Three Papers

DAVID HENDERSON

These three short papers are among many more that I wrote over the past five years while my outlook on mathematics (and the world) was changing and growing. They came from my experiences with mathematics and my experiences with people I have taught in first grade and at Cornell University

MATHEMATICS AND LIBERATION*

Mathematics has a major and beneficial role in our society. It has enabled us, as human beings, to participate in and understand more and more of our universe (By "universe" I mean "all experience".) And this role is more widespread and more powerful today than ever before in the history of human kind

But much is wrong. The majority of people today are scared of mathematics (and mathematicians) and feel powerless in the presence of mathematical ideas. Many people learn and view mathematics in rigid, rote ways that lock those persons into conditioned responses that limit their creativity. This situation has been systematically reinforced by our culture which views mathematics as only accessible to a talented few These views and attitudes, besides affecting individuals, have become part of what separates and holds down many oppressed groups, including women, working class, and racial minorities

I believe that this situation is not necessary. I believe that mathematics can be a part of every person's understanding and can have an important role in the liberation of human beings. I define liberation as the removal of all barriers to a person's full creativity. With that as our goal there are steps that we can take now to move in that direction

1. Mathematics can be understood

So why the fear? — Why the rigid, rote responses to mathematics? Let me relate what happened to me when I started teaching calculus for the first time (after I was already an established mathematician)

I tried to listen to the people in the class. I tried to understand what their questions were I found that some people were not thinking clearly because of emotional problems or because of rigid reactions that came from previous conditionings But other people were obviously thinking clearly and I tried to understand what they were seeing In many cases I found this terribly difficult — my gut reaction was that it couldn't possibly be right — it felt like nonsense I felt threatened — here was something which I couldn't see in an area I felt certain about.

Gradually, after much persistence and with the help of friends, I began to sense that I had blinders on — that my ways of understanding calculus had blinded me to other ways of perceiving. I saw that many of the people in the class had real questions about the meaning of limits and derivatives — questions which I could not answer or questions which I then started to explore for the first time I lost a certain narrow feeling of certainty but gained a broader perspective. Now I perceived calculus in a different way

What was happening to the people in my class who were asking a real question I couldn't understand? Some correctly sized up the situation and blamed my blinders, but this was rare. Most blamed themselves.

It is a hurtful experience to have someone whom you see as an authority not understand a real question of yours When this and other distressful mathematics experiences happen to people enough times over the years, they feel stupid, they feel they can't think about mathematics. They then react to mathematics through fear or in rigid, rote ways. Their reactions are reinforced by the cultural view that mathematics can only be understood by a select few. This view becomes, in part, a self-fulfilling prophecy.

I make the assumption that mathematics is accessible to everyone Specially, my assumption, is that *every person who needs some part of mathematics in order to understand some aspect of their experience can grasp that part of mathematics in a very short time.* All that is needed is confidence in their thinking and in their perception. This assumption applies, I believe, regardless of the person's mathematical background.

The assumption has had a liberating influence on me and on the people I contact. As I listen to people in my classes,

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the same gut reaction of "that's nonsense and can't possibly be right" still comes up But now I know that, if I do not act on the basis of that reaction, and instead persist, I will perceive something new or in a new way. Both the other person and I will gain

2 What is correctness?

For me the overall goal of mathematics is to further each person's liberation by expanding their ways of understanding and perceiving reality There are many diverse ways of trying to understand our universe and we all perceive our experience differently. Nevertheless, it seems possible for us to get a sense of when our understanding and perception is expanding and when it isn't

I relate correctness to the goal by saying that something is correct to the extent it moves an individual or group of individuals in the direction of an expanded understanding and perception of reality. I claim that my above assumption that mathematics is accessible to everyone is correct according to this criterion

I apply this same criterion to any statement in mathematics In particular, an argument is correct to the extent that it expands a person's understanding and perception. So what's correct depends both on reality and on the individual.

