Philosophical and Epistemological Aspects of Mathematics and their Interaction with Theory and Practice in Mathematics Education*

HANS-GEORG STEINER

*This article is based on a paper presented at PME 8 in Sydney (1984) and talks given at the Free University of Berlin (1984) and the University of Georgia, Athens, U S A (1985).

Philosophical positions and epistemological theories related to mathematics, such as logicism, formalism, constructivism, structuralism, empiricism, have always had a significant influence on the guiding ideas and leading principles in mathematics education. This not only holds for curriculum development and teaching methodology but also for theoretical work and empirical research related to the mathematical learning process. As Seymour PAPERT has pointed out in "Mindstorms": BOURBAKI's *theory of mother-structure* is learning theory. Whether it is a good or bad one is another question. In a similar way PLATO's and PROCLUS' philosophy of EUCLID's elements with its idealistic ontology and its emphasis on the *dialectic between analysis and synthesis* being the "capstone of the mathematical sciences" comprises the elements of a mathematical pedagogy and of a theory of learning. [42], [50], [66] This phenomenon can be pursued through the history of mathematics and its concomitant philosophy. Of course, the observation not only holds for global mathematical philosophies but also for epistemological views of particular parts or concepts of mathematics, such as the set-theoretical foundation of the function concept, the logical interpretation of variables as place holders, the interpretation of geometry in terms of Felix KLEIN's Erlanger Program etc.

On the other hand, considering the inverse relation, what was said by René THOM in his 1972 Exeter speech is also true: "In fact, whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." Such "philosophies" may consist in a teacher's "private" opinion on the nature of mathematics and mathematical knowledge (often indirectly acquired in his own academic studies), and in his thoughts about how this is related to his teaching and to the learning of his students. They are also inherent in didactical principles such as the spiral approach (BRUNER), the deep-end principle (DIENES: [16]), the operative principle (AEBLI), etc. They underlie theories of stages in the learning process (PIAGET, VAN HIELE) or theories of transfer, and they also stand behind hypotheses of empirical studies on how children learn or fail. As was pointed out by Thomas S. KUHN, each *research paradigm* has philosophical and ontological assumptions among its different components, and these are parts of the paradigm's bias, its strength and weakness. Often they are a source of profound discrepancies between competing approaches and of controversies between research groups or individual researchers.

Recent developments in mathematics education show a *new dynamics in the field*. New philosophies and epistemological theories have entered the scene: the theory of epistemological obstacles (BACHELARD, BROUSSEAU [10], CORNU, SIERPINSKA), a synthesis of Kuhnian theory dynamics and Piagetian genetic epistemology (v.GLASERSFELD, CAWTHRON & ROWELL) a new quasi-empiricist philosophy of mathematics (LAKATOS [43], SNEED, JAHNKE), a complementarist view of mathematics (KUYK, OTTE [51]), the epistemology of "microworlds" and of the "society of mind" based on cognitive studies within research on artificial intelligence (PAPERT, MINSKY). They claim to provide a better background for an investigation of pupils' real learning behaviour both from a cognitive and a social point of view. They also claim to give a more adequate foundation for the theory and practice of mathematics teaching. Based on these developments, and supported by new empirical data, strong criticisms have been put forward especially against positions elaborated in close connection with the "new-math philosophy". Deficiencies of this reform are being characterized by terms like "Jourdain effect", "Topaze effect", "metacognitive shift", etc. (BROUSSEAU [91]), and by exhibiting its one-sided "mentalistic" orientation guided by the assumption of the autonomous, universal, coherent, and homogeneous natures of both mathematical structures and the human mind. Other failures are related to a "purist" or "static" view that neglects the applications which belong to mathematical theories and concepts as they do to physical theories, and neglects the representational, the social, the procedural and the processual dimensions of mathematics. This criticism says that the structural concepts of mathematics have been overemphasized in trying to organize and understand mathematical learning.

In the following I shall explain my position a little further by formulating and briefly commenting on *half a dozen theses,* the first two of which have already been touched upon in the introductory remarks.

> *Thesis 1*
> Generally speaking, all more or less elaborated conceptions, epistemologies, methodologies, philosophies of mathematics (in the large or in part) contain — often in an implicit way — ideas, orientations or germs for theories on the teaching and learning of mathematics.

