Understanding: Instrumental, Relational, Intuitive, Constructed, Formalised ...? How Can We Know?

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If we, as mathematics educationalists, are to devise effective teaching strategies, make sense of pupils' actions, provide experiences which enable children to construct their own mathematical concepts, we must first have a viable model of understanding on which to build This statement, however, is of course too simplistic. In actuality, we can never fully comprehend "understanding" itself. As Piaget [1980] claims for all knowledge, with each step that we take forward in order to bring us nearer to our goal, the goal itself recedes and the successive models that we create can be no more than approximations, that can never reach the goal, which will always continue to possess undiscovered properties. What we can, however, do is attempt to categorise, partition and elaborate component facets of understanding in such a way as to give ourselves deeper insights into the thinking of children Skemp [1976, 1978, etc.] proposed a distinction between relational and instrumental understanding, Buxton [1978] introduced the idea of a formal or logical understanding, Backhouse [1978] elaborated on symbolic understanding, Haylock [1982] suggested that it was useful to consider understanding as the creation of connections between new and existing knowledge and Herscovics and Bergeron [1983] defined four levels of understanding Further examples abound. Such theoretical categorisations and metaphors are undoubtedly valuable, but we must take especial care when, in practice, we attempt to describe the thinking of a particular pupil as evidenced by her actions.

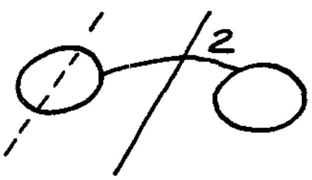
This paper arises from an incident which occurred during a pupil interview while collecting data for a project entitled "Discussion: is it an aid to mathematical understanding?" Since talk within a classroom can be of a very varied nature, both in quality and content, a definition of discussion has been constructed and then taken as axiomatic throughout the research: discussion is purposeful talk on a mathematical subject in which there are genuine pupil contributions and interactions A brief description of part of the project is necessary to set the context for the arguments which follow

Four mathematics teachers, each of whom consciously uses pupil discussion as part of his personal teaching style, form the core of the project Each teaches a first year secondary class (11-12 years old) and a small group of pupils in each class is the focus of the research During the first six months, four weeks were spent observing each class throughout every mathematics lesson and recording all the interactions of the chosen groups. The first of these weeks was considered an acclimatisation period for both the observer and the pupils The small groups were later interviewed away from the classroom.

All four teachers have evolved quite distinctive styles of behaviour in their classrooms both for themselves and for their pupils. That of the teacher at Alsworth, the school where the incident currently being considered took place, is best described as "promoting whole class discussion" Typically he will start the lesson by offering the pupils an idea, a problem or a piece of mathematics, pupils are encouraged to offer their thinking to the rest of the class and, frequently, lively open discussion takes place The teacher virtually excludes himself from this forum, both physically, by sitting unobtrusively to one side, and orally by deflecting direct questions back to the pupils His intention is that the pupils shall form their own intuitions about the structures of mathematics Such whole-class discourse may last for up to half an hour or may be refocussed by a problem from the teacher, contrived to bring out or refute the substance of the discussion At this point pupils will spontaneously group themselves in two's and three's to work on the problem, but are permitted to move freely to another group to check their ideas The bedrock on which this style is built is the belief that knowledge must be actively constructed and not passively received if the pupils are to truly learn mathematics.

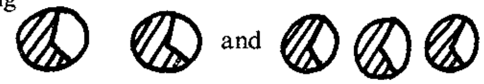
Visits by the observer were timed so that the beginning of the main three-week observation period would coincide with the beginning of a new topic. During the acclimatisation week at Alsworth the pupils were working on division of and by a fraction before moving on to work on graphs It must be stressed that there was absolutely no intervention by the researcher in the content, sequence, or method of teaching, and therefore the topics and strategies used carry no outside endorsement as to their worth. In the previous week, pupils had worked on multiplication of fractions. Throughout the week on division the pupils used no concrete materials, and the only non-verbal, non-mathematical symbolisation used was pictures of circles appropriately shaded.

