Colophon

Perhaps, one general lesson to be learned (relevant to our topic, but also applicable beyond it), is to face questions and dilemmas in all their depth and scope, to sharpen them, and to continue the systematic search for more answers. A cautionary note is due here: even when many questions may be solved by research, still others are strongly imbued by deep epistemological beliefs and philosophical standpoints. Thus, we should continue our dialogues within ourselves and with all people genuinely concerned with mathematics education who are willing to learn the trade and enter the fray. We need to be mindful in order not to confine ourselves to the voicing of strongly held beliefs – such a practice only serves the pushing of the pendulum into either one of the extreme positions of the past.

I hope this communication serves to fuel the continuation of a productive dialogue which unfolds subtleties and complexities and avoids extremes.

Notes

[1] $x + (1/7)x = 19$

[2] As opposed to misconceptions.

[3] Other intermediate models worth discussing in this context are the *Lab Gear* (Wah and Picciotto, 1994, p. 211) and a proto-symbolic approach (Bruckheimer and Arcavi , 1999).

[4] A further discussion can be found in the papers, and subsequent reactions to them, at the Research Forum on "Early Algebra" in van den Heuvel-Panhuizen, M. (ed.), 2001, *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands, Freudenthal Institute, **1,** pp. 129-159.

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Early algebra: perspectives and assumptions

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In FLM **24**(2),

> *Dickson and Eade* describe how a number line for solving linear equations can be used with 11-year-old children.
>
> *Brizuela and Schliemann* present a function approach to ten-year-old students as part of a method for solving linear equations.

Both articles suggest that students were successful with these methods, not only in applying them but understanding the rationale behind them as well.

Certainly, the authors of these articles present compelling arguments for placing algebraic topics earlier in the elementary mathematics curriculum. The methods for solving linear equations presented here are appropriate for younger children and offer the opportunity to gain an understanding of what solving an equation means.

However, both articles make the assumption that students understand equalities, quantified relationships, and units. Without these three understandings, the approaches suggested by the authors would result in either a rather algorithmic application of a function or a diagrammatic method for solving equations. Additionally, the articles intimate that solving linear equations is an add-on to what could be considered a traditional curriculum. That is, students' background in mathematics in the earlier grades would have focused on number and operations presented in a manner consistent with a conventional approach.

Let's step back for a moment and ask 'what if. . .?' What if students began their mathematical journey at age six in a different place other than arithmetic? What if students had the opportunity to explore quantified relationships intuitively and develop conjectures about operations on them? What if number was delayed so that students could focus on bigger ideas first before they tried working with specific cases, like natural numbers?

My comments stem from the research currently underway at the University of Hawai'i's *Curriculum Research and Development Group* in a project called *Measure Up* (Dougherty, Okazaki, Zenigami, Slovin, Venenciano and Thatcher, 2004; Dougherty and Zilliox, 2003) This project, based on the work of Davydov and colleagues (1975), promotes a generalized approach that is algebraic in nature and seated within a measurement context.

For much of the work done in mathematics with the youngest children, we have placed ourselves into a box that dictates that number and counting are the first topics children must experience if they are to be successful in any future endeavors in mathematics. But despite the general agreement that this is the place to start, there is also the recognition that this beginning does not necessarily promote success when these same students confront a formal algebra class when they are older. Dickinson and Eade made multiple references to well-publicized discussions of issues related to depressed success, including the view of the equals sign as an action symbol.

In order, then, for older children to solve equations with meaning, we have to first 'undo' their ideas about the equals sign before any approach to solving equations makes sense. Then we have to teach the new method, hopefully in a way that promotes understanding. However, any approach brings new mathematical baggage into the picture.

Take, for example, the diagrammatic method that Dickinson and Eade suggest. A number line, empty or not, is based on the use of a unit to establish a length. In the very early years, few children have the opportunity to create a number line; they are usually given one to model the actions of addition or subtraction or to see the sequence or progression of numbers. When asked to find 15 on the number line, students point to the number 15, not realizing that 15, in this case, is the length from 0 to 15.

In the equation $3x + 4 = 19$, students have to realize that, in using the Dickinson and Eade model, there are two lengths, one that measures $3x + 4$ and one that measures 19. The variable represents a quantity, iterated three times. Each iteration of that quantity has the same number of units of length. If children as young as six years old have had experience with length Y (rather than only with lengths that are specified with standard units), they can visualize why and how a length such as $3x + 4$ can be constructed. In fact, six- and seven-year-old students can solve equations such as this with arguments as to the reasonableness of its solution in a similar fashion as stated above. A length of x units would have to have 5 units in order for the two quantities or lengths to be equal.

In the Brizuela and Schliemann model, 19 would have to be thought of in a part-whole way as 15 and 4 so that 4 would 'match up' with the 4 on the left side of the equation. The remaining 15 would be thought of as three parcels of five and so each parcel would match up with one of the x's. This approach seems readily understandable by students but, as with the other model, there are assumptions about students' thinking. Most importantly, this approach requires that students have the ability to think of a number or other quantities in part-whole relationships. That is, 19 can be thought of as being made up of the parts 18 and 1, 19 and 0, 8 and 11 and so on with students selecting the most appropriate part-whole relationship for the problem. More importantly, however, is the notion that $3x + 4$ is a quantity, such as length, that can be thought of as a length or quantity of $3x$ and another length or quantity of 4. The function and the diagrammatic approach are thus similar in that there are underlying assumptions about what students understand with regard to relationships of quantities.