As has already been indicated, it would be an interesting study to verify this thesis in a more systematic way throughout history and thus to exhibit how philosophies of mathematics have actually pre-determined didactical approaches. It is another aspect to evaluate theories on the teaching and learning of mathematics by means of empirical investigations and in this way to find out about the adequacy of the underlying philosophy of mathematics as a founding element in the construction of *empirically oriented* mathematical pedagogy. Such empirical investigations have only recently become a matter of coherent and progressive research. Related research findings are now successively exposed to be matched with a fitting mathematical epistemology which allows a coherent and adequate interpretation of the findings. Here we should emphasize again the explanatory capacity of *recent mathematical epistemologies* which are less normative than many traditional ones and more of a descriptive and empirical nature, such as LAKATOS' quasi-empirical position, or JAHNKE's transfer from physics to mathematics of SNEED's interpretation of theories as consisting of a theory-kernel plus a set of intended applications. Sneed's position underlines the role of constraints, of domain-specific knowledge and experience, and the importance of tools and means of representation, which according to post-Piagetian studies also play a significant role in children's cognitive behaviour and development in mathematics. [31], [34], [35]

> *Thesis 2*
> Concepts for the teaching and learning of mathematics — more specifically: goals and objectives (taxonomies), syllabi, textbooks, curricula, teaching methodologies, didactical principles, learning theories, mathematics education research designs (models, paradigms, theories, etc.), but likewise teachers' conceptions of mathematics and mathematics teaching as well as students' perceptions of mathematics — carry with them or even rest upon (often in an implicit way) particular philosophical and epistemological views of mathematics.

This is a generalization of THOM's statement quoted above and represents a kind of inverse to the first thesis. The two are bound together by a deep correspondence between intellectual perception and learning, between *knowledge structures and learning structures* (v. HENTIG, FICHTNER).

One area in which recent research has given particular support to Thesis 2 is research on *conceptions and beliefs teachers hold* about mathematics and mathematics teaching, the origins of these conceptions and how they are related to teachers' work and instructional practice. Examples of such conceptions are reported by Alba G. THOMPSON who compiled case studies on three high school mathematics teachers. Here is a selection from Thompson's summaries of their professed views of mathematics:

Teacher Jeanne:
- Mathematics is an organized and logical system of symbols and procedures that explain ideas present in the physical world.
- Mathematics is a human creation, but mathematical ideas exist independently of human ability to discover them.
- Mathematics is mysterious — its broad scope and abstractness of some of its concepts make it impossible for a person to understand it fully.

Teacher Kay:
- The primary purpose of mathematics is to serve as a tool for the sciences and other fields of human endeavour.
- Except in statistics, conclusions and results in other branches of mathematics are certain.
- The validity of mathematical propositions and conclusions is established by the axiomatic method.

Teacher Lynn:
- Mathematics is an exact discipline — free of ambiguity and conflicting interpretations.
- The content of mathematics is "cut and dried". Mathematics offers few opportunities for creative work.
- Mathematics is logical and free of emotions. Its study trains the mind to reason logically. Mathematical activity is like "mental callisthenics".

In a similar way, Thompson has also documented the three teachers' conceptions of mathematics teaching. Matching the two kinds of conceptions with the teachers' observed actual teaching practice, she came to the following conclusion:

> Teachers' beliefs, views, and preferences about mathematics and its teaching, regardless of whether they are consciously or unconsciously held, play a significant, albeit subtle, role in the teachers' characteristic pattern of instructional behavior. In particular, the observed consistency between the teachers' professed conceptions of mathematics and the manner in which they presented the content strongly suggests that the teachers' views, beliefs, and preferences about mathematics do influence their instructional practice. (Thompson [75], p. 124/125)

As should be clear from these connections, teachers' conceptions of mathematics can have positive but also very *negative effects* on their teaching, and in particular on their ability and readiness to try out and develop new approaches. The strong emphasis which — as many teachers think — has to be laid on rigor, precision and deductivity in mathematics, makes it difficult for them to justify and realize genetic approaches, to orient teaching towards problem-solving and mathematical modeling which ask for a view of mathematics as being open, flexible and developing rather than closed, fixed and ready-made. When probability and statistics were introduced recently into the curriculum of the German Gymnasium, the voices of many teachers were heard saying that all their lives they had spent educating their students to be precise and to think rigorously, and now they were supposed to teach something in which absolute security had been abandoned — a concern which also seems to be indicated in one of Kay's judgements, listed above.