I claim that this is what we all naturally try to do whenever we are involved in understanding or communicating mathematics. How do we view mathematical arguments? When do we call an argument good? When do we consider it convincing? — When we're convinced! — Right? — When the argument causes us to see something we hadn't seen before. We can follow a logical argument step by step and agree with each step but still not be satisfied We want more. We want to perceive something

My shortest mathematical paper, one page long, contains a very short logical argument that can be easily followed But this paper triggered the most questions of any of my papers — questions of "Why?", "Where did that come from?", "How did you see it?", etc. The logical argument is not the goal, the ultimate goal is to perceive something new or to understand something in a new way.

3. Our view of mathematics is biased

Recently, I was thinking back over the times that my perception of mathematics had been changed by the insights or questioning of a person in my class Suddenly, I realized that in almost all of those cases the other person was a woman or from a different culture than my own I don't think that this is just a coincidence.

Over the recent centuries the people in charge of mathematics, as we culturally define it, have been mostly Western (white), upper/middle class, males. So it should not be surprising if this has instilled a bias into our conception of mathematics I see evidence for this, but I do not claim to see it all clearly.

Evidence of this bias I see in the fact that most histories of mathematics downplay or ignore the role of non-Western cultures (for example, the Muslim culture during the West's "Dark Ages"). There are mathematical ideas (such as "Saccheri quadrilateral") which are named after the person (an Italian Christian) who first translated the idea into a Western language instead of being named after the person (Omar Khayyam, a Persian) who first introduced the ideas to the world (in Arabic). Though less often than in the past, women are still being told that they can't understand mathematics as well as men can. Working class people are traditionally considered to not know mathematics; but I have seen from personal experience that a thinking carpenter or cabinet maker knows and uses a lot of geometry and understands it in a way that is different from, but just as correct as, what is normally taught in school. I'm sure that there are lots more examples.

As I indicated above when I listen to how other people view mathematics my understanding of mathematics changes I am certain that as women, and members of the working class and other cultures participate more and more in the established mathematics, our societal conception of mathematics will change and our ways of perceiving our universe will expand This will be liberating to us all

4 Mathematics is more than a technical tool

Mathematics is not merely techniques for solving analytic problems, as much of our culture supposes Mathematics is also ways of perceiving beauty, order, and unity. For example, the notions of symmetry and pattern are a part of mathematics The techniques and theory of mathematics are analogous to the techniques and the theory of music. And, like music, there is beauty and meaning behind the techniques and theory.

In the ancient Greek and Muslim cultures mathematics was viewed broadly as relating to almost all areas of human understanding including religion, art, music, metaphysics, in addition to science But as our Western culture split science and technology away from religion and the arts, our culture has viewed mathematics as on the science/technology side. There has been a related split within mathematics between pure and applied. Here "applied mathematics" means almost always "applied to science and technology" Even pure mathematics, which is viewed by parts of our culture as merely a frivolous pasttime, is defended and funded for its actual or hoped-for benefits to science and technology. Nevertheless, most mathematicians see beauty and aesthetic enjoyment as a part of mathematics (which may relate to the fact that mathematicians are noted for being good musicians.) Many mathematicians, including myself, view mathematics as closer to art, music and poetry than to science and technology

There is, of course, a major difference between mathematics and the arts today. Most people derive meaningful pleasure from art and music, but very few people derive meaningful pleasure from mathematics. Try to imagine how it would be if we treated mathematics in our culture the same way we treated music — as something everyone could make and enjoy. Hard to imagine, isn't it? But I think it is possible.

In the future, I see our conceptions of mathematics expanding to touch all the ways of understanding human experience. I see mathematics as bridging the gap between religion and the arts, science and technology, and human liberation.

SUE IS A MATHEMATICIAN

"I could *never* do that I could *never* do that"

"Oh, I think you can." I answered as Sue got embarrassed and giggled — but didn't move away.

"*No I can't?*"

"Sure you can What needs to be done?" I said expectantly, and Sue responded:

"Well. . ahh . . . I sort of need to count how many numbers between 439 and 535."