The initial problem on the board for discussion was $(1/2) \div 2$, although the teacher suggested that they first clarify their thoughts on the meaning of division, by considering $6 \div 2$. Expressions such as "six cut in half", "six split into two pieces", "six shared between two people", "how many two's in six?", were exchanged and applied to the original problem. The idea of halving and sharing led easily to the answer of a quarter, but the class struggled with the "how many?" approach for some time before all professed to being convinced by arguments such as that of Robert "there is only half a two in one, so there is a quarter in half of one", accompanied by this diagram on the board.



The second problem that the pupils were asked to consider was $1 \div (1/2)$. Initially, analogy with previous explanations proved difficult: "cut into a half piece" was rejected and "shared" offered no clarification although one pupil tentatively tried "if one is a half then 2 is a whole". It seemed as if, perhaps since it did not prove illuminating in the previous problem, the class had not thought of using the "how many?" approach. The teacher suggested they review their meanings for division, leading them to formulate "how many halves in one?" which produced a comprehensible, "draw-and-count"-able method for solution. It was clear that many of the pupils derived great satisfaction and a feeling of power from accurately dividing integers by unit fractions and fractions by unit fractions with the same denominator. After allowing them their euphoria, the teacher wrote "$1 \div (2/3)$" on the board. This caused no obvious problem — the answer was "1". It has been widely reported [Carpenter et al, 1983; Schoenfeld, in press] that 29% of the American students taking part in a National Assessment of Educational Progress mathematics test gave the response "31 remainder 12" when faced with the question: "An Army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?" Far from descending into the meaningless rubric of remainders, the Alsworth pupils considered the real problem of "how many (2/3)s can you cut from a whole one?".

The teacher extended this to $2 \div (2/3)$ and $3 \div (2/3)$, drawing



in response to their answers.

At this point one of the pupils, Katie, noticed that if it could be "in bits" there were further two-thirds to be had. Discussion broke out, which did not seem to have been resolved to the satisfaction of all by the end of the lesson. In the final lesson the teacher wrote numerically on the board a short list of unit fraction divisor problems; he added to this as certain pupils became ready for more. The pupils worked alone or in pairs and when requesting help from the teacher were often told to refer their problem to another, specified, pupil. There was suprisingly little "telling" of answers but a fair amount of checking and arguing over different solutions. Katie and Anna worked together, drawing circular diagrams to represent each calculation, as an aid to solution, and using Robert as a referee. By half way through the lesson Katie had noticed that "you keep multipling by that bottom number". The teacher suggested at this point that she and Anna go back to considering the problem of $1 \div (2/3)$. Discussion ranged around "the bit left over" which suddenly crystalised as "a half of the two-thirds piece" and "there are one and a half two-thirds in one". $3 \div (2/3)$ became "three and three half two-thirds", in fact "three and another whole two-thirds and a half two-thirds". A little later, having solved

$$6\tfrac{1}{3} \div \tfrac{2}{3}$$

in the following way:

$$6+3+\frac{1}{2} = 9\frac{1}{2}$$

Katie pronounced this as "boring, all these little circles" and suggested that given 6 whole ones there would be 6×3 thirds and half that number of two-thirds, so for $6\tfrac{1}{3}$ there would be 19 thirds and $9\tfrac{1}{2}$ two-thirds. The concrete, visual representations of the fractions and the operations were being discarded, to be replaced by mental images and numerical symbolisations of fractions. "Let's see if it works for other numbers" (Katie) produced the problem $2\tfrac{1}{4} \div \tfrac{2}{3}$. "Times by 3" gave $6\tfrac{3}{4}$ and "half of it" gave rise to a discussion which led to agreement that since half of a quarter was an eighth, half of three-quarters was one-and-a-half eighths. An instant problem arose when trying to write this down as Anna would not allow

$$\frac{1\frac{1}{2}}{8}$$

and Katie pronounced "that's silly it must be $\tfrac{3}{8}$". Here the lesson ended with the teacher saying they were to do six of their own "fraction divisions" as homework, and that they "might find a pattern, or maybe not". At the beginning of the next lesson the observer asked Katie how she had got on with her homework and was told "Oh, they're easy, you just times the bottom number and see how many 2's or 4's or whatever go into it, like if it was divided by $\tfrac{4}{5}$ then times by 5 and see how many 4's or whatever go into it and if there is a bit left you divide it into fourths, well quarters, or something". She had consolidated the tentative method of solution which she had evolved, towards the end of the last lesson, and appeared no longer to need the medium of the diagrams to aid her thinking. Between then and the interview eight weeks later, no further work was done on fractions, nor did the pupils cover any topics in which incidental use of fractions might have occurred.