I think we will get further evidence on the role of teachers' views of mathematics when we go into more detail and investigate their understanding of different domains of mathematics, of specific components such as the meaning of mathematical concepts, proof, definition, theorem, conjecture, variable, symbols, rule, formula, axiom, problem, problem solving, application, model, computation, graphical representation, visualization, metaphor, etc., both with respect to the various sub-domains of mathematics as well as in a more general sense.

I would also suggest we need to study in a similar way the views of mathematics and of mathematics teaching and learning held by students, how these views change in the course of schooling and how they affect students' mathematical learning behavior (see [60]). From interviews with pupils I have some evidence that most students have developed a personal position towards mathematics which will probably, in the end and on the average, contribute to the overall public opinion on mathematics, or to what one might call "folk-mathematics". Finding out about the image students have of and about their personal relation to mathematics could give mathematics educators and teachers important hints on possible corrections to be made with respect to the curriculum and the actual work in the classroom.

As another domain related to Thesis 2, I would like to emphasize the *goals* and *objectives of mathematics education* as they are especially formulated in mathematical syllabi or stated in connection with mathematical abilities or competences.

With respect to the development of the discussion and identification of objectives for mathematics education in the German Gymnasium, an extensive study has been made by H. LENNE. [46] For the period between 1945 and 1965 he distinguishes three main directions in the didactics of mathematics: (I) traditional mathematics; (II) genetic approaches according to Wittenberg and Wagenschein; (III) modern mathematics. His comparative analysis is particularly concerned with the meaning and importance given by each of these directions to the relations between mathematics and concepts like:

- intuitive abilities; geometrical imagination
- scientific way of thinking and operating
- intellectual initiative — fantasy — creativity
- ability to make adequate linguistic representations
- ability to be systematic and to concentrate
- objectivity
- self-criticism
- tolerance
- autonomy and responsibility
- philosophy
- the arts
- society
- religion
- ethics
- modern civilization

Behind each of these points one can find philosophical aspects related to mathematics which were differently viewed and operationalized by the three directions and which are matters of interpretation for every mathematics curriculum and its related pedagogy.

As a third domain of specific relevance for Thesis 2, I want to comment on *research in mathematics education*. In almost every research paper and study there are explicit or implicit assumptions about the nature of mathematics or about particular mathematical concepts, theories, methodologies, etc., which shape the research design and the leading research question. Often important aspects of a concept or a method are neglected which might well play a significant role in the way children actually use and should be allowed — if not encouraged — to use the concept or method. H. FREUDENTHAL complained "that researchers conduct subtle investigations to find out whether students understand variables as polyvalent names or as mere place-holders while it never crossed their mind that variables should and could be understood as variable objects." [23, p. 1705] R. KARPLUS criticized some studies on the development of the concept of function in children: "In those investigations, the principal attention was on the mapping of one set on to another, distinguishing between a relation and a function, determining inverse relations and finding functions of functions. Virtually all their examples lacked a context that might have provided an intuitive basis for a functional relationship." [38, p. 397]

In a Unesco Report on Mathematics in Primary Education, Z.P. DIENES [17] summarized the positions taken by a variety of authors (Suppes, Rosenbloom, Hull, Dienes, etc.) and projects (Greater Cleveland, SMSG, Ball State, etc.) with respect to the concept of natural numbers: "What these attempts all have in common is that the workers believe that the foundation on which the idea of number is based is explicit knowledge of the properties of sets, because it is assumed that since number is a property of sets, the fundamental notions relating to sets must be learned first because number is superordinate to set and therefore number cannot properly be understood without the subordinate concepts of set being understood first." [17, p. 73] Accordingly, research activities by those authors and studies in these projects devoted to children's learning of numbers and arithmetic were highly influenced and biased by the underlying philosophy. Recent empirical research doesn show that set-theoretical foundations and measures have limited use in explaining children's actual ability to add and to subtract and lead to "underestimating the significance of such basic quantitative skills as counting, estimating, and subitizing. There is a growing body of research that suggests that the development of basic number concepts involves the integration or increasingly efficient application of such skills." Carpenter & Moser [12, p. 13]

What has just been said about Dienes' philosophy of natural numbers can be stated more generally with respect to the influence of PIAGET's view of mathematics and its relation to cognitive psychology. In many Piagetian studies one can observe a certain *dominance of a structural philosophy of mathematics* which is due to the strong relations Piaget had established between his concept of schema and structural concepts in mathematics (see [5]). This went together with a characteristic overweight of assimilation as compared to accommodation in Piaget's work. It had the effect that the universality and homogenicity of structural concepts predominated and that the role of domain specifity and the impact of modes and means of representation and operation in children's dealing with concepts such as number, arithmetic, proportionality, and functions were not adequately observed (see also [6], [34], [35], [38]). If one tries to integrate such non-Piagetian findings into a new modular and *society-of-mind philosophy of mathematics* (see [54], [4]) we should express the warning that it might create a new one-sidedness insofar as the underlying complementarities between the two positions are not sufficiently understood and respected (see e.g. [53]). To further clarify this is apparently a strong challenge for future research.