"That's fine — see, you do know what to do"

"But it's too big. I can't count that much. — I don't have enough fingers" — And still she didn't move away

It was Sue, a kindergartener in the combined kindergarten-first grade class I was visiting The class together had come up with a problem that involved numbers and Sue's words were saying that it was too much for her. But Sue knew I was a mathematician and the way she came close to me said that she really thought that maybe she could do it — only she was scared

The class had built a tall tower out of blocks and the question came up: "How tall is it?" It was taller than the yard stick, and taller than the principal who at 6′5″ was the tallest person around. But someone, Bill I think, suggested using a number roll — strips of paper taped together into a long ribbon with a line every inch. Many people in the class had been exploring counting by making their own number roll and writing the numbers in order in the inch spaces

"But will a number roll be long enough?"

"Jason's will Jason's number roll goes almost across the gym!"

This class took delight in what each other did. They were proud that *their* Jason had such a *long* number roll In a scurry of movement the class mobilized: I was led over to the tower — Jason came up with his number roll and climbed on my shoulders. He held the end (marked "535") at the top of the tower and let the roll unroll to the floor At the floor someone read "439" as the others crowded around and kibitzed.

"But — how many inches is it?"

That's how the question came about and several people in the class (mostly first graders) started working on it The first graders had learned to add small numbers. Sue had only experienced simple addition that could be done by slowly counting on her fingers No one in the class had gotten to subtraction in their mathematics program. But, like Sue, everyone could see clearly what the problem was and what needed to be done.

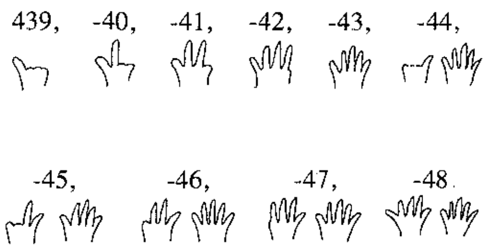
"I could sort of count how many numbers between 439 and 535 But it's too big. I don't have enough fingers."

"Can you get more fingers?" I asked, and off Sue went to round up some of her classmates so she could borrow their fingers. She had to give up that project when she discovered that they wouldn't stand still So she had only my fingers besides her own

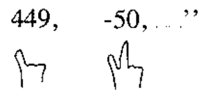
"Well, I could do it if I had enough fingers, but they won't stay still." Now Sue understood how to do it and knew she understood It still seemed like too much, but she was breaking through — and she stayed by me

"Is there some way, Sue, that you can use just my fingers?" I held out the expectation and Sue responded by giggling and nuzzling up next to me *and thinking*. After a little experimentation she soon came up with an idea:

"You put up one of your fingers every time I count through my fingers.



Now put up *your* finger.



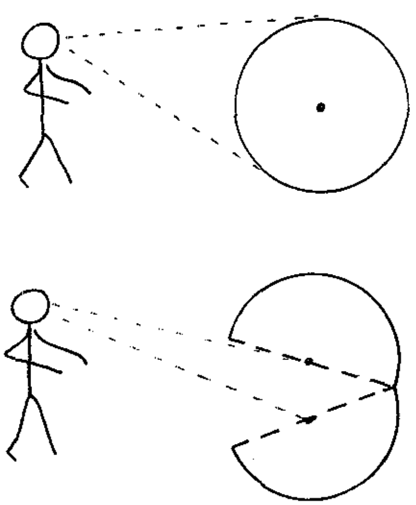
Sue continued slowly — but triumphantly

Sue is a mathematician.

MATHEMATICS AS IMAGINATION

Think of something that you would consider mathematical and that is meaningful to you. Now there are many aspects of your mathematical thing — many ways to view it from. But I want you to look at the aspect of how it allows you to imagine something that need not exist in any concrete sense. For example, if you were thinking of "2 + 2 = 4" then this allows you to imagine two apples plus two apples gives four apples without actually taking two apples and two apples and physically putting them together. Or if you were thinking of "triangle" then you are imagining something which doesn't physically exist. We can represent (or suggest) triangles by using 3 sticks or 3 pencil lines, but we know that they are not the same as the triangle we imagine (which has perfectly straight sides, for example) Remember imagination is only one way of looking at mathematics, but a very important way and a way that I would like to explore with you