Katie and Anna were interviewed on the topic of graphs, the content focus of the observed, three-week teaching session, and as a last, irrelevant question were asked what they rememebered of the work they had done on fractions. The expected reply was either a lack of recall (unlikely) or a reasoned description of the way they had been solving the problems in class. Katie's totally unanticipated response on division "You just turn it upside down and multiply" left the crestfallen observer with no time to follow up the remark!

It would seem that Katie, who had not, as is common in teaching this topic, been presented with an algorithm, nor yet overtly formulated it for herself in concise language while working on fraction division, somehow, during the intervening weeks, crystallised understanding into a verbally explicit, operational rule. What then is her understanding? What is her knowledge of the operation of division when applied to fractions? How was this constructed and internalised? Might it be fruitful at this point to pursue Piaget's distinction between knowledge abstracted from action—the drawing, dissecting and counting applied to circle diagrams—and that abstracted from the properties of fractions themselves? Certainly one might enquire into her current concept of a fraction. Katie, it would seem, had abstracted a "doing strategy" from her personally constructed understanding of fraction division. It would be valuable if one could discover more about the way in which she did this, although the realisation of her thinking could always be only from the viewpoint of the observer. An algorithm is not of itself knowledge, it is a tool whose use is directed by mathematical knowledge and care must be taken not to confuse evidence of understanding with the understanding itself.

It became possible to interview Katie and Anna again some two months later. A colleague of the observer, who had not attended any of the classes, was also present at the interview with a view to discouraging Katie from making tacit assumptions of shared memory of incidents during the lessons. The interview was not tightly structured since it was essential to be able to follow any leads that Katie offered. The intention was to elicit her definitions of multiplication, division and fractions and to probe her understanding through a "think aloud" technique. Another perspective on her understanding was sought using her intuitions and estimations of the likely size of an answer. Extracts and an analysis of the interview follow.

Int: How would you do this?

(*Writes 2/3×2/5*)

Anna: Hang on. First, nothing goes either way, does it?

Katie: No. Two times two on the top and three times five on the bottom

(*Writes = 2×2/3×5*)

Anna: Then it's four on the top and fifteen on the bottom

(*Writes = 4/15*)

Some clarification sought of Anna's first statement and general discussion of cancelling.

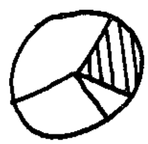
Int: Right, so that's the final answer. Can you tell me what this (*points to written 2/3×2/5*) actually means? What does two-thirds times two-fifths mean?

Katie: Two divided by three, times two divided by five

The two girls have a "how to do it" understanding of a written multiplication of fractions problem and this is underlined by the definition of a fraction as an operation rather than an object. A study of secondary pupils in Israel [Sfard, 1987] proposes the theory that "an operational conception is for most pupils the first step in acquisition of a new mathematical idea, even if it is not deliberately fostered at school". Knowing that the pupils had met at least one representation of fractions: the traditional "fractions as pieces of a pie", the interviewer asked whether the pupils could "draw a picture that would help to explain it ($^2/_3 \times ^2/_5$)". The following commentary accompanied the drawing of these pictures:

Katie: It's just two-thirds of something times two-fifths of something. and that (*the answer*) is just that bit there. Oh no, hang on, it's not. No it's not. It should be more than that so I'll make it...

No, it shouldn't be that bit there. I'll draw it again.



Int: How did you work out its size?

Katie: Well, 3 goes into 15 five times, so if you divide it roughly into 3 then each bit's got to have like 5 bits, so you just take off about one-fifth of those bits.