My third thesis is based on the *increasing relativity* of the validity of philosophies of mathematics as observed in the foundations, epistemology, and philosophy of mathematics as well as in the *sociology of knowledge* and the *sociology of the sciences* (see [73]). We may refer here to Imre LAKATOS' "quasi-empiricism" ([43]), René THOM's philosophy of mathematics as a combination of a realist-platon-ic with an empirico-sociological view ([75]), Hao WANG's "substantial factualism" within his "new philosophy of knowledge" [78], GOODMAN's "knowing mathematicians" [25], or PUTNAM's "mathematics without foundations" [52]. As an example we quote WANG:

> While we are skeptical of oversimplified accounts of the foundations of human knowledge, factualism is much interested in how we know in the sense of desiring to consider the basic aspects of the factual process of knowing. An attention to these not only helps to uncover shortcomings of oversimplified pictures, but also promises to lead to a balanced and appropriately anthropocentric overview of how stable and how structured actual knowledge is. The most important single aspect is the process by which a proposition or a theory becomes accepted as part of human knowledge or a particular individual's knowledge. This factor of acceptance is the central anthropocentric component of factualism which is related to intellectual comfort, understanding, coherence, and perspicuity. Wang [79, p. 19/20]

From the point of view of the *sociology of knowledge,* knowledge (including mathematical knowledge) is socially constituted, negotiated in social interactions, carried and transferred by means of social norms and institutions. Here we quote ESLAND who refers to MILL [49]:

> It should be emphasized that questions of "truth" and "validity" are also problematic. The problems which are thought to reside in a "body" of knowledge and the rules for their effective solution or verification are themselves socially constructed. The cognitive tradition which generates the problems also, through its relevance system, legitimates the inferential structure which is activated in their solution ..... Thus, as Mills suggests: "The rules of the game change with a shift in interest". Mills goes on to argue that zones of knowledge, through their human constitution, have careers in which the norms of truth change: "Criteria, or observational and verificatory models are not transcendental. There have been, and are, diverse canons and criteria of validity and truth, and these criteria, upon which determination of the truthfulness of propositions at any time depend, are themselves, in their persistence and change, open to socio-historical relativization." Mills explicitly did not exclude the post-Renaissance scientific paradigm from this. This is another way of saying that epistemologies are institutionalized. It is important to emphasize that the cognitive tradition which forms an epistemology can exist only through a supporting community of people. Its members are coproducers of reality and the survival of this reality depends on its continuing plausibility to the community. Esland [19, p. 71f]

This does not mean that relativism is abandoning all references. It rather makes references debatable and allows for pragmatic attitudes in the presence of alternatives. Thus my next thesis is formulated as follows:

> *Thesis 3*
> There is no distinguished, constant, universal philosophy of mathematics. One should evaluate philosophies of mathematics according to their fruitfulness for particular goals and purposes and develop criteria for evaluation.

A specially important matter of study would be the identification and elaboration of *relations between different philosophies*. Here we may particularly underline the *role of complementarity*. In many domains of human experience and thinking we find characteristic dualisms, indicated by pairs of seemingly opposing concepts such as: subject and object, a priori and a posteriori, rationalism and empiricism, structure and process, mind and body, determinism and free will, etc. Some of them seem to be of a rather general epistemological nature, others appear to be more related to specific domains. In his Pre-Kuhnian studies on discontinuities and epistemological breaches and obstacles in scientific and individual cognitive development, G. BACHELARD has observed,

> . . that in a very general manner the obstacles to the formation of a scientific mind always appear in pairs. One could even call it a psychological law of bipolarity of errors. Such regularity in the dialectic of errors does not originate from the objective world...... In my opinion it results from the polemic attitude of scientific thinking towards the scientific community. Bachelard [2, transl. by the author]

