be able, in the process of learning, to test the limitations and constraints of a given piece of knowledge. This can be enhanced by developing learning environments functioning as feedback systems within which the learner is free to explore his beliefs and obtain specific feedback on his actions.

b) In cases where the learner receives unexpected feedback, if not condemned for it, he will be intrigued and motivated to pursue an inquiry.

c) The teacher cannot fully predict the effect of the student's earlier knowledge system in a new environment Therefore before he completes his instruction, he should provide opportunity to the student to manifest his misconceptions, and then relate his subsequent instruction to these misconceptions.

d) Misconceptions are usually an outgrowth of an already acquired system of concepts and beliefs wrongly applied to an extended domain. They should not be treated as terrible things to be uprooted since this may confuse the learner and shake his confidence in his previous knowledge. Instead, the new knowledge should be connected to the student's previous conceptual framework and put in the right perspective

e) Misconceptions are found not only behind erroneous performance, but also lurking behind many cases of correct performance. Any instructional theory will have to shift its focus from erroneous performance to an understanding of the student's whole knowledge system from which he derives his guiding principles

f) The diagnostic items that discriminate between proper concepts and misconceptions are not necessarily the ones that we traditionally use in exercises and tests in schools. A special research effort should be made to construct diagnostic items that disclose the specific nature of the misconceptions.

I have tried to examine instructional issues from the misconception angle The examination consisted of more than an analysis of pedagogical problems; it had to penetrate epistemological questions concerning truth and falsehood. Delving into questions of knowledge has traditionally been the prerogative of philosophy, particularly epistemology. Mental representation and the acquisition of knowledge, on the other hand, have been dealt with in the field of cognitive psychology Obviously, each discipline adopts a different stance when dealing with the study of knowledge. While philosophers are concerned with questions related to the sources of knowledge, evidence and truth, cognitive scientists are mainly interested in questions related to the representation of knowledge within human memory and to understanding the higher mental activities

The educational questions are quite different. The agenda in education is to facilitate the acquisition and construction of knowledge by the younger members of society While scholars of cognitive science and recently of artificial intelligence are interested mainly in the performance of experts who are already skilled in various domains, educators, on the contrary, are interested in naive learners, or novices and how they develop into experts. My claim is that the road to a state of expertise is paved with errors and misconceptions Each error has the potential to become a significant milestone in learning. Let these errors be welcomed.

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Appendix A

A Random Comparison Decimal Test
(Numbers up to three decimal digits)

.66	.154	Discriminating Jeremy's rule
.254	.045	Not discriminating
.122	002	Not discriminating
101	067	Not discriminating
885	.106	Not discriminating
.238	.433	Not discriminating
.233	244	Not discriminating
.713	838	Not discriminating
.245	.885	Not discriminating
806	.702	Not discriminating

A Random Comparison Decimals Test
(Unequal lengths of numbers up to three decimal digits)

.15	.114	Discriminating Jeremy's rule
.185	.06	Discriminating Ned's rule (Not discussed here)
.51	.446	Discriminating Jeremy's rule
.31	438	Discriminating Ruth's rule
.861	.33	Discriminating Ruth's rule
.606	82	Discriminating Jeremy's rule
.72	722	Discriminating Ruth's rule
.08	.822	Discriminating Ned's rule
814	.46	Discriminating Ruth's rule
404	33	Discriminating Ruth's rule

Appendix B

Knowledge of Decimal Fractions: Identifying Place Value of Individual Digits

Elements of Decimal Knowledge	Corresponding Elements of Whole Number Knowledge	+ or -*
A Column Values:	A Column Values:	
1 Correspond to column names	1 Correspond to column names	+
2 Decrease as move l to r	2 Decrease as move l to r	+
3 Each column is 10 times greater than column to r	3 Each column is 10 times greater than column to r	+
4 Decrease as move away from decimal point	4 Increase as move away from ones column (decimal point)	-
B Column Names:	B Column Names:	
1 End in <ths>	1 End in <s>	-
2 Start with tenths	2 Start with units	-
3 Naming sequence (tenths, hundredths. .) moves l to r	3 Naming sequence (tens, hundreds...) moves r to l	-
4 Reading sequence is tenths hundredths thousandths	4 Reading sequence is thousands hundreds, tens, ones	- -
C Role of Zero:	C Role of Zero:	
1 Does not affect digits to its left	1 Does not affect digits to its right	-
2 Pushes digits to its right to next lower place value	2 Pushes digits to its left to next higher place value	-
D Reading Rules:	D Reading Rules:	
1 The number can be read *either* as a single quantity (tenths for one place, hundredths for two places etc) *or* as a composition (tenths *plus* hundredths etc)	1 The number can be read as a single quantity and as a composition at the same time (e g , *seven hundred sixty two* means seven hundreds plus six tens plus two)	-

Knowledge of Decimal Fractions: Identifying Place Value of Individual Digits

Elements of Fractional Decimal Knowledge	Corresponding Elements of Ordinary Fraction Knowledge	+ or -*
E Fraction Values:	E Fraction Values:	
1 Expresses a value between 0 and 1	1 Expresses a value between 0 and 1	+
2 The more parts a whole is divided into the smaller is each part	2 The more parts a whole is divided into the smaller is each part	+
3 There are infinite decimals between 0 and 1	3 There are infinite fractions between 0 and 1	+
F Fraction Names:	F Fraction Names:	
1 The number of parts divided into is given implicitly by the column position	1 The number of parts divided into is given explicitly by the denominator	-
2 The number of parts included in the fractional quantity are the only numerals explicitly stated	2 The number of parts included in the fractional quantity are the numerator of the fraction	-
3 The whole is divided only into powers of 10 parts	3 The whole is divided into any number of parts	-
4 The ending "-th" ("tenth") is typical for a fractional part	4 The ending "-th" ("fourth") is typical for a fractional part	+

* Supports (+); contradicts (-)