This final explanation reveals that at least by this stage Katie is simply drawing a picture to fit the already calculated answer of 4/15. Her earlier attempts, however, suggest that her initial, intuitive, reaction to the pictures was to select the intersection as the answer, and then a feeling of "multiplication makes it bigger" intruded and she did some kind of visual addition of the shaded pieces. Her response to the question of whether she would be able to do a problem presented only in pictures was, "Yes, if you could see the sum in your head. You'd turn pictures into numbers in you head". It would seem that the pictures were merely descriptions of the fractions and not an aid to the calculation of an answer. She was no longer able to retrace her steps along the route she had taken when evolving her strategy for working with the numerical symbolism. Her actions were among the class of those described by Piaget [Beth and Piaget, 1961] as "destined to become interiorisd as operations" and the connection between the manipulation of objects and the subsequent abstraction of a "doing strategy" seems to have atrophied.

Orey and Underhill [1987] in some preliminary work on the evolution and exploration of a paradigm for studying mathematical knowledge have also encountered this phenomenon with a third grade pupil who, given manipulatives to aid his working, solved each problem algorithmically and then represented this solution with the manipulatives. No matter in what context the problem was presented he reduced it to a written algorithm, calculated the solution and reinterpreted the answer. *But*, are these not part of the sequence of steps necessary to successful mathematical modelling and solution of real-life problems? Teachers present pictures and cut up paper in an attempt to relate mathematical theory to reality for their pupils, but what *is* reality for 11 year old Katie? Is  more "real" than $^{2}/_{3}$? The teacher sees the pie diagrams as real objects and "$^{2}/_{3}$" as a symbolisation, whereas Katie may well see both the pie diagrams and the numerical form as comparable symbolic representations of fractions. It is possible that for her  is an abstract diagram and not a picture of a pie. How are pupils to know which version of "reality" applies to any particular classroom calculation? The successful use of pie diagrams to solve integer and common-denominator fraction divisions provided the "reality" for the Alsworth class in which $1 \div {}^{2}/_{3} = 1$. Hart [1987] suggests that, contrary to most teachers' intentions, pupils exposed to lots of "doing" do not in fact themselves synthesise this into a rule or algorithm and that when finally presented with the algorithm they accept or forget it independently from their experience. Certainly Katie seemed to have evolved her algorithm for division of fractions by noticing the number patterns ("you keep multiplying by that bottom number") emerging when calculating, not by reference to the "reality" of cutting up pies. In fact in the context of multiplication she was unable to refer back to the pie diagrams in any useful manner. The interviewer therefore attempted to probe her understanding of division as it applied to fractions.

Int: Let's look at a different one. Let's look at this one: $^{2}/_{3} \div {}^{1}/_{5}$

Katie: What do you want it in, pictures or numbers?

Int: Just do it first of all.

Katie: (*Writing*) $^{2}/_{3} \div {}^{1}/_{5} = {}^{2}/_{3} \times {}^{5}/_{1} = {}^{10}/_{3} = 3^{1}/_{3}$
You change it from the divide into a times and turn the second number upside down so the five is on top and the one is on the bottom, then just do five times two and three times one.

What is Katie's current thinking? Is it instrumental or relational? Taken on its own, this citation of the rule would seem to suggest it is instrumental, yet, knowing what went before in the classroom, one is encouraged to say that it is relational. Katie has come to her intuitively derived "child method" by a similar route to that shown by pupils discussed in Hart [1984], Booth [1984], and Romberg et al [1985], but whereas for those pupils the methods were often not transferable, Katie's is precisely the powerful algorithm used by mathematicians. It is of course possible that in the intervening period a parent or other child might have told her "the rule" which, in Hart's terms, she just accepted. The interviewer continued:

Int: Right. do you know what division *means*?

Anna: See how many times it goes into it. One-fifth goes into two-thirds.

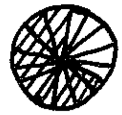
Int: Could you draw me a picture for that?

Katie: (drawing below her writing)

Int: O.K. You've drawn me 3. Can you tell me what you would be doing if you use those first two pictures, to find out how many fifths there were in two-thirds?

Katie: Well, say you had the diagram and you did it into fifteenths ... 1,2,3,...

...like that and you had ten of them shaded in.



then you had divided by..
.. then you had three of those

..then shaded in and then the answer is to see how many times three-fifteenths goes into ten-fifteenths and then it would go ..5,6,7,8 ..you say three goes in once, then twice, then three times, then a bit. If you had two colours you could colour the two-thirds in first then with another colour go over the top and see how many times it fitted it.