This reminds us of some *characteristic dichotomies* in mathematics education and of related waves of fashion in the history of curriculum reform which swing between polaristic positions, such as: skill vs. understanding, structure building vs. problem solving, axiomatics vs. constructivism, pure vs. applied mathematics, etc. In his plenary talk at the Karlsruhe ICME congress, Peter HILTON discussed these phenomena as false dichotomies:

> It will be argued that many of the prevailing dichotomies are false, that is to say that the two concepts which are set in opposition to each other do not form part of an either/or situation; that while the two concepts under scrutiny are different, they have an essential overlap, and that, when properly understood and applied, they can in fact mutually reinforce each other. Hilton [29]

Apparently, a deeper understanding of the kind of "overlap" and "mutual reinforcement" between the two interrelated concepts under consideration must be of great importance both from an epistemological and didactical point of view. BACHELARD [3], who particularly speaks of the "obligatory alternation between a priori and a posteriori" and the "peculiar tie which in scientific thinking links empiricism and rationalism", suggests "a dialectic approach that comprehends a concept from two different philosophical points of view in a complementarist manner" and "places itself into the epistemological domain between theory and practice".

In his interpretation of quantum physics Niels BOHR [7] had already indicated deeper underlying mechanisms behind complementarity when referring to the involvement of the subject being himself a part of nature and the impossibility of a strict separation between the subject and the object. From Bohr's principle of complementarity it became clear in a broader sense that

> every relevant piece of theoretical knowledge, being part of some idea or model of the real world, will in some way or another have to take into account that the person having that knowledge is part of the system represented by the knowledge. All knowledge presupposes a subject, an object and relations between them (which are established by means of the subject's activity). Therefore, all knowledge has an incoherent structure with metaphorical and strictly operative connections. Otte [52]

Along these lines a theory of *human object-related activity* including its social and cooperative conditions viewed as an *interactive system* seems to be an adequate basis for understanding complementarity and for discovering and investigating more complementarities in various domains of human experience and thinking (see e.g. PATTEE [55], JANTSCH [36]). For mathematics and mathematics education (e.g. JAHNKE [31], OTTE & BROMME [53]) this has led to the study of a variety of interrelations such as those between concepts as methods and as objects, between the representational and socio-communicative on one hand and the instrumental and operational momentum of a concept on the other hand, between the descriptive and the explorative function of models (including texts, visualizations, etc.), but also between knowledge and meta-knowledge. A further elaboration which is also related to the development of a comprehensive approach to mathematics education as an interactive system comprising research, development and practice (see Thesis 6), is a challenging research and developmental program.

Hao WANG, who also refers to a "concept of complementarity" and adds pairs such as justice and charity, contemplation and volition, the pragmatic and the mystical, different national cultures, argues: "The highly suggestive idea seems to be awaiting more careful analysis and elaborations at the present stage." Wang [78, p. 341]

The next thesis is more or less a natural consequence of the preceding considerations:

> *Thesis 4*
> For mathematics education one should prefer and elaborate philosophies of mathematics which especially respect the following aspects: different forms and conditionalities of mathematical knowledge, means and modes of representation and activities, relations between subjective and objective developments of knowledge (complementarity, obstacles, dynamics), relation of mathematical knowledge to other knowledge, special fields and applications; the personal, social and political dimension of mathematics.

The inherent intention of this thesis is that mathematics education and especially teachers' knowledge and practice should on the one hand be guided by an *adequate philosophy of mathematics* and on the other hand be freed from unnecessary and fruitless confinements. This is explicitly stated in the next thesis which also includes the *students' right to enlightenment and participation*.

> *Thesis 5*
> Such philosophies of mathematics should become an ingredient of a form of reflective mathematics teaching and learning, and contribute to the development of an adequate meta-knowledge not only for teachers but also for students.

I conclude by referring to the program *Theory of Mathematics Education* (TME) which was started at the 5th International Congress on Mathematical Education, 1984, in Adelaide, Australia, and is concerned with a needed comprehensive approach to basic problems in the orientation, foundation and methodology of mathematics education as an interactive system comprising research, development, and practice (see [70], [71], [72]). Here I state:

Thesis 6

Mathematics education needs comprehensive approaches and meta-theories which should comprise an adequate philosophy of mathematics. For a meta-theory which is built on a systems approach based on human activity and social interaction, an adequate philosophy of mathematics should view mathematics itself as a system from the point of view of human object-related cooperative activities.

References

[1] Aebli, H. *Didactique psychologique — Application à la didactique de la psychologie de Jean Piaget.* Neuchâtel, 1951

[2] Bachelard, G *La formation de l'esprit scientific* Paris, 1938(1977)

[3] Bachelard, G. *La philosophie du non.* Paris, 1940(1973)

[4] Bauersfeld, H. Subjektive Erfahrungsbereiche als Grundlage einer Interaktionstheorie des Mathematiklernens und -lehrens. In: H. Bauersfeld et al. *Lernen und Lehren von Mathematik* IDM-Series "Untersuchungen zum Mathematikunterricht", Vol. 6. Köln 1983

[5] Beth, E.W., Piaget, J. *Mathematical epistemology and psychology* Dordrecht, 1966

[6] Behr, M.J., Lesh, R., Post, I R., Silver, E A Rational Number Concepts. In [47], 91-126

[7] Bohr, N. *Atomtheorie and Naturbeschreibung.* Göttingen, 1931

[8] Bloch, J B., Künzli, R., Lang, M. (Eds.) *Grundlagenknozepte der Wissenschaftskritik als unterrichtsstrukturierende Momente.* IPN-Arbeitsberichte 29. Kiel, 1978

[9] Brousseau, G The Central Role of the Didactical Contract for the Analysis and Construction of Didactical Situations in the Teaching and Learning of Mathematics. In [71]

[10] Brousseau, G. Les obstacles epistémologiques et les problèmes en mathématiques *Recherches en Didactique des Mathématiques.* Vol. 4. 2(1983)

[11] Bruner, J S. *The process of education.* New York, 1963

[12] Carpenter, T.P., Moser, J M. The Acquisition of Addition and Subtraction Concepts. In [47], 7-44

[13] Cawthron, E & Rowell, J A. Epistemology and Science Education *Studies in Science Education.* 5(1978), 31-59

[14] Cornu, B Apprentissage de la notion de limite — conceptions et obstacles. Thèse de Doctorat, 3ème Cycle Univ de Grenoble, 1983

[15] Damerow, P. Anmerkungen zum Begriff "Abstrakt" — Philosophiegeschichtliche und mathematikdidaktische Aspekte. In [68]

[16] Dienes, Z.P *Building up mathematics* London 1960

[17] Dienes, Z.P *Mathematics in primary education.* Unesco Instit, Hamburg, 1965

[18] Engfer, H J (Ed) *Philosophische Aspekte schulisher Fächer und pädagogischer Praxis.* München, 1978

[19] Esland, G M. Teaching and Learning as the Organization of Knowledge In [82]

[20] Fessel, A Die Auffassung mathematischer Theorien und ihr Einfluss auf den Unterricht. *Int. Zeitschrift f. Erziehungsw.* Bd IV(1959), 176ff

[21] Fichtner, B. Das Verhältnis von Wissensstruktur und Lernstruktur als Grundproblem der Didaktik. Münster, 1976

[22] Frech, H.W. Berufsvorbereitung und Fachsozialisation von Gymnasiallehrern *Studien u. Berichte des MPI f. Bildungsforschung* Vol 34A. Berlin, 1976

[23] Freudenthal, H. The Implicit Philosophy of Mathematics History and Education. *Proc. Int Contr Math.*. Warsaw, 1983 (Amsterdam, 1984)

[24] V. Glasersfeld, E. The Concept of Equilibration in a Constructivist Theory of Knowledge. In: F Benseler et al. (Eds) *Autopoiesis, communication, and society.* New York, 1980

[25] Goodman, N.D The knowing mathematician *Synthese* 60(1984), 21-38

[26] Heintel, P. *Modellbildung in der Fachdidaktik. Eine Philosophisch-wissenschaftstheoretische Untersuchung.* Klagenfurt, 1978

[27] v. Hentig, H (Ed.) *Wissenschaftsdidaktik. Sonderheft der Neuen Sammlung.* Göttingen, 1969

[28] van Hiele, P.M *Begrip en Inzicht.* 1973

[29] Hilton, P. Education in Mathematics and Science Today: The Spread of False Dichotomies. *Proc 3rd. Int. Congr Math. Ed.*. Karlsruhe, 1977, 75-97

[30] IDM-Arbeitsgruppe Mathematiklehrerbildung: Perspektiven für die Ausbildung des Mathematiklehrers. *IDM-Unters. z MU.* Vol. 2. Köln, 1982

[31] Jahnke, H.N. Zum Verhältnis von Wissensentwicklung und Begründung in der Mathematik — Beweisen als didaktisches Problem. *IDM-Materialien u. Studien,* Vol. 10 Bielefeld, 1978