So far we have only mentioned imagining things which, though we need not experience them physically, we could if we wanted to or (as in the case of the triangle) could experience close physical approximations. But many common things in mathematics we cannot experience physically at all For example, we can in mathematics imagine (use, think about) numbers such as 1,236,748,172 which we can't possibly experience physically. Now note that I say "can't experience *physically*" for I do consider that we experience 1,236,748,172 as an image when we, for example, write it down and add it to another number. Another example: Consider a ball; we can directly physically experience only one side of the ball at a time, but we can imagine the whole ball all at once. Even more, if we try a little we can image the center of the ball even though we can't see the center without cutting the ball Now you try it yourself



Do you see what I'm saying? Can you imagine the center of the ball? In fact, can you almost say you can see the center in your imagination? Now I don't mean imagining taking a knife and slicing the ball. I mean imagining the whole ball, center and all, all at once. Now consider what you have just done You have in effect stepped out of 3-dimensional space! Let me explain what I mean. Imagine a 2-dimensional (flat) person living on a plane and looking at a disk (a ball to him) He would only see one side at a time. He could walk around it and see all sides but not all at once And the only way our 2-dimensional man can see the center of the disk is to slice the disk in half. Now of course, we, being able to see in 3-dimensions, can see the disk and its center all at once. So we could say that the 2-dimensional man is imagining from a 3-dimensional perspective if he imagines the disk and its center all at the same time. Thus, in the same way, you just imagined from a 4-dimensional perspective when you imagined the ball and its center Convinced? Well, even if you don't quite grasp the 4-dimensional bit, remember that you are imagining something you can't see physically (in 3-dimensions)

More about imagining later. For now let's look at what it says about mathematical proofs (or arguments) if we want to pay attention to imagining things. That is, instead of asking of a mathematical statement, "Is it true?" suppose we ask, "Can I imagine it (or see it) as true?" Now, first of all, notice that we no longer have the Law of the Excluded Middle (LEM) (The LEM says that a mathematical statement is either true or false.) If you give me a mathematical statement I might imagine (see) it as true or I might imagine (see) it as not true or (a third or middle possibility) I could not be able to imagine it as either true or not true. Some examples: I can see that $2 + 2 = 4$ is true — I experience some image like .... transforming into : : . And I can see that "All triangles are equilateral" is not true. But I can't see or imagine if "All maps (in the plane) can be colored with four colors so that no two countries which share a common border are colored the same" is true or not true. Someone has recently claimed a proof that the above statement is true So presumably I'll never be able to see the statement as not true. However, it is certainly different to be able to say "I can see (imagine) that it is true" as opposed to "There is a proof that it is true". Haven't you had the experience of listening to or reading a proof and following every step and being logically convinced that it must be true but still not being able to "see" it or "imagine" it? That's the difference I want to focus on. Notice also that once I ask "Can I imagine that it is true", I am putting myself into the situation. It is no longer an objective question about some ultimate right or wrong, but rather it is asking something about *me* Can I imagine it as true? And clearly the imagining can vary with time. I may not be able to imagine it as true now but tomorrow I might be able to.

Now a question may occur to you: How can we know if what I am imagining is the same as what you are imagining? Good question! Ultimately we can't ever be entirely sure However, we can communicate with each other, ask questions, draw pictures, etc , until we have a feeling that we are imagining the same thing. Also, here's where the formal theories help. The formal theory is a precise definite thing which we can each relate (or try to relate!) to our imagination (or experience) The theory can help show us ways to stretch our imagination and our imagination can point out ways to expand the theory. And if we both are relating our imagination to the same theory, then it helps us feel that we are imagining the same things or, at least, that the theory represents some element common to both of our imaginations.

I find that in the above I have been fumbling some over the word "experience". By "experience" I mean more than physical experience. We experience ideas and images. We experience "seeing" the whole ball with its center all at once. We experience ourselves growing (changing). How do we do it? All I can physically sense is some aspects of myself at a given instant in time. Yet I have an image of (we often say "have a sense of") my life as a whole changing from one point in time to another. It's very similar to imagining the ball as a whole. It's a non-physical image in the sense that we can't sense it with our physical senses all at once Yet they are certainly real experiences, in that they have meaning and affect us

With this notion of experience and imagination I believe that mathematics has meaning that can be experienced and imagined. And I believe that the meaning of mathematics can be found in (or based on) these experiences and imaginations.