The equality relationships presented in the problems in both articles assume that students understand properties of equivalent relationships. As Brizuela and Schliemann note, students cancel out, or match up, like quantities. In the Dickinson and Eade model, there is the implication of cancelling out from above and below the line. While we might suggest that students do this intuitively, it is not always the case that they see the fundamental reason why this is a legitimate method.

If students begin very early, in looking at quantities, say volume, they notice that two volumes are equal. Furthermore, they can explain that the equivalent relationship is maintained if equal amounts are added to (or subtracted from) the original quantities. Rather than imposing a number-based rationale in later years, students can begin their mathematical thinking about equivalent quantities with volume, area, length, or mass and consequently develop visual pictures that help them model situations with understanding.

The idea of cancelling out is replaced with an algorithmic method in that students may only see two like representations without establishing the meaning of the 'cancelling' action. The notion that 'if equal amounts are taken from two quantities that have been established as equal then the new quantities are also equal' is the conceptual foundation upon which methods for solving equations are built. This idea extends to working with inequalities that present even more difficulties for students.

Solving equations is a large part of a formal algebra program and so it is appropriate that these articles focus on them. However, it is also important to consider ways in which early mathematics can be structured so that these and other methods are readily accessible to students. Without a different start in the early grades, teachers will be forced to find ways to help students create a much deeper understanding of quantitative relationships before methods such as the ones proposed in these articles can be used appropriately.

Showing that students can solve equations with different methods at an earlier age is encouraging. If they are capable of using these methods, even after coming from an almost strictly numerical perspective in their early beginnings in mathematics, what would be possible if students started with a focus on the structure of mathematics within a measurement context? The measurement context would give students the opportunities to model and visualize the interactions across and within quantified relationships and would form the basis for developing algebraic methods related to skill acquisition. Students could confidently approach routine and non-routine problems, modelling them in multiple ways, with an understanding of why their method is appropriate.

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What is wrong with the teaching of algebra?

ISTVÁN LÉNÁRT

> [I]f we can present evidence of younger, elementary school children engaging with algebra, and using and understanding the syntactic rules of algebra, we have to ask ourselves why so many adolescents face difficulties with algebra. Perhaps [...] the teaching or curriculum to which the students have been exposed has been preventing them from developing mathematical ideas and representations they would otherwise be capable of developing.
>
> It is our belief that, as previously stressed by many, the difficulties middle- and high-school students have with algebra result from their previous experiences with a mathematics curriculum that focuses exclusively on arithmetic procedures and computation rules. (Brizuela and Schliemann, p. 33)

Brizuela and Schliemann, in FLM **24**(2), give a convincing argument, with documentation, in support of their standpoint. Far from casting any doubt on their reasoning, I want to describe a complementary approach to the same problem. Hopefully, the interference of different opinions will lead not to a weakening, but a strengthening of our mutual efforts to reach the truth!

My focus is that adolescents' problems with algebra originate not only in their *previous* experiences, but also in their *simultaneous* middle- and high-school experiences within their algebra curriculum. In other words, the problem lies not only with the preliminary, introductory stage, but also with the actual form of presenting these ideas in the classroom.

What does 'algebra' mean for general education? What does a mathematics teacher want to teach under the name 'algebra' for middle- or high-school students? What is the central message of school algebra to the students?

It would appear, from my reading of the same article, that many educators and researchers apparently believe that the main objective of teaching algebra is to reach a well-understood and precisely performed handling of equations with constants denoted by numbers, and unknown quantities denoted by letters.

Another method of teaching the basics of abstract algebra lies in the algebraic formalisation of geometric transformations, that is, through the concepts of transformation groups. However, my personal experience is that, for the majority of high-school students (and also for many college and university students), this way to abstract algebra is not a viable one. For a beginner, it is hard, often impossible, to accept that a transformation of infinitely many different geometric shapes into infinitely many other geometric shapes might be considered as a single element of an algebraic set.

In my own case, I had problems in my university years understanding the first step of vector algebra, namely, all vectors with the same length and direction constitute one and the same element in a certain set of algebraic objects. I thought this theory stupid if it could not tell two vectors apart that I found clearly distinguishable from each other!

So, what does algebra mean for me? If I agree – as I do – that algebra is among the most powerful thinking tools of modern mathematics, what is my main (but not by any means the only) aim in teaching abstract algebraic concepts to an average high-school student (or first-year college student or university student)?

I think that, under the name of 'algebraic concepts', we should not teach one fixed algebraic system that takes its origin from sets of numbers and the operations amongst them. Instead, we should focus on the ways and means of creating a given world of algebra, changing to another world if necessary. The educational task is to teach about the art of creation in algebra, or, in a broader sense, in any branch of mathematics.

This method of teaching has a message even for a future *non*-mathematician who does not care much about the algebraic language of equations or the associative property of group theory.

Our goal can perhaps best be achieved by alternative models, given that the ancient tradition of introducing algebraic concepts via the language of arithmetic has given problems. I think that abstract algebraic concepts are best introduced by alternative, non-arithmetical micro-worlds.

Such a micro-world should be close enough to the topics usually labelled as 'algebra', but, at the same time, distant enough from students' former algebraic experiences to be rewarding enough to make the hard abstraction to algebra. Probably, one of the reasons why adolescents do not recognise abstract algebraic concepts is that these abstract concepts are not really striking enough for them, compared with the well-known (to them) properties of numbers. Students may feel that abstract algebra only speaks about the properties of numbers under funny pseudonyms.

Still another requirement, perhaps the hardest of all, is that the alternative micro-world should not be more complex, more difficult to grasp, than the traditional worlds that are usually offered in the curriculum.