[32] Jahnke, H.N., Otte M. (Eds.) *Epistemological and social problems of the sciences in the early Nineteenth Century* Dordrecht, 1981

[33] Jahnke, H.N., Otte, M., Schubring, G Mathematikunterricht und Philosophie In [68]

[34] Jahnke, H.N., Seeger, F Proportionalität. In: G. v Harten et al., *Funktionsbegriff und funktionales Denken.* Köln, 1986(a)

[35] Jahnke, H.N., Seeger, F Piaget und Selz — Logische Strukturen vs. Erkenntnismittel. In: H G. Steiner (Ed) *Grundfragen der Entwicklung mathematischer Fähigkeiten.* Köln, 1986(b)

[36] Jantsch, E *Die Selbstorganisation des Universums* München, 1982

[37] Karplus, R. Continuous Functions: Student's Viewpoints. *Europ J. Sci Ed.* 1(1979), 39-415

[38] Karplus, R., Pulos, S., Stage, E.K. Proportional Reasoning of Early Adolescents. In [47], 45-90

[39] Kuhn, T S. *The structure of scientific revolutions.* Chicago, 1962

[40] Kuyk, W *Complementarity in mathematics.* Dordrecht, 1977

[41] Lakatos, I. *Proofs and refutations* Cambridge, 1976

[42] Lakatos, I The method of analysis-synthesis In: Lakatos, I *Mathematics, science and epistemology.* Cambridge, 1978(a)

[43] Lakatos, I. A renaissance of empiricism in the recent philosophy of mathematics. In: Lakatos, I. *Mathematics science and epistemology* Cambridge, 1978(b)

[44] Lakatos, I., Musgrave, A (Eds) *Criticism and the growth of knowledge.* London, 1970

[45] Laugwitz, D. Der Streit um die Methode in der modernen Mathematik. *Neue Sammlung* 5(1965), H. 1

[46] Lenné, H *Analyse der Mathematikdidaktik in Deutschland.* Stuttgart, 1969

[47] Lesh, R., Landau, M. *Acquisition of mathematics concepts and processes.* New York, etc., 1983

[48] Minsky, M. K-lines: A Theory of Memory. *Cognitive Schience* 4(1980), No. 2, 117-133

[49] Mills, C W Language, Logic and Culture. *Am. Soc Rev.* 4(1939)

[50] Morrow, G R *Proclus. A commentary on the first book of Euclid's Elements* (Engl Transl. with Introd. and Notes) Princeton, 1970

[51] Otte, M. Komplementarität *IDM-Occ Paper* No. 42 Bielefeld, 1984

[52] Otte, M. The Work of E G. Judin (1930-1976) on Activity Theory in the Light of Recent Tendencies in Epistemological Thinking In: M. Hedegaard et al (Eds.) *Learning and teaching on a scientific basis.* Aarhus Universität, 1984

[53] Otte, M., Bromme, R. Der Begriff und die Probelme seiner Aneignung. In [8]

[54] Papert, S. *Mindstorms Children. computers and powerful ideas.* New York, 1980

[55] Pattee, H.H The Need for Complementarity in Models in Cognitive Behavior In: W.G Weimer & D.S Palermo (Eds.) *Cognition and the symbolic process*

[56] Piaget, J *The psychology of intelligence* New York, 1960

[57] Putnam, H. Mathematics without Foundation. *J. of Phil.* 64(1967), 5-22

[58] Reiss, M , Steiner, H G (Eds) *Mathematikkenntnisse — Leistungsmessung — Studierfähigkeit.* IDM-Reihe "Untersuchungen zum Mathematikunterricht" Vol. 9. Köln, 1984

[59] Richenhagen, G Kinder, Computer und eine neue Epistemologie? — Bemerkungen zu Seymour Paperts "Mindstorms" *IDM-Occ. Paper* No. 66 Bielefeld, 1985

[60] Schoenfeld, A H Beyond the Purely Cognitive: Belief Systems, Social Cognitions, and Metacognitions as Driving Forces in Intellectual Performance. *Cognitive Science* 7(1983), 329-363

[61] Sierpinska, A The concept of epistemological obstacle in teaching mathematics. Paper read at the CIEAEM-conference at Leiden (Holland), 1985

[62] Sneed, J.D. *The logical structure of mathematical physics* Dordrecht, 1971

[63] Steiner, H G Mathematische Grundlagenstandpunkte und die Reform des Mathematikunterrichts *Maht.-phys Semesterberichte* Bd. XII(1965), 1-22

[64] Steiner, H.G. Zur Methodik des mathematisierenden Unterrichts. *Anwendungsorientierte Mathematik in der Sekundarstufe* II, ed by H. Dörfler and R. Fischer, Klagenfurt, 1976, 211-245

[65] Steiner, H.G. What is Applied Mathematics? *Indian Journal of Mathematics Teaching* Vol. III(1976), No 1, pp 1-18

[66] Steiner, H.G. Relations between elementary and advanced mathematics In: H.G. Steiner (Ed.): The Education of Mathematics Teachers Proceedings of an ICMI-Symposium during the Intern. Congr. of Mathematicians, Helsinki, 1978. *IDM-Materialien und Studien* Vol. 15 Bielefeld, 1979

[67] Steiner, H.G. School Curricula and the Development of Science and Mathematics. *Int J. Math. Ed. Sci. Techn* 11(1980), 97-106

[68] Steiner, H G (Ed) Mathematik-Philosophie-Bildung. *IDM-Seires "Untersuchungen zum Mathematikunterricht"*. Vol. 4, Köln, 1982

[69] Steiner, H G The Process of Mathematization and the Social Dimension of Mathematics — Epistemological and Didactical Considerations. *IDM-Occ. Pap.* No. 41 Bielefeld 1984. Also: *Séminaire de Didactique des Mathématiques et l'Informatique* No. 44 Grenoble, May 1983

[70] Steiner, H G. Theory of Mathematics Education (TME): An Introduction *For the Learning of Mathematics* 5(1985), No 2 11-17

[71] Steiner H G et al Theory of Mathematics Education (TME). ICME 5 - Topics and Miniconference. *IDM-Occ Paper* No 54 Bielefeld, 1984

[72] Steiner, H G. Implications for Scholarship of a Theory of Mathematics Education *IDM-Occ. Paper* No. 69. Bielefeld 1985

[73] Steiner, H.G. Philosophische Aspekte der Mathematik und ihre Wechselwirkungen mit Theorie und Praxis des Mathematikunterrichts. In: *Beiträge zum Mathematikunterricht 1985* Bad Salzdefurth, 1985, 305-308

[74] Steiner, H.G. Philosophie und die Bedingtheit der Mathematik *Math. Semesterberichte* 32(1985), 110-130

[75] Thom, R Modern Mathematics: Does it Exist? In: A.G. Howson (Ed.) *Developments in mathematical education.* Cambridge, 1973, 194-209

[76] Thompson, A G. The Relationship of Teachers' Conceptions of Mathematics and Mathematics Teaching to Instructional Practice. *Ed. Stud. Math* 15(1984), 195-127

[77] Volk, D Philosophie im Mathematikunterricht. Der Mathematikunterricht als Anwendungsgebiet der Philosophie. *ZDM* 9(1977), 121-130

[78] Wand, H. *From mathematics to philosophy* London, 1974

[79] Welzel, A. Zur Spezialisierung allgemeiner kognitiver Fähigkeiten. Das Affinitätskonzept - entwickelt am Beispiel "Affinität zur Mathematik". In [68]

[80] Wittenberg, A.I. *Bildung und Mathematik.* Stuttgart, 1963

[81] Wuchterl, K. Die neuesten wissenschaftstheoretischen Entwicklungen und ihre Auswirkungen auf die Philosophie der Mathematik sowie deren didaktische Realisierung. In [68]

[82] Young, M.F.D. (Ed.) *Knowledge and control: new directions for the sociology of education* London, 1971

There is a philistine and self-deceptive assumption among the scientifically inclined that all avenues other than their own amount to "anecdote". But if a psychologist, however self-consciously scientific, is in need of personal insight — if his marriage has collapsed or his children hate him — he does not turn to the professional journals in search of it. He goes to Donne or Chekhov, Freud or Laing. As receptacles of our knowledge about people, in other words, both literary and clinical modes at present show every sign of being superior to science; and it must in principle be the case, I think, that each of these rival routes of access to knowledge about people is valid after a fashion. What we do not know is how their respective merits can be weighed, nor what hostages we give as we choose to follow one route as opposed to another. To put the matter in crudely economic terms, there are patterns of cost and benefit associated with the experiment, the clinical enquiry, the novel, the poem — and, of course, the play and film too. But we do not yet know what these are.

Liam Hudson