Here Katie is able to articulate explicitly her meaning for division by fractions using the medium of pie diagrams. It is evident that she has extracted the knowledge from her actions; there is evidence that she was working towards a way of dividing fractions which was independent of the action materials; it is clear that she now has an algorithm for this operation with which she is comfortable to work. What is not clear is where, or whether, there lie links between fractions as structural objects together with the physical division of these objects and p/q as a symbol for division of p by q together with a manipulative rule for division *which produces the same answer*. Katie is in no doubt that the answer will be the same, and had she no links between these approaches there would be no basis for her belief. It is this "view from inside" (Schoenfeld, 1987) which we must strive to comprehend, because Katie's concepts will be formed, not by her experiences but by her interpretation of her experiences. Sinclair (1987) refers to Piaget's normative facts as "ideas, concepts or modes of reasoning" which are slowly and laboriously constructed, finally becoming "obvious" and releasing the mind for other things. "The subject feels such ideas to be both evident and necessary, and often can no longer imagine that at some earlier time they were not present in his mind." These assimilated facts can then act as a springboard for further reasoning. Although she is referring in her address to such "truths" as the commutativity of addition, one can argue that algorithms derived from personal experiences and ostensibly acceptable to the outside mathematical world could also merit the label "normative facts". A later inability to reproduce the "doing strategies" from "first principles" does not indicate a diminished understanding, nor is it necessary or even desirable when problem solving. If called upon to investigate the maxima and minima of a particular function a mathematician will "know" how to differentiate the function, and going back to the derivation of a derivative through consideration of infinitesimals would add nothing to the solution of the problem.

In an attempt to get at this "view from inside", the interviewer asked Katie to predict whether $50 \div {}^{1}/_{3}$ (*written*) would be bigger or smaller than 50. The reply came without hesitation: bigger.

Int:	Why do you think it is going to be bigger?
Katie:	Because a third will go into 50 quite a lot of times.
Int:	Now how about (*writes*) ${}^{4}/_{5} \div {}^{1}/_{2}$? Is that going to be bigger or smaller than 1?
Katie:	Bigger or smaller? .. Bigger than 1.
Int:	Why do you think it is going to be bigger?
Katie:	Because the four-fifths is more than a half. If it was going to be one or smaller it would be smaller than two-and-a-half-fifths. You've got to see how many times a half goes into four-fifths. If it was two-and-a-half-fifths it would be one and if it is less it is still going to go less but if it is more than two-and-a-half it is going to go in more than one times.

The links which bind the concepts of fractions and division in Katie's mind clearly do exist.

One conclusion which can be drawn from the foregoing analysis is that Katie is a mathematically bright, articulate 11 year old who "understands" division of fractions whatever definition or categorisation of understanding is used. The second is that a refined version of the interviewer's exploratory path could provide a useful protocol for exploring other pupil's understanding of division of fractions. Neither of these is the purpose for which this paper was written. The intention is to illustrate the caution with which *any* label must be attached to a child's understanding. Taken alone, Katie's citation of an algorithm and her subsequent inability to relate it to "reality" might lead one to assume her knowledge to be rule-bound and highly instrumental. It was the privileged preview of her working which prompted an attempt to examine her understanding more closely. It is important that psychologists and educationalists should continue to investigate the multiple meanings and categorisations of understanding which allow us to attempt to make sense of pupils' concept formations. It is, however, equally important that we acknowledge that currently our "tools for *measuring* mathematical understanding are primitive" (Parker, 1987) and we should be reluctant to apply labels to children based on how they "do" mathematics unless we explicitly state that the labels are concerned only with the pupils' ability to perform particular mathematical tasks. A child may demonstrate an apparently formalised understanding dependent on context. Indeed, Bergeron et al (1987) working within the theoretical framework provided by the model of understanding which defines four levels in a child's construction of a mathematical concept: intuition, procedural understanding, abstraction and formalisation, concluded that "it would be a mistake to perceive it as a linear model.... the child evolves simultaneously at many levels". Much research needs still to be done by those who would catch and assess a child's potentially nebulous and fluctuating mathematical understanding